Simplified Method for Estimating Refraction and Shoaling Effects on Ocean Waves

by

C.M. McClenan

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The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.
This report presents a nomogram for the computation of combined refraction-shoaling coefficients for straight and parallel bottom contours. Its use is illustrated by two examples. The nomogram permits a rapid solution of idealized refraction phenomena. The technique provides a useful first estimate to the true solution, and for many problems, it provides as accurate a solution as other time-consuming methods.
This report is published to provide coastal engineers with a more convenient and rapid method for estimating refraction and shoaling effects in the study of ocean wave phenomena. The work was carried out under the wave mechanics research program of the U.S. Army Coastal Engineering Research Center (CERC).

The report was prepared by C.M. McClenan, a former CERC Oceanographer, with the assistance and supervision of Dr. D.L. Harris, Chief, Oceanography Branch, Research Division, CERC.

Comments on this publication are invited.

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JAMES L. TRAYERS
Colonel, Corps of Engineers
Commander and Director
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SYMBOLS AND DEFINITIONS

\( b \) = distance between wave orthogonals as measured along wave crest

\( b_c \) = distance between wave orthogonals as measured along bottom contour

\( b_o \) = distance between wave orthogonals as measured along wave crest in deep water

\( C \) = local wave speed

\( C_o \) = wave speed in deep water

\( C_G \) = local group velocity of wave train

\( C_{G0} \) = group velocity of wave train in deep water

\( d \) = local depth

\( K_s \) = shoaling coefficient

\( H \) = local wave height

\( H_o \) = wave height in deep water

\( K_R \) = refraction coefficient

\( L \) = local wavelength

\( L_o \) = wavelength in deep water

\( P \) = local power transmitted by waves in the direction of wave travel

\( P_o \) = power transmitted by waves in deep water

\( T \) = wave period

\( \alpha \) = local angle between wave crest and bottom contour (bottom contours parallel to coastline)

\( \alpha_o \) = angle between wave crest and bottom contour in deep water

\( \gamma \) = specific weight of water
SIMPLIFIED METHOD FOR ESTIMATING REFRACTION AND
SHOALING EFFECTS ON OCEAN WAVES

by

C.M. McClenan

I. INTRODUCTION

Most schemes for calculating wave refraction effects are based on linear wave theory, and most are also based on the assumptions that the only water motions present are those due to the waves being refracted and that no reflection or attenuation of wave energy occurs as the wave approaches the coast. Many refraction procedures are also based on the assumption that bottom contours are parallel, at least within a small region. When all of these assumptions are accepted, wave refraction can be computed by Snell's Law; the wave direction at any point depends only on the wave direction in deep water; the wave period, and the water depth at the shallow water point of interest. The wave shoaling coefficient depends on the local water depth. Thus, since local wave velocity depends on the wave period and the local depth, both the direction of the refracted wave and the ratio of the height of the refracted wave to the height of the deepwater wave are functions of the deepwater wave direction, the wave period, and the local water depth. When these three parameters are known, the effects of refraction and shoaling can be quickly determined from the simple nomogram described in this report (Fig. 1a).

The assumptions required to derive the nomogram preclude considering the effects of currents, whether generated by tides, winds, or the waves; the effects of interaction between waves of different frequency or phase; and any effects of wave energy reflection from nearshore structures or attenuation by friction or backscattering due to irregularities in the seabed. Although all of these phenomena occur in nature, they are neglected in most wave refraction calculation schemes.

The nomogram technique also neglects convergence or divergence of wave energy due to curvature of the bottom contours, an additional simplification not required in many of the more sophisticated techniques for using linear wave theory in the calculation of refraction effects. However, the effects of curvature in the contours are, in many cases, no more important than the effects of currents, reflection, backscattering, and the exchange of energy between waves of different periods, which are neglected in most refraction models. Therefore, refraction calculations from the nomogram approximate the natural phenomena nearly as accurately as many other available calculation techniques.

