THE INVERSE TSUNAMI PROBLEM FOR SYMMETRIC ISLANDS
OF SIMPLE SHAPE

Charles E. Knowles
and
Robert O. Reid

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Texas A&M University
College of Geosciences
Department of Oceanography
College Station, Texas

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The Texas A & M Research Foundation

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ABSTRACT

The problem investigated in this paper is that of estimating the deep water signature of a tsunami based on an observed marigram in the immediate vicinity of an island. The basic assumption is made that the incident tsunami in deep water is represented by a plane wave but that its signature in time at a fixed point in deep water is unknown. This implies that the distance of the earthquake epicenter is large compared with the horizontal scale of the island at its base on the ocean floor. The present study is limited to the linear theory for long waves and accordingly its application requires that the observed water level signatures be at locations where non-linear effects and dispersion are minimal. The method is numerical.

For a given direction of the input wave train in deep water and a given observation point (P) near the island, the solution of the problem as posed rest on the determination of the transfer function for the response at P due to the input. If the transfer function can be established from a known pair of input-output time sequences having a broad band spectrum, then
in principle, one can estimate the deep water input from other measured time sequences at the same point P.

The transfer function at P is estimated from a numerical model in which the island bathymetry is represented on a discrete grid and the wave equations are solved numerically for a given deep water input and the response at the point in question is obtained, subject to appropriate boundary conditions.

