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Theory of Densification of Dry Snow on High Polar Glaciers

U. S. ARMY SNOW ICE AND PERMAFROST RESEARCH ESTABLISHMENT Corps of Engineers
Theory of Densification of Dry Snow on High Polar Glaciers

by Henri Bader
PREFACE

This paper presents a theory of densification of dry snow on high polar glaciers, which could be useful in an analysis of the data obtained from snow pits in Greenland, Ellesmere Island, and Antarctica. The study was made by Dr. H. Bader, chief scientist, USA SIPRE.*

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SUMMARY

The relations between rate of snow accumulation, snow density, snow age, and depth below the surface are formulated in terms of a compactive viscosity factor, which is assumed to be a function of only density and temperature, with parameters which will vary with the snow-type sequences of different climatic areas of high polar glaciers. The theory could be useful in an analysis of the data obtained from the large number of snow pits in Greenland, Ellesmere Island, and Antarctica.
THEORY OF DENSIFICATION OF DRY SNOW
ON HIGH POLAR GLACIERS

by
H. Bader

Introduction

The process of densification of snow in the dry-snow zones of glaciers is one of the interesting problems of polar glaciology. Measurements of snow densities and temperatures to depths of from 3 to 5 m have been made at a large number of points in Antarctica, Greenland, and Ellesmere Island during the last few years, particularly during the International Geophysical Year. Measurements to 30 m depth are available from a dozen localities, and to between 200 m and 400 m from three or four places.

To obtain generalized knowledge from this wealth of data, a great deal of computational work will have to be done, based on an adequate theory of snow densification. The following is an attempt to formulate such a theory.

The Sorge depth-density curve (See Sorge, 1935, 1938; Bader 1954)

Sorge's law states that, at constant rate of snow accumulation and constant temperature, the function $\gamma = F(h)$ is invariant with time.

Notation:

- $h =$ depth below snow surface (cm)
- $\gamma =$ snow density (g/cm$^3$)
- $A =$ rate of snow accumulation (g/sec-cm$^2$)
- $t =$ time elapsed since deposition on surface, i.e. age of snow (sec)
- $\sigma =$ load of overlying snow (g/cm$^2$)
- $V =$ vertical velocity downwards in relation to snow surface (cm/sec)
- $v =$ specific velocity of densification (sec$^{-1}$)

Sorge's law leads to the following relations:

$$\int_{0}^{h} \gamma dh = At$$

(1)

$$\gamma = \frac{d\sigma}{dh}$$

(2)

$$V = \frac{dh}{dt} = \frac{A}{\gamma}$$

(3)

$$V = \frac{dv}{dh} = \frac{A}{\gamma^2} \frac{d\gamma}{dh} = -\frac{1}{\gamma} \frac{d\gamma}{dt}$$

(4)

Laboratory work shows that, for stresses smaller than 1000 g/cm$^2$, the assumption of validity of Newtonian viscosity is quite reasonable in snow mechanics. The following analysis can then be used when $\sigma < 1000$ g/cm$^2$, which usually occurs at a depth of between 15 and 20 m.

We define a "compactive viscosity factor".$$
\eta_c = -\frac{\sigma}{v}$$

(5)

$\eta_c > 0$, because $v$ is always negative, except at the surface, where it is zero. For a given snow-type sequence, $\eta_c$ is a function of only density and temperature. For the present we consider the conditions at constant temperature

$$\eta_c = f(\gamma).$$

(6)

As snow densifies under load, it changes not only in density, but also in grain size and structure. The main purpose of the analysis of the many depth-density profiles must be to determine the function (6), and to study how it varies with climatic area, rate of accumulation, and, reduced to a standard temperature, how it varies with mean local temperature. The function can then perhaps become one of the defining characteristics
of dry-snow sequences. Once it is known, the average rate of accumulation can be calculated in the many instances, especially in Antarctica, where it cannot be directly measured in pits.