The nomogram gives a quick estimation of the effects of refraction at many locations and can be used to obtain insight into the effects of refraction and shoaling on a sea state containing wave energy at different frequencies.
Figure 1a. Refraction nomogram. Dashed curves refer to the angle of wave propagation; solid curves refer to (H/H₀).
II. DESCRIPTION OF NOMOGRAM

The nomogram (Figs. 1a and 1b) combines estimates of the refraction coefficient which could be obtained from the graph developed by Johnson, O'Brien, and Isaacs (1948), and the shoaling coefficient which could be obtained from the tables in the Shore Protection Manual (U.S. Army, Corps of Engineers, Coastal Engineering Research Center, 1973).

The nomogram may be regarded as a "refraction slide rule." Figure 1a contains two sets of curves. The solid curves, which are concave upward, are plots of the ratio of local wave height to the deepwater wave height \( \frac{H}{H_0} \), as functions of \( \frac{d}{L_0} \), the ratio of the local water depth to deepwater wavelength. Each solid curve is associated with a constant deepwater wave angle \( \alpha_0 \) between the wave crest and the bottom contour. All of these curves converge toward unity in deep water. As waves move from deep water to the shore from any direction, the height first decreases and then increases.

The dashed curves in Figure 1a, which are concave downward, are plots of the local angle \( \alpha \) between the bottom contour and wave crest, shown along the right vertical axis, as a function of the ratio \( \frac{d}{L_0} \). Each dashed curve is associated with a constant \( \alpha_0 \) value. As the wave moves into shallower water (i.e., \( \frac{d}{L_0} \) gets smaller), \( \alpha \) also becomes smaller. The horizontal scale on the top of Figure 1a is the local water depth \( d \) and is used in conjunction with Figure 1b.

Figure 1b, when constructed on transparent material, becomes an overlay which can be considered as the cursor or "D scale" of the refraction slide rule. When Figure 1b is properly positioned over Figure 1a, solutions to refraction problems can be rapidly obtained. The scale on the top of Figure 1b is wave period in seconds. When the index line, the heavy line at 14 seconds (actually 13.975 seconds) is superimposed over the local water depth, as given in the top of Figure 1a, the vertical line for any period will cross the curves for local relative wave height and local wave angle at the proper point for waves in that period and depth. Thus, if deepwater conditions are known, \( \alpha \) and \( \frac{H}{H_0} \) for waves having a period between 1 and 30 seconds at a designated depth, can be obtained from one setting of the slide rule. Conversely, if wave parameters are known at a fixed shallow water point, deepwater wave conditions can be estimated. A simple extension of this procedure permits wave conditions at one shallow water point to be estimated from known wave conditions at some other shallow water point. The known shallow water conditions can provide the deepwater conditions; and then, with known deepwater conditions, conditions can be evaluated at any other depth. Example calculations are presented in a later section.

III. DERIVATION

The curves plotted in Figure 1a are based on the standard approach to wave refraction and involve the following assumptions:
Figure 1b. Refraction slide rule overlay (cursor).
(a) The small amplitude wave theory is applicable.

(b) The waves refract according to Snell's Law.

(c) The coastline and bottom contours are straight and parallel.

(d) There is no transfer of energy along wave crests.

(e) The effects of currents and nonlinear processes on refraction are negligible.

To solve for the relative wave height \( \frac{H}{H_0} \) from the combined effects of shoaling and refraction, it has been assumed that the power transmitted shoreward remains constant between orthogonals to the wave crests. According to linear wave theory, the power transmitted in deep water is:

\[
P_d = C_{Go} H_0^2 b_0 \gamma / 8 ,
\]

where

\[C_{Go} = \text{deepwater group velocity},\]
\[H_0 = \text{deepwater wave height},\]
\[b_0 = \text{length of crest between orthogonals in deep water},\]
\[\gamma = \text{specific weight of the water};\]

the power transmitted in shallow water is:

\[
P = C_G H^2 b \gamma / 8 ,
\]

where

\[C_G = \text{group velocity},\]
\[H = \text{wave height},\]
\[b = \text{length of crest between orthogonals in shallow water}.\]

The above assumption requires:

\[
C_G H^2 b \gamma / 8 = C_{Go} H_0^2 b_0 \gamma / 8 .
\]

Solving for \( \frac{H}{H_0} \) yields:

\[
\frac{H}{H_0} = \sqrt{\frac{C_G}{C_{Go}}} \sqrt{\frac{b_0}{b}} ,
\]
Figure 2. Wave refraction with coastline and bottom contours (straight and parallel).
where, by definition,

\[ \sqrt{\frac{C_{Gc}}{C_G}} = K_S = \text{shoaling coefficient, and} \]  

\[ \sqrt{\frac{b_0}{b}} = K_R = \text{refraction coefficient} \]  