The results indicate that the method will give a good estimate of the response function for more realistic islands only if the spatial resolution is sufficiently good to render the spectra accurately and the input-output time sequences are very long. Both conditions will result in very time consuming operations.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER I</th>
<th>INTRODUCTION AND STATEMENT OF PROBLEM</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>General Discussion</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Problem to be Investigated</td>
<td>3</td>
</tr>
<tr>
<td>CHAPTER II</td>
<td>MATHEMATICAL DEVELOPMENT OF THE PROBLEM</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>The Fundamental Wave Equation and Boundary Conditions</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Finite Difference Analogues</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Inversion Equations</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>Inputs to be Used to Obtain R(f)</td>
<td>19</td>
</tr>
<tr>
<td>CHAPTER III</td>
<td>SPECIFIC ISLAND MODELS AND RESULTS</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>Simple Cylindrical Island - Model I</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>Cylindrical Island with Step-Sill - Model II</td>
<td>33</td>
</tr>
<tr>
<td>CHAPTER IV</td>
<td>SUMMARY AND SUGGESTIONS FOR FURTHER RESEARCH</td>
<td>54</td>
</tr>
<tr>
<td>REFERENCES</td>
<td></td>
<td>57</td>
</tr>
<tr>
<td>APPENDIX A</td>
<td>THE ANALYTIC SOLUTION OF THE STEP-ISLAND PROBLEM</td>
<td></td>
</tr>
<tr>
<td>APPENDIX B</td>
<td>A TEST OF THE STEP-Boundary CONDITION AND THE WAVE EQUATION NUMERICAL MODEL WITH A GAUSSIAN PULSE</td>
<td>64</td>
</tr>
</tbody>
</table>
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapter I</td>
<td>Introduction and Statement of Problem</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>General Discussion</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Problem To Be Investigated</td>
<td>3</td>
</tr>
<tr>
<td>Chapter II</td>
<td>Mathematical Development of the Problem</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>The Fundamental Wave Equation and Boundary Conditions</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Finite Difference Analogues</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Inversion Equations</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>Inputs to be Used to Obtain $R(f)$</td>
<td>19</td>
</tr>
<tr>
<td>Chapter III</td>
<td>Specific Island Models and Results</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>Simple Cylindrical Island - Model I</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>Cylindrical Island with Step-Sill - Model II</td>
<td>33</td>
</tr>
<tr>
<td>Chapter IV</td>
<td>Summary and Suggestions for Further Research</td>
<td>54</td>
</tr>
<tr>
<td>References</td>
<td></td>
<td>57</td>
</tr>
<tr>
<td>Appendix A</td>
<td>The Analytic Solution of the Step-Island Problem</td>
<td>58</td>
</tr>
<tr>
<td>Appendix B</td>
<td>A Test of the Step-Boundary Condition and the Wave Equation Numerical Model with a Gaussian Pulse</td>
<td>64</td>
</tr>
<tr>
<td>Vita</td>
<td></td>
<td>70</td>
</tr>
</tbody>
</table>
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 2.1</td>
<td>Schematic of symmetry condition about line $\theta = 0, \pi$ for Models I and II.</td>
<td>13</td>
</tr>
<tr>
<td>Figure 2.2</td>
<td>Schematic of island system, input $X(t)$ at Q and response $Y(t)$ at P due to $X(t)$.</td>
<td>15</td>
</tr>
<tr>
<td>Figure 3.1</td>
<td>Schematic of simple cylinder island, Model I.</td>
<td>21</td>
</tr>
<tr>
<td>Figure 3.2</td>
<td>Plot of kernel function $G(t)\Delta t$ versus $t/\Delta t$ for Model I.</td>
<td>28</td>
</tr>
<tr>
<td>Figure 3.3</td>
<td>Schematic of cylindrical island with a step-sill, Model II.</td>
<td>34</td>
</tr>
<tr>
<td>Figure 3.4</td>
<td>Comparison of numerical and analytic frequency spectra for wave mode zero ($l = 0$) and $\Delta \theta = \pi/16$, Model II.</td>
<td>42</td>
</tr>
<tr>
<td>Figure 3.5</td>
<td>Comparison of results of Gaussian pulse input tests (Model II), at the step-sill.</td>
<td>44</td>
</tr>
<tr>
<td>Figure 3.6</td>
<td>Comparison of results of Gaussian pulse input tests (Model II), on the sill.</td>
<td>45</td>
</tr>
<tr>
<td>Figure 3.7</td>
<td>Comparison of numerical and analytic spectra for wave mode zero ($l = 0$) and $\Delta \theta = \pi/50$, Model II.</td>
<td>49</td>
</tr>
<tr>
<td>Figure 3.8</td>
<td>Comparison of numerical and analytic spectra for wave mode one ($l = 1$) and $\Delta \theta = \pi/50$, Model II.</td>
<td>50</td>
</tr>
<tr>
<td>Figure 3.9</td>
<td>Comparison of numerical and analytic spectra for wave mode two ($l = 2$) and $\Delta \theta = \pi/50$, Model II.</td>
<td>51</td>
</tr>
<tr>
<td>Figure 3.10</td>
<td>Comparison of numerical and analytic spectra for wave mode three ($\lambda = 3$) and $\Delta \theta = \pi/50$, Model II</td>
<td>52</td>
</tr>
<tr>
<td>Figure 3.11</td>
<td>Comparison of numerical and analytic spectra for wave mode four ($\lambda = 4$) and $\Delta \theta = \pi/50$, Model II.</td>
<td>53</td>
</tr>
<tr>
<td>Figure B.1</td>
<td>Presentation of results of Gaussian pulse input test 2, Model II, in deep water.</td>
<td>66</td>
</tr>
<tr>
<td>Figure B.2</td>
<td>Comparison of results of Gaussian pulse input tests (Model II), at the inner boundary.</td>
<td>67</td>
</tr>
<tr>
<td>Figure B.3</td>
<td>Schematic of typical ray path used to calculate refraction factors for Model II; inbound ray.</td>