Eq 5 and 6 are written as:
\[ \sigma = -v f(y) \]  
(7)
and using eq 1 and 4:
\[ \sigma = A t = \frac{f(y)}{y} \frac{dy}{dt} \]  
(8)
which integrates to:
\[ \frac{A t^2}{2} = \frac{\sigma^2}{2A} = \int_{\gamma_0}^{\gamma} \frac{f(y)}{y} dy \]  
(9)
so that:
\[ t = \sqrt{\frac{2}{A} \int_{\gamma_0}^{\gamma} \frac{f(y)}{y} dy} \]  
(10)
\( \gamma_0 \) is the density at the surface, where \( t = 0 \).

Eq 10 relates the age of the snow, since it fell, to its density. We can also formulate the relation between \( h \) and \( t \) by integrating eq 3:
\[ h = A \int_0^t \frac{1}{y} \frac{dy}{dt} dt \]  
(11)
and if eq 10 can be written (tabulated) as \( y = G(t) \),
\[ h = A \int_0^t \frac{1}{G(t)} dt \]  
(11a)
Finally, from eq 4, 7, 1, and 9 we deduce:
\[ \sigma = \frac{A f(y)}{\gamma} \frac{dy}{dh} = \frac{\sqrt{2A}}{\sqrt{\gamma - \gamma_0}} \int_{\gamma_0}^{\gamma} f(y) dy \]  
(12)
Separating the variables and integrating between corresponding limits:
\[ h = \sqrt{\frac{A}{2}} \int_{\gamma_0}^{\gamma} \frac{f(y) dy}{\sqrt{\gamma^2 - \int_{\gamma_0}^{\gamma} f(y) dy}} \]  
(13)
This is the relation between \( h \) and \( \gamma \), which is the Sorge depth-density curve (modified by a temperature correction to be treated later) in terms of snow mechanics through the use of \( n_c = f(y) \). Since \( A, \gamma_0 \), and \( f(\gamma_0) \) are all greater than 0, it follows from eq 12 that \( \left( \frac{dy}{dh} \right)_{h=0} = 0 \), because \( \sigma = 0 \) at the snow surface.
A theoretically satisfactory Sorge curve must therefore intersect the surface at right angles, because the slope $dy/dh$ decreases to zero there. One of the important features of eq 13 is that it contains the value of $A$, the rate of accumulation. Consider, for instance, two Sorge curves from different localities. At one locality $A_1$ is known, but not $A_2$ at the other. If the temperature is similar, and $f(Y)$ is the same, then by picking $h_1$ and $h_2$ for a given density from the two curves, it follows from eq 13 that

$$h_1 = \sqrt{\frac{A_1}{A_2}}$$

from which the unknown $A_2$ can be calculated. But the relation 13a is unlikely to be very useful, because the snow-type sequence, and consequently $f(Y)$, must vary with $A$. The smaller the rate of accumulation, the longer it takes a snow to reach a given density; and time is an important factor in snow metamorphism.

**Generalization for time-variable rate of accumulation**

In some cases, where the annual layers can be identified on the face of a snow pit, or on the cores from a drill hole, it will be seen that the rate of accumulation changes with time.

Let the time function be $E(t)$. Sorge's law, based on a constant $A$, now no longer holds, and the depth-density curve will change with time. Now $h$ and $t$ are independent variables, and $\gamma$ is the dependent variable. At time $t$, the velocity of flow of snow is $V$ through the horizontal plane at depth $h$, and $(V + \gamma V)$ through the plane at depth $(h + \delta h)$. The mass flow is $\gamma V$ at depth $h$, and $(\gamma + \delta \gamma)(V + \gamma V)$ at depth $(h + \delta h)$. The mass of snow lying between $h$ and $(h + \delta h)$ is $\gamma \delta h$, and we can write its change with time:

$$\frac{\partial (\gamma \delta h)}{\partial t} = \delta h \frac{\partial \gamma}{\partial t} = \gamma V - (\gamma + \delta \gamma)(V + \gamma V) = -\gamma \delta V - \gamma \delta \gamma$$

or

$$\frac{\partial \gamma}{\partial t} + \gamma \frac{\partial V}{\partial h} + V \frac{\partial \gamma}{\partial h} = 0$$

or

$$\frac{\partial \gamma}{\partial t} = -\frac{\partial}{\partial h} (\gamma V).$$