The shoaling coefficient is a measure of the change in wave height resulting solely from a change in water depth. The refraction coefficient is a measure of the change in wave height resulting from curvature in the wave rays. If the shoreline and bottom contours are straight and parallel, as assumed here, all wave rays for a given wave period (T), and a given angle between wave crest and bottom contour in deep water (\(\alpha_0\)), can be obtained from one orthogonal by simple translation without rotation in a direction parallel to the bottom contours. The distance between two adjacent orthogonals, \(b_0\), measured parallel to the bottom contours, is constant along the entire length of the two orthogonals (Fig. 2).

Near the shore the wave rays are nearly orthogonal to the bottom contours, and the distance between the designated rays, \(b\), tends toward \(b_0\). Right triangles can be used to obtain an expression for \(b\) in terms \(\alpha\) and \(\alpha_0\). First consider triangle ABC in Figure 2. Side AC (the hypotenuse) has a length \(b_0\). Side AB has length \(b\). For triangle ABC:

\[ b = b_0 \cos \alpha \]  

Similarly for triangle DEF:

\[ b_0 = b_0 \cos \alpha_0 \]  

Solving both equations for \(b_0\) and equating them:

\[ \frac{b_0}{\cos \alpha_0} = \frac{b}{\cos \alpha} \]  

which can be rearranged as

\[ \frac{b_0}{b} = \frac{\cos \alpha_0}{\cos \alpha} \]  

Taking the square root of both sides given two expressions for the refraction coefficient:

\[ \sqrt{\frac{\cos \alpha_0}{\cos \alpha}} = \sqrt{\frac{b_0}{b}} = K_R \]
Since $\alpha_o$ is known, $\alpha$ can be obtained by using Snell's Law:

$$\frac{\sin \alpha_o}{\sin \alpha} = \frac{C_o}{C},$$  \hspace{1cm} (12)$$

where

$$C = \text{wave speed},$$

$$C_o = \text{wave speed in deep water}.$$

Solving for $\alpha$ yields:

$$\alpha = \arcsin \left( \frac{C}{C_o} \sin \alpha_o \right).$$  \hspace{1cm} (13)$$

The shoaling coefficient, $K$, equation (5), may be determined from the small amplitude wave theory equation for group velocity:

$$C_G = \frac{C}{2} \left[ 1 + \frac{4\pi d/L}{\sinh (4\pi d/L)} \right],$$  \hspace{1cm} (14)$$

where

$$d = \text{water depth},$$

$$L = \text{wavelength}.$$

Linear wave theory also provides the following relationship between group and phase velocity in deep water:

$$C_{G_o} = \frac{C_o}{2}.$$  \hspace{1cm} (15)$$

Equations (14) and (15) can then be used to obtain an expression for the shoaling coefficient:

$$K = \left( \frac{C_{G_o}}{C_G} \right)^{\frac{1}{2}} = \left( \frac{C_o}{C} \right)^{\frac{1}{2}} \left[ 1 + \frac{4\pi d/L}{\sinh (4\pi d/L)} \right]^{-\frac{1}{2}}.$$  \hspace{1cm} (16)$$

IV. DISCUSSION

This section is devoted to an explanation of the reasons why both plots, $H/H_0$ and $\alpha$ versus $d/L_o$ for constant values of $\alpha_o$, have the straight line and constant slope characteristics in the region of small $d/L_o$ values.

As shown by Figure 1a, when $d/L_o$ becomes less than 0.1, the $\alpha$ versus $d/L_o$ plots (the dashed lines) are reasonably straight and parallel with a slope of about $+\frac{1}{2}$. This result can be derived from the equations presented in Section III, Derivation.
According to linear theory, the surface wave speed is:

\[ C = \sqrt{\frac{gL}{2\pi}} \tanh \left( \frac{2\pi d}{L} \right) . \]  (17)

For large values of \( \frac{2\pi d}{L} \), \( \tanh \left( \frac{2\pi d}{L} \right) \) approaches unity. Therefore, the wave speed in deep water is given by:

\[ C_0 = \sqrt{\frac{gL_0}{2\pi}} . \]  (18)