<td>68</td>
</tr>
<tr>
<td>Figure B.4</td>
<td>Schematic of typical ray path used to calculate refraction factors for Model II; outbound ray.</td>
<td>69</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>----------</td>
<td>------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>Figure 3.10</td>
<td>Comparison of numerical and analytic spectra for wave mode three ( (\ell = 3) ) and ( \Delta \theta = \pi/50 ), Model II</td>
<td>52</td>
</tr>
<tr>
<td>Figure 3.11</td>
<td>Comparison of numerical and analytic spectra for wave mode four ( (\ell = 4) ) and ( \Delta \theta = \pi/50 ), Model II.</td>
<td>53</td>
</tr>
<tr>
<td>Figure B.1</td>
<td>Presentation of results of Gaussian pulse input test 2, Model II, in deep water.</td>
<td>66</td>
</tr>
<tr>
<td>Figure B.2</td>
<td>Comparison of results of Gaussian pulse input tests (Model II), at the inner boundary.</td>
<td>67</td>
</tr>
<tr>
<td>Figure B.3</td>
<td>Schematic of typical ray path used to calculate refraction factors for Model II; inbound ray.</td>
<td>68</td>
</tr>
<tr>
<td>Figure B.4</td>
<td>Schematic of typical ray path used to calculate refraction factors for Model II; outbound ray.</td>
<td>69</td>
</tr>
</tbody>
</table>
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table 3.1</th>
<th>Parameters of Model I</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 3.2</td>
<td>Comparison of amplitudes and phases for numerical and analytic results, Model I</td>
<td>26</td>
</tr>
<tr>
<td>Table 3.3</td>
<td>A measure of reproduction of input from the lee side output for the cylindrical island, Model I</td>
<td>29</td>
</tr>
<tr>
<td>Table 3.4</td>
<td>Parameters of Model II</td>
<td>34</td>
</tr>
<tr>
<td>Table 3.5</td>
<td>Test parameters for Gaussian input, Model II</td>
<td>43</td>
</tr>
<tr>
<td>Table 3.6</td>
<td>A comparison of numerical and ray theory arrival times and wave heights for plane Gaussian pulse input, Model II</td>
<td>46</td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>Coefficient from (2.6)</td>
</tr>
<tr>
<td>$A(j)$</td>
<td>Fourier coefficient used to define $H_i(k)$, where $i = 1,2$</td>
</tr>
<tr>
<td>$a$</td>
<td>Radius of inner boundary, Model II</td>
</tr>
<tr>
<td>$B$</td>
<td>Coefficient from (2.6)</td>
</tr>
<tr>
<td>$b$</td>
<td>Radius of step-sill, Model II</td>
</tr>
<tr>
<td>$c_i$</td>
<td>Lagrangian wave speed: $i = 1$, sill; $i = 2$, deep water; $i = 0$, limiting velocity of wave front</td>
</tr>
<tr>
<td>$CF(j)$</td>
<td>Cosine transform of $J(j)$</td>
</tr>
<tr>
<td>$D_2(k)$</td>
<td>Hanning window, (3.21)</td>
</tr>
<tr>
<td>$E(k)$</td>
<td>Exponential window, (3.17)</td>
</tr>
<tr>
<td>$F_X(f)$</td>
<td>Fourier transform of $X(t)$</td>
</tr>
<tr>
<td>$F_Y(f)$</td>
<td>Fourier transform of $Y(t)$</td>
</tr>
<tr>
<td>$f$</td>
<td>Frequency in hertz</td>
</tr>
<tr>
<td>$f_N$</td>
<td>Maximum (Nyquist) frequency</td>
</tr>
<tr>
<td>$G(t)$</td>
<td>Kernel function, (2.13)</td>
</tr>
<tr>
<td>$g$</td>
<td>Acceleration of gravity</td>
</tr>
<tr>
<td>$H_i(k)$</td>
<td>Generalize input sequences: $i = 1$, (3.1); $i = 2$, (3.15)</td>
</tr>
<tr>
<td>$\mathcal{K}(k)$</td>
<td>&quot;Box-car&quot; window, (3.16)</td>
</tr>
<tr>
<td>$h_i$</td>
<td>Water depths ($i = 1$, sill; $i = 2$, deep)</td>
</tr>
<tr>
<td>$h_\infty$</td>
<td>Constant depth for $r \geq R_m$</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$J(j)$</td>
<td>Numerical version of $F_x, F_y$</td>
</tr>
<tr>
<td>$j$</td>
<td>Integer, $j = f/\Delta f$</td>
</tr>
<tr>
<td>$K_i$</td>
<td>Constants of outer boundary condition, (2.9), $i = 1, 2$</td>
</tr>
<tr>
<td>$K(t)$</td>
<td>Kernel function, (2.14)</td>
</tr>
<tr>
<td>KS</td>
<td>Constant indicating wave start point</td>
</tr>
<tr>
<td>$k$</td>
<td>Integer, $k = t/\Delta t$</td>
</tr>
<tr>
<td>$M_1$</td>
<td>Real amplitudes of $F_x$</td>
</tr>
<tr>
<td>$M_2$</td>
<td>Real amplitudes of $F_y$</td>
</tr>
<tr>
<td>$M$</td>
<td>Integer number of frequency intervals, $\Delta f$</td>
</tr>
<tr>
<td>$MM$</td>
<td>Integer number of angular intervals, $\Delta \theta$</td>
</tr>
<tr>
<td>$m$</td>
<td>Integer, $(m-2) = \theta/\Delta \theta$</td>
</tr>
<tr>
<td>$N$</td>
<td>Integer of step boundary, $r = N\Delta r$</td>
</tr>
<tr>
<td>$NB$</td>
<td>Integer of inner boundary, $r = NB\Delta r$</td>
</tr>
<tr>
<td>$NN$</td>
<td>Integer of outer boundary, $r = NN\Delta r$</td>
</tr>
<tr>
<td>$NT$</td>
<td>Integer number of time intervals, $\Delta t$</td>
</tr>
<tr>
<td>$n$</td>
<td>Integer, $n = r/\Delta r$</td>
</tr>
<tr>
<td>$P$</td>
<td>Point near island shore where $Y(t)$ is measured</td>
</tr>
<tr>
<td>$Q$</td>
<td>Point in deep water where $X(t)$ is measured</td>
</tr>
<tr>
<td>$R(f)$</td>
<td>Response function at a point</td>
</tr>
<tr>
<td>$R_1$</td>
<td>$M_2/M_1$</td>
</tr>
<tr>
<td>$R_m$</td>
<td>Radius outside of which depth is constant</td>
</tr>
<tr>
<td>$r$</td>
<td>Radial polar coordinate</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>$r_0$</td>
<td>Inner boundary radius, Model I</td>
</tr>
<tr>
<td>SF(j)</td>
<td>Sine transform of $J(j)$</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
</tr>
<tr>
<td>$W$</td>
<td>Term representing angular portion of $(2.5)$</td>
</tr>
<tr>
<td>$X(t)$</td>
<td>$\zeta$ at point $Q$</td>
</tr>
<tr>
<td>$x$</td>
<td>$r \cos \theta$</td>
</tr>
<tr>
<td>$Y(t)$</td>
<td>$\eta$ at point $P$</td>
</tr>
<tr>
<td>$Z(k)$</td>
<td>Numerically simulated version of $\zeta$</td>
</tr>
</tbody>
</table>