Now $\gamma = \frac{\partial \sigma}{\partial h}$, from $\sigma = \int_0^h \gamma dh$

so that $\frac{\partial \gamma}{\partial t} = \frac{\partial^2 \sigma}{\partial t \partial h} = \frac{\partial}{\partial h} \left( \frac{\partial \sigma}{\partial t} \right)$

and eq 14 takes the form

$$\frac{\partial}{\partial h} (\gamma V + \frac{\partial \sigma}{\partial t}) = 0.$$ 

This means that $(\gamma V + \frac{\partial \sigma}{\partial t})$ is constant with respect to $h$, i.e. it is a function of time only. But we know that $(\frac{\partial \sigma}{\partial t})_{h=0} = 0$, and $(\gamma V)_{h=0} = \gamma_0 V_0 = E(t)$ where $E(t)$ is the time function of the rate of accumulation (continuous function $> 0$). The conclusion is that

$$\frac{\partial \sigma}{\partial t} = E(t) - \gamma V.$$
This is the statement that the time rate of change of snow load at a given depth is equal to the rate of accumulation minus the rate at which snow mass flows through the horizontal plane at the given depth.

Incidentally, the same result can be obtained by writing the identity:

$$\frac{d\sigma}{dt} = \frac{\partial \sigma}{\partial t} + \frac{dh}{dt} \frac{\partial \sigma}{\partial h}$$

or

$$\frac{d\sigma}{dt} = \frac{\partial \sigma}{\partial t} + \frac{dh}{dt} \frac{\partial \sigma}{\partial h}$$

(17)

Since $$\sigma = \int_{t_0}^{t} E(t) dt$$, $$\frac{d\sigma}{dt} = E(t)$$, $$\frac{dh}{dt} = V$$, and $$\frac{\partial \sigma}{\partial h} = \gamma$$, eq 17 is seen to be identical with 16.

The flow rate $$V$$ cannot be written in terms of partial derivatives of the independent variables, but we know that:

$$V = \frac{\partial V}{\partial h} = -\frac{\sigma}{f(\gamma)}$$

(18)

which integrates to:

$$V - V_0 = -\int_0^h \frac{\sigma}{f(\gamma)} dh.$$  

Since $$(V)_{h=0} = V_0 = \frac{E(t)}{\gamma_0}$$

$$V = \frac{E(t)}{\gamma_0} - \int_0^h \frac{\sigma}{f(\gamma)} dh.$$  

(19)

Let the surface density $$\gamma_0$$ be constant, independent of the rate of accumulation. The second term of eq 19 represents the rate of thinning by densification of the layer of snow lying between the initial surface and depth $$h$$, while the first term is the rate of deposition of new snow onto the initial surface, i.e., the upward velocity of the $$h = 0$$ reference plane. The velocity $$V$$ of a particle at depth $$h$$ (in reference to the plane $$h = 0$$) is the difference between the two mentioned rates. $$V$$ can be positive, zero, or negative. It will for instance be negative (motion upwards) if it stops snowing.

Substituting from eq 18 and 19 into eq 14, we obtain:

$$\frac{\partial \gamma}{\partial t} = \frac{V \sigma}{f(\gamma)} - \frac{\partial \gamma}{\partial h} \left( \frac{E(t)}{\gamma_0} - \int_0^h \frac{\sigma}{f(\gamma)} dh \right).$$  

(20)

This equation can be useful in estimating the change of the depth-density curve with time. If $$f(\gamma)$$ is known, the rate of change of density at any given depth, i.e. the rate of horizontal displacement $$\frac{\partial \gamma}{\partial t}$$ of any point on the depth-density curve, can be calculated from pit data.

Path of a particle on the surface $$\gamma = B(h, t)$$

We will now follow the path of a particle, or better said, of a small volume of snow which fell at time $$t_0$$. Time is the single independent variable, and both density and depth are expressible in terms of time.
The snow load pressing on the particle is not only equal to the integral of density over depth, but also of rate of accumulation over time.

\[
\sigma = \frac{h}{t_0} \gamma dh = \int_{t_0}^{t} E(t) \, dt
\]

(21)

and using the previously encountered relation:

\[
-\nu = \frac{1}{\gamma} \frac{d\gamma}{dt} = \frac{\sigma}{f(\gamma)}
\]

we can write:

\[
\frac{f(\gamma)}{\gamma} \, d\gamma = \sigma dt = dt \int_{t_0}^{t} E(t) \, dt.
\]

(22)

While time changes from \(t_0\) to \(t\), density changes from \(\gamma_0\) to \(\gamma\), so that:

\[
\int_{\gamma_0}^{\gamma} \frac{f(\gamma)}{\gamma} \, d\gamma = \int_{t_0}^{t} \int_{t_0}^{t} E(t) \, dt \, dt.
\]

(23)

This is implicitly the function \(\gamma = F_1(t)\).