For small values of \( \frac{2\pi d}{L} \), \( \tanh \left( \frac{2\pi d}{L} \right) \rightarrow 2\pi d/L \), and

\[ C = \sqrt{gd} . \]  (19)

Combining equations (18) and (19) for small \( d/L_0 \) yields:

\[ \left( \frac{C}{C_0} \right) \sin \alpha_0 = \left[ \sqrt{\frac{2\pi d}{L_0}} \sin \alpha_0 \right] = \left[ \sqrt{2\pi \sin \alpha_0} \right] \left( \frac{d}{L_0} \right)^{\frac{1}{2}} . \]  (20)

Substitution from equation (20) into equation (13) yields

\[ \alpha = \arcsin \left[ \sqrt{2\pi \sin \alpha_0} \left( \frac{d}{L_0} \right)^{\frac{1}{2}} \right] . \]

For sufficiently small \( \alpha \), the trigonometric function may be represented by the first term in the power series expansion, or

\[ \alpha = \left[ \sqrt{2\pi \sin \alpha_0} \right] \left( \frac{d}{L_0} \right)^{\frac{1}{2}} , \]  (21)

and

\[ \log(a) = \log \left( \sqrt{2\pi \sin \alpha_0} \right) + \frac{1}{2} \log \left( \frac{d}{L_0} \right) . \]  (22)

It is seen that for constant values of \( \alpha_0 \), and small \( d/L_0 \), the plot of \( \alpha = \alpha(d/L_0) \) on a log-log scale should tend toward a slope of 1/2.

It can be seen from Figure 1a, that the plot of \( H/H_0 \) as a function of \( d/L_0 \) on a log-log grid tends toward a value of -1/4 for small values of \( d/L_0 \). The reasons for this limit can be established as follows:

From equations (4), (5), and (6) it is seen that \( H/H_0 = K_R K_S \). From equation (21) it can be seen that \( \alpha \rightarrow 0 \) as \( (d/L_0) \rightarrow 0 \). It is well known that \( \cos \alpha + 1 \) as \( \alpha \rightarrow 0 \). Therefore, for small values of \( (d/L_0) \), the refraction coefficient, as given by equation (11) tends toward \( \sqrt{\cos \alpha_0} \).

The refraction coefficient in equation (16) may be expressed in the form:

\[ K_S = \left\{ \left( \frac{C_0}{C} \right) \left[ 1 + \left( \frac{(4\pi d/L)}{\sinh(4\pi d/L)} \right) \right]^{-1} \right\}^{\frac{1}{2}} . \]  (23)
For sufficiently small values of \(4\pi d/L, \sinh(4\pi d/L) \to 4(\pi d/L)\) and equation (23) tends toward \(K_s = \left(\frac{C_0/C}{2}\right)^{1/2}\). Substitution from equations (18) and (19) for \(C_0/C\) gives:

\[
K_s = \left[\frac{1}{2} \sqrt{\frac{1}{2\pi}} \left(d/L_0\right)^{-1/4}\right]^{1/2}
\]

Thus,

\[
K_s = (8\pi)^{-1/4}(d/L_0)^{-1/4}.
\]  

(24)

Thus,

\[
\frac{H}{H_0} = \sqrt{\cos\alpha_0} \left(8\pi\right)^{-1/4}(d/L_0)^{-1/4}.
\]  

(25)

Taking logarithms of both sides yields:

\[
\log(H/H_0) = \log\left[\sqrt{\cos\alpha_0} (8\pi)^{-1/4}\right] - 1/4 \log(d/L_0).
\]  

(26)

Thus, it is seen that a log-log plot of \((H/H_0)\) as a function of \((d/L_0)\) must tend toward a slope of \(-1/4\) for small values of \(d/L_0\).

V. EXAMPLES OF APPLICATION

Two applications of the refraction slide rule are illustrated by the following examples:

a. Example 1.