**Greek Symbol**

- $\alpha$: Non-dimensional parameter, $(2.10)$
- $\beta$: $9/2M^2$, $(3.4)$
- $\Gamma(m,k)$: Water level anomalies at all inner boundary points
- $\gamma$: Constant, $(2.6)$
- $\Delta f$: Frequency increment
- $\Delta r$: Radial increment, 1 km
- $\Delta \theta$: Angular increment, $\pi$/MM
- $\Delta t$: Time increment, 4 sec
- $\epsilon_1$: Real phase of $F_x$
- $\epsilon_2$: Real phase of $F_y$
- $\zeta$: Water level anomaly above mean sea level
- $\zeta_s$: Scattered portion of $\zeta$
<table>
<thead>
<tr>
<th>Greek Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$</td>
<td>Incident wave in deep water</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Angular polar coordinate</td>
</tr>
<tr>
<td>$l$</td>
<td>Integer indicating wave mode</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Time lag</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Standard deviation Gaussian pulse</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Non-dimensional frequency parameter $2\pi fr_0/c$</td>
</tr>
<tr>
<td>$\varphi(f)$</td>
<td>Phase lag, $\epsilon_1 - \epsilon_2$</td>
</tr>
<tr>
<td>$\psi_j$</td>
<td>Random phase angle</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$2\pi f$</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION AND STATEMENT OF PROBLEM

General Discussion

Tsunami is the Japanese name given to those disturbances of sea level originating from a localized molar disturbance of the earth's crust. The primary source of such waves is an earthquake (generally of magnitude 6.5 on the Richter scale or greater with a focal depth of 50 km or less). In certain cases landslides, bottom slumping and volcanic eruptions have been cited as causes. Not all earthquakes cause tsunamis, however, indicating that there is a great deal of local variation in the generating mechanisms. Once a tsunami is formed, the wave system resembles the disturbances produced by a pebble tossed into a placid pond, but on quite a different scale. A tsunami propagates with a limiting velocity \( c_0 = (gh)^{1/2} \) for free waves in water of depth \( h \) (\( g \) is the acceleration of gravity), but is dispersive in that the shorter period waves are 'sorted

The citations on the following pages follow the style of the *Journal of Marine Research.*
out' and left behind by the longer period waves.

In the mid-ocean it is generally agreed that tsunamis have periods that vary from several minutes to hours and heights which usually do not exceed 1 meter. They travel over water of say 4000 m in depth (the average depth of the ocean) at a speed of 720 km/hr. Thus, in spite of their small amplitude in deep water, the amount of energy transmitted by these waves can be considerable. Upon reaching the shallow coastal regions of islands or continents, this energy become concentrated both by the effect of decreased depth and by local focusing due to the configuration of the coastline, and results in a considerable buildup in wave amplitude (as much as tens of meters). Unfortunately for coastal inhabitants, what was once a small amplitude and virtually undetectable wave in mid-ocean, frequently becomes a high destructive surge of water which can inundate large regions of the coastal terrain and inflict great losses of both life and property.

The deep water characteristics of tsunamis mentioned above have not as yet been directly determined by quantitative mid-ocean measurements (deep water wave gages are
presently being employed for the study of deep sea tides and could detect a tsunami (Miller, personal correspondence), so what is known has been derived almost entirely from theory. Records obtained on small islands in the mid-Pacific have given the best information, but theoretical deductions imply that the waves can be significantly modified by the transformation that takes place when they interact with the island shoreline and surrounding sea-floor topography. Included in this interaction are modifications and amplifications due to linear and non-linear transformations related to shoaling, scattering, diffraction, refraction and possible resonance phenomena created when wave energy is trapped around the island by the bathymetry (see Longuet-Higgins, 1967). Studies of this transformation should take into account each of these factors.

Problem To Be Investigated

The inverse problem. The problem investigated here is directed towards the goal of estimating the deep water signature of a tsunami, based upon an observed marigram (a tsunami recording) in the immediate vicinity of an island. The basic assumption is made that the
incident tsunami in deep water is represented by a plane progressive wave, but that its signature in time at a fixed point in deep water is unknown. This implies that the distance from the source is large compared with the horizontal scale of the island (at its base on the ocean floor). The present study is limited to the linear theory for long waves and accordingly its application requires that the observed water level signatures be recorded at locations where non-linear effects and dispersion are minimal.

For a given direction of the input wave train in deep water and a given observation point (P) near the island, the solution of the problem as posed (i.e., to estimate the deep water signature of an incident tsunami) rests on the determination of the transfer function for the response at P. The transfer function is the ratio of the Fourier transform of the response to the Fourier transform of the input. In general it is a complex function of frequency. If the transfer function can be established from a known pair of input-output time sequences having a broad band spectrum, then in principle one can estimate the deep water input from other measured time sequences at the point P for the
same direction of wave propagation.

Related studies. The attempt to recover the incident wave train is not unique to this study. The transfer function at the selected point \( P \) of a given island and for a given direction of the input plane waves can be estimated in at least three different ways:

1. From field measurements in which both deep water and island near field data are obtained simultaneously for a given tsunami. This is the method currently being explored by Vitouseck and Miller (1970).

2. From laboratory model experiments with measurements analogous to those of (1), and for which the island bathymetry is duplicated in the model on an undistorted scale (Van Dorn, 1970).

3. From a numerical model in which the island bathymetry is represented on a discrete grid and the wave equations are solved numerically for a given deep water input and the response at the point in question is obtained, subject to appropriate boundary conditions (Vastano and Reid, 1967).

This investigation is concerned with the latter approach and consists of several phases. Specifically the different phases of the overall numerical study include:
(1) A test of the numerical model for symmetric islands of simple geometry for which analytical solutions are known and for which the input train is simple harmonic (monochromatic).

(2) A test of the numerical model for simple islands as in (1), but for an input sequence of long duration which possesses a broad band spectrum with respect to frequency.

(3) A test of the ability to reproduce the known input function from the output for the conditions of phase (1).
CHAPTER II

MATHEMATICAL DEVELOPMENT OF THE PROBLEM

The Fundamental Wave Equation and Boundary Conditions

Consider an island of given bathymetry within a circle of radius $R_m$, outside of which the depth is assumed constant. For simplicity attention will be confined to axially symmetric conditions for which the depth $h$ depends upon $r$ at most, the island shore being a circle of radius $r_0 << R_m$. It is assumed that the tsunami source is far enough away from the island that the prescribed input disturbance can be approximated by a plane progressive wave of permanent form. Specifically this input represents the water level anomaly which would occur in the absence of the island.

The equations controlling the water level variations in the presence of the island are given by Vastano and Reid (1967) and are repeated below for specific reference.

Wave equation. The fundamental linearized long wave equation for the two cylindrical islands considered in this investigation is
\[
\frac{\partial^2 \zeta}{\partial t^2} = g \left[ r^{-1} \frac{\partial}{\partial r} \left( hr \frac{\partial \zeta}{\partial r} \right) + hr^{-2} \frac{\partial^2 \zeta}{\partial \theta^2} \right]
\] (2.1)

where \( \zeta \) is the water level anomaly relative to mean water level, \( h \) is the depth at \( r \), \( r \) and \( \theta \) the polar coordinates with origin centered in the island, \( g \) is gravity and \( t \) is time.

**Inner boundary.** At the shoreline, \( r = r_o \), the islands considered extend vertically through the sea surface as impermeable fixed circular cylinders. The inner boundary condition requires a zero component of radial flow, or in terms of \( \zeta \),

\[
\frac{\partial \zeta}{\partial r} = 0.
\] (2.2)

**Outer boundary.** The outer boundary condition is a condition of radiation for scattered waves. It is a statement that the energy contained in the scattered portion of the \( \zeta \) field, \( \zeta_s \), is progressing radially outward to infinity in the far field. The radiational condition (Vastano and Reid, 1967) is of the form
\[
\frac{\partial}{\partial t} \left( r^{1/2} \zeta_s \right) + c_2 \frac{\partial}{\partial r} \left( r^{1/2} \zeta_s \right) \to 0 ,
\]

as \( r \) approaches infinity, assuming the depth \( h_\infty \) is a constant beyond some finite region of \( r \). The scattered portion of \( \zeta \) is in essence the residual of the total field \( \zeta \) and is defined by

\[
\zeta_s = \zeta - \eta(t + x/c_2) ,
\]

where \( \eta \) is the incident plane wave which is taken as propagating at speed \( c_2 = (gh_\infty)^{1/2} \) in the negative \( x \) direction, where \( x = r \cos \theta \).

**Finite Difference Analogues**

The finite difference analogue of (2.1) is taken as a centered difference form in respect to \( r, \theta, \) and \( t \) at all \( r \) except the outer boundary of the system. The finite difference form of (2.2) is simply a symmetry condition for \( \zeta \) at \( r = r_o \); i.e., \( \zeta \) at \( (a - \Delta \eta) \) equals \( \zeta \) at \( (a + \Delta r) \) for a given \( \theta \) and \( t \), thus allowing the finite difference version
of (2.1) to be applied at \( r = r_0 \). The radiational condition (2.3) is applied in a manner identical to that described by Vastano and Reid (1967).

**Wave equation.** Let \( Z^k(n,m) \) represent the numerically simulated version of \( \zeta \) at \( r = n\Delta r, \ \theta = (m-2)\Delta\theta, \ t = k\Delta t, \) where \( n, m, k \) are integers. The centered difference form of (2.1) is taken as

\[
Z^{k+1}(n,m) - 2Z^k(n,m) + Z^{k-1}(n,m) = \gamma \left[ A \left\{ Z^k(n+1,m) - Z^k(n,m) \right\} 
+ B \left\{ Z^k(n,m) - Z^k(n-1,m) \right\} 
+ h(n) W \right],
\]

(2.5)

where

\[
\gamma = g \left( \frac{\Delta t}{\Delta r} \right)^2,
\]

\[
A = \frac{(1 + 1/2n)[h(n+1) + h(n)]}{2},
\]

\[
B = \frac{(1 - 1/2n)[h(n) + h(n+1)]}{2},
\]

\[
W = (n\Delta\theta)^{-2} \left[ Z(n, m+1) - 2Z(n,m) + Z(n, m-1) \right].
\]

(2.6)
Using (2.5) one can evaluate \( Z^{k+1}(n,m) \) for all points interior of the outer boundary from the field of \( Z \) at previous time levels.

**Inner boundary.** The numerical analogue of (2.2) in the present notation is simply

\[
Z^k(NB - 1, m) = Z^k(NB + 1, m), \tag{2.7}
\]

where \( r_0 = NB\Delta r \).

**Outer boundary.** The analogue of (2.3) is a special prognostic relation for \( Z \) at a large value of \( n \), say \( NN \) \( (NN\Delta r > R_m) \). Equation (2.3) asserts that \( r^{1/2} \zeta_s \) is a constant for constant value of \( (r - c_2 t) \) at large \( r \), holding \( \theta \) fixed. In finite difference notation, following the interpolational procedure discussed by Vastano and Reid (1967) this condition yields

\[
Z^{k+1}(NN, m) = H^{k+1}(NN, m) + [Z^k(NN, m) - H^k(NN, m)]K_1
\]

\[+ K_2 [Z^k(NN-1, m) - H^k(NN-1, m)], \tag{2.8}
\]

where
and \( H^k(n,m) \) is the value of the incident plane wave, \( \eta \), at \( r = n\Delta r \), \( \theta = (m-2)\Delta \theta \), \( t = k\Delta t \). Specifically, \( H \) is a function of the non-dimensional parameter, \( \alpha \), defined by

\[
\alpha = k + n\Delta r \left( c_s \Delta t \right)^{-1} \cos(m-2)\Delta \theta - KS, \tag{2.10}
\]

where \( KS \) is a constant taken such that \( \alpha \approx 0 \) at \( k = 0 \) for \( m = 2 \) and \( n = NN \). Hence,

\[
KS = NN \Delta r/c_s \Delta t. \tag{2.11}
\]

The input function at a point (to be considered below) is taken as the value of \( H \) at \( \theta = \pi/2 \); i.e., at \( \alpha = k - KS \). A different \( H \) will be employed for each model and will be designated \( H_1 \) for Model I and \( H_2 \) for Model II.

**Symmetry condition.** Since the island models are symmetric, calculations are made only for \( 0 \leq \theta \leq \pi \) by using the following symmetry boundary condition for all \( k \), (see Fig. 2.1),
\( Y(n, MM+1) = Y(n, MM+3) \) \( Y(n, 1) = Y(n, 3) \)

Figure 2.1 Schematic of symmetry condition about line \( \theta = 0, \pi \) for Models I and II.
\[ Z(n, l) = Z(n, 3) , \]

\[ Z(n, MM + l) = Z(n, MM + 3) , \]

where \( n = NB, \ldots, NN \) and MM is the number of \( \theta \) intervals.

**Inversion Equations**

Let \( X(t) \) represent the input water level time sequence at a fixed point \( Q \) associated with a tsunami arriving from a stipulated epicenter (Fig. 2.2). Actually \( X(t) \) represents \( \eta \) at a specific point.

Let \( Y(t) \) represent the response to the input \( X(t) \) at some fixed point \( P \) near the island, either measured or estimated by the application of a numerical integration of the long wave equations with appropriate boundary conditions at the island and in the far field as given above. Thus \( Y(t) \) is simply \( \zeta \) at a particular point.