In order to obtain the function \(h = F_2(t)\), we take the last term of eq 19 and write the identity:

\[
\int_{t_0}^{h} \frac{\sigma}{f(\gamma)} \, dh = \int_{t_0}^{t} \frac{\sigma}{f(\gamma)} \, dh = \int_{t_0}^{t} \frac{\sigma}{f(\gamma)} \, dh
\]

(24)

Substituting eq 24 into 19, deriving with respect to \(t\), and using \(\frac{dV}{dt} = \frac{d^2h}{dt^2}\), we obtain the differential equation:

\[
\frac{d^2h}{dt^2} + \frac{\sigma}{f(\gamma)} \frac{dh}{dt} \frac{E'(t)}{\gamma_0} = 0.
\]

(25)

\(E'(t)\) is the first derivative of \(E(t)\) with respect to \(t\). Eq 25 is a textbook example of integration, with the following solution:

Let \(\frac{\sigma}{f(\gamma)} = H(t)\)

\[
h = \frac{1}{\gamma_0} \int_{t_0}^{t} \left[ \exp - \int_{t_0}^{t} H(t) dt \right] \int_{t_0}^{t} E'(t) \left[ \exp - \int_{t_0}^{t} H(t) dt \right] dt + C \int_{t_0}^{t} \left[ \exp - \int_{t_0}^{t} H(t) dt \right] dt
\]

(25a)

This is the desired relation \(h = F_2(t)\).

When \(E'(t) = 0\), i.e. \(E(t) = A = \text{constant}\), then the first term of eq 25a is zero and the solution is

\[
h = C \int_{t_0}^{t} \left[ \exp - \int_{t_0}^{t} H(t) dt \right] dt
\]

(25b)
Figure 1.

Eq 25b must be identical with eq 11, and, because $\frac{\gamma_0}{Y} = \exp \int H(t) dt$ from eq 7 and 4, it follows that $C = \frac{A}{Y_0}$ in eq 25a and 25b. Eq 25a can now be rewritten as

$$h = \frac{1}{Y_0} \int \int \frac{1}{t_0} Y E'(t) dt dt + A \int \frac{1}{t_0} Y dt. \quad (25c)$$

Here $E(t)$ is given the form $E(t) = A + D(t)$.

The surface $\gamma = B(h, t)$ could be constructed by tracing the paths, which are curves on the surface, of a number of snow particles falling at successive time increments (see Fig. 1).

**Effect of temperature.**

If $\eta_{c_1}$ is the compactive viscosity factor at temperature $T_1$, and $\eta_{c_2}$ the factor at another temperature $T_2$, then

$$\frac{\eta_{c_1}}{\eta_{c_2}} = \exp \frac{E}{R} \left( \frac{1}{T_2} - \frac{1}{T_1} \right). \quad (26)$$

This form, adopted from chemical reaction kinetics, is justified by the results of laboratory snow mechanics. $E$ is an activation energy, $R$ the gas constant, and $T$ the absolute temperature.

Near the surface there is a large annual variation of temperature. If $T_m$ is the mean annual snow temperature (the temperature measured at 10-m depth at any time of the year is quite close to $T_m$) and $\Delta T$ the annual amplitude, $T$ varies from $T_m + \Delta T$ to $T_m - \Delta T$. 
By using a sine function as a useful approximation of the temperature cycle, we can calculate an equivalent temperature \( T_e \) such that:

\[
\text{exp} - \frac{E}{RT_e} = \frac{1}{t_1} \int_{0}^{t_1} \left( \text{exp} - \frac{E}{R(T_m + \Delta T \sin 2\pi \frac{t}{t_1})} \right) dt
\]

(27)

where \( t_1 \) is one year, and \( 2\pi t/t_1 \) is expressed in radians.