**GIVEN:** Deepwater wave conditions:

\(T = 12\) seconds,
\n\(\alpha_0 = 70^\circ\),
\n\(H_0 = 2.0\) feet

**FIND:** \(H\) and \(\alpha\) for the wave when it has moved into water 30 feet deep.

**SOLUTION:** Place the overlay, Figure 1b, on top of the nomogram, Figure 1a, so that the index (14-second-period line) covers the vertical line corresponding to a 30-foot depth as designated along the top of the nomogram. Locate the vertical line for a 12-second period on the overlay, shown at point A in Figure 3. Locate the \(\alpha_0\) value of 70° on the right axis shown at B in Figure 3 and follow this dashed curve to the point at which it intersects the 12-second line on the overlay, shown at C. Then move from the intersection horizontally to the right and read an \(\alpha\) value of 27° at point D.
Figure 3. Use of refraction slide rule (12-second period)
Without moving the graph and overlay, again move down the 12-second line on the overlay until it intersects with the plotted solid curve which is labeled \( \alpha_O = 70 \). This point is shown at E in Figure 3. From this point move horizontally to the left and read from the \( H/H_O \) axis a value of 0.66, shown at F. To obtain \( H \), multiply \( H/H_O \) (0.66) by the given \( H_O \) (2 feet). This yields a local wave height of 1.32 feet.

Therefore, a wave whose deepwater characteristics are:

\[
T = 12 \text{ seconds}, \\
\alpha_O = 70^\circ, \\
H_O = 2.0 \text{ feet}
\]

will have the following shallow water characteristics after moving shoreward to a water depth of 30 feet:

\[
T = 12 \text{ seconds}, \\
\alpha = 27^\circ, \\
H = 1.32 \text{ feet}.
\]

b. Example 2.

**GIVEN:** Wave conditions observed nearshore:

\[
T = 9.6 \text{ seconds}, \\
\alpha = 5^\circ, \\
H = 4 \text{ feet}, \\
d = 5 \text{ feet}.
\]

**FIND:** \( H_O \) and \( \alpha_O \) for the wave in deep water and \( H \) and \( \alpha \) for the wave in a water depth of 30 feet.

**SOLUTION:** Place the overlay on the graph so that the index (14-second-period line) is covering the 5-foot-depth line extending from the top of the graph. Locate the vertical line for a 9.6-second period on the overlay shown at A in Figure 4 (the 9.6-second line is not shown in the overlay but may be estimated between the 9- and 10-second period lines). Locate the \( \alpha \) value of 5° on the right axis shown at B and move horizontally to the left to the point at which it intersects with the approximated 9.6-second line (shown at C). A deepwater angle is obtained by following the dashed line which passes through this intersection to the right axis and reading an \( \alpha_O \) value of 20° at D. With the graph and
Figure 4. Use of refraction slide rule (9.6-second period).
overlay in the same position, locate the intersection of the vertical 9.6-second period line with the solid line having an \( \alpha_0 \) value of 20°. Since there is no slide line for this \( \alpha_0 \) value, it must be interpolated between the \( \alpha_0 \) equal 0° and \( \alpha_0 \) equal 40° line and is shown at E. Move horizontally to the left from this intersection and read an \( H/H_0 \) value of 1.37, at F. \( H_0 \) may then be obtained by dividing \( H(4 \text{ feet}) \) by the \( H/H_0 \) value of 1.37. This will yield a value of 2.92 feet for \( H_0 \). Using the values:

\[
T = 9.6 \text{ seconds}, \\
\alpha_0 = 20.0°, \\
H = 2.92 \text{ feet}
\]

and the procedure illustrated in Figure 3, \( H \) and \( \alpha \) at a depth of 30 feet are found to be:

\[
\alpha = 11.6, \\
H = 0.96 (2.92) = 2.80 \text{ feet}
\]

VI. SUMMARY

The refraction method presented in this paper is the fastest method presently available to the engineer for obtaining information on wave height variation due to the combined effects of shoaling and refraction. Insight on wave height reaction to angle of approach and change in depth may be quickly extracted from the nomogram.

The simplifying assumptions limit application to relatively simple bottom topography. However, in many cases this method will provide useful and timely information on wave height variation in the absence of results from more sophisticated refraction techniques.
LITERATURE CITED


McClenan, C.H.
This report presents a nomogram for the computation of combined refraction-shoaling coefficients for straight and parallel bottom contours. The nomogram permits a rapid solution of idealized refraction phenomena. The technique provides a useful first estimate to the true solution, and for many problems, as accurate a solution as other time-consuming methods.


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