Under the restriction of a linear response at point \( P \) and the assumption of a plane progressive wave input at \( Q \), the response \( Y(t) \) can be
Figure 2.2 Schematic of island system, input $X(t)$ at Q and response $Y(t)$ at P due to $X(t)$. 
expressed as an appropriate convolution of \( X(t) \) and vice versa, i.e.,

\[
Y(t) = \int_{-\infty}^{\infty} K(\lambda) X(t - \lambda) d\lambda , \tag{2.13}
\]

or,

\[
X(t) = \int_{-\infty}^{\infty} G(\lambda) Y(t - \lambda) d\lambda , \tag{2.14}
\]

where the integrations extend nominally over all time lags \( \lambda \). Actually the kernel function \( K(\lambda) \) should vanish for \( t < 0 \), but this is not necessarily true for \( G(\lambda) \). Both kernel functions depend upon the particular island bathymetry, the location of \( P \) and the direction of wave propagation, but not on \( X(t) \). Thus, if \( G(\lambda) \) can be estimated from a known input-output pair, based upon numerical integration of the long wave equation (2.1), then in principle (2.14) can be used to estimate any unknown input sequence \( X_2(t) \) derived from measurements \( Y_2(t) \) at \( P \) for the same direction of wave propagation.

If the input \( X(t) \) is localized in time (this implies that \( \int_{-\infty}^{\infty} |X(t)| dt \) exists) then it can be represented by a Fourier integral
\[ X(t) = \int_{-\infty}^{\infty} F_x(f) e^{-i2\pi ft} df, \quad (2.15) \]

the inverse of which is

\[ F_x(f) = \int_{-\infty}^{\infty} X(t) e^{i2\pi ft} dt; \quad (2.16) \]

i.e., (2.15) and (2.16) are Fourier transform pairs and \( f \) is the frequency in hertz. Transforms of \( Y(t) \) and \( K(t) \) can be defined by expressions similar to (2.16). Specifically, let

\[ X(t) \quad \text{--} \quad F_x(f), \]

\[ Y(t) \quad \text{--} \quad F_y(f), \quad (2.17) \]

\[ K(t) \quad \text{--} \quad R(f), \]

be corresponding Fourier transform pairs. Then according to the convolution theorem, (2.15) yields

\[ F_y(f) = R(f)F_x(f). \quad (2.18) \]
From an examination of (2.14) and (2.18), \( R(f)^{-1} \) is clearly the transform of \( G(t) \). In general, \( F_x \), and \( F_y \), and \( R \) are complex and can be expressed in the polar form

\[
F_x(f) = M_1(f) e^{i\epsilon_1(f)} \\
F_y(f) = M_2(f) e^{i\epsilon_2(f)} \\
R(f) = R_1(f) e^{-i\varphi(f)},
\]

where \( M_1, M_2, \) and \( R_1 \) are real amplitudes, \( \epsilon_1 \), \( \epsilon_2 \) are real phases and \( \varphi \) is a phase lag. From (2.18) and (2.19) it follows that

\[
R(f) = (M_2/M_1) e^{i(\epsilon_2 - \epsilon_1)},
\]

and therefore,

\[
R_1(f) = M_2/M_1,
\]

and

\[
\varphi(f) = \epsilon_1 - \epsilon_2.
\]
Moreover, \( G(t) \) can be calculated as the inverse transform of \( R(f)^{-1} \), i.e.,

\[
G(t) = \int_{-\infty}^{\infty} R(f)^{-1} e^{-i2\pi ft} \, df.
\] (2.22)

In practice \( G(t) \) will be only as estimate since the original sequences of \( X(t) \) and \( Y(t) \) used to define \( R(f) \) are generally discrete sequences at time step \( \Delta t \) and are of limited duration. The discrete time step imposes an upper limit in the frequency domain \( (f_N = 1/2 \Delta t) \) and the finite duration of the record restricts the resolution of the estimate of \( R(f) \).

Inputs to be Used to Obtain \( R(f) \)

In the special case where the input \( X(t) \) is simple harmonic as can be realized in the numerical model, the output will be simple harmonic, at least after the initial transients have subsided. In this case, the response \( R(f) \) for the particular frequency of the input can be obtained without the necessity of Fourier Analysis. However, this procedure does
require many inputs at different frequencies in order to obtain adequate resolution of \( R(f) \) versus \( f \). It is employed only for those cases where sharp resonant peaks do not exist in the spectral response.

An alternate input employed to obtain \( R(f) \) is a stationary time sequence of long duration