Thus if a snow layer is subjected to an annual temperature cycle of amplitude \( \Delta T \) around \( T_m \), its increase in density will be equal to that of an identical snow layer held at the constant temperature \( T_e \). It turns out that \( T_e > T_m \). At a point on the Greenland ice sheet, for instance, \( T_m = -24.5^\circ C \) and \( T_e = -19^\circ C \) at the snow surface, a very significant difference.

In order to determine the viscosity factor as a function of density, it is necessary to make the temperature correction from \( \eta_{cT_e} \) to \( \eta_{cT_m} \). Given the depth-density curve, the thermal diffusivity as a function of density \( \eta_{cT_m} \) and \( \Delta T \) at the surface, then \( \Delta T \) can be calculated as a function of density. \( T_e \) can then also be calculated as a function of density using eq 27, and \( \eta_{cT_e} \), obtained from snow pit data, can be corrected to \( \eta_{cT_m} \), by means of eq 26. \( \eta_{cT_m} = f(y)T_m \) can then be formulated, and the structure of the function and the values of its parameters are the first fruit of the investigation. When a sufficient number of cases have been calculated, they must all be reduced to an arbitrary standard temperature \( T_s \) for a study of the significance of the parameters:

\[
\eta_{cT_s} = \eta_{cT_m} \text{exp} - \frac{E}{R} \left( \frac{1}{T_m} - \frac{1}{T_s} \right)
\]

(28)

A source of error in the temperature correction must be mentioned. This results from \( T_e \) being the equivalent temperature at a given depth for one whole annual cycle, during which time snow layers move down with respect to the surface. Thus a layer deposited in the summer may already be protected from the extreme low temperature of the following winter. The effect is one of damping of amplitude, and will produce an error, positive or negative, which is the larger, the greater the rate of accumulation.

**Computational procedure**

As an idealization of the natural process, the foregoing theory is a first approximation analog. It does not, for instance, consider the facts that snowfall is not continuous, that winter and summer snows can be quite different, that the density of the surface snow varies with weather conditions, that there is sometimes some melting in the dry-snow regions, and that the annual mean temperature fluctuates. A plot of the points on a depth-density graph covers a rather wide band, giving some choice in tracing a mean depth-density curve.

Considering the rather large amount of computational labor to be expended in obtaining the function

\[
\eta_{cT_s} = f(y)T_s
\]

for even a single profile, the development of a good routine computational procedure will be a major task in itself. One will have to work with a set of profile data and proceed according to its aspect.
As a guideline, it is mentioned that the occurrence of the expression \( \int_{y_0}^{y} f(y) \, dy \) in the important equations 10, 13 and 23 will make it desirable to choose \( f(y) \) in a form such that \( f(y) \) is integrable. Since we are yet quite unable to derive a theoretically satisfactory expression for the compactive viscosity factor from a model of the snow structure, there is a freedom of choice here.

The value of \( \frac{\sigma^2}{2A} \) is equal to the above mentioned integral, according to eq 9, and can be accurately determined. Thus eq 9 can serve as a starting point. One cannot a priori determine at which stage of computational procedure the important temperature corrections are to be made.

SIPRE laboratory work indicates that a good value for the activation energy \( E \) is 13,400 cal per mol. The gas constant \( R = 1.987 \) cal/mol degree.

Densification under high loads

The present theory is probably applicable only if the snow load is smaller than about 1000 g/cm\(^2\). At the SIPRE station in Greenland (77\(^\circ\) N 56\(^\circ\) W), the corresponding depth is 20 m and the density 0.59 g/cm\(^3\), \( T_m = -24.5 \)C and \( A = 40 \) g/cm\(^2\)-yr. At the South Pole the corresponding depth is 21 m, the density 0.56 g/cm\(^3\), \( T_m = -51 \)C and \( A = 61.5 \) g/cm\(^2\)-yr.

Thus, the theory will be applicable to all data except that from very deep pits and corings. Under higher loads the rate of densification is no longer proportional to the load, but is some stronger function. The theory then becomes more complicated. When the density exceeds approximately 0.82 g/cm\(^3\), the air in the pores becomes compressed and this pressure must probably be subtracted from the load stress in order to obtain the effective pressure.

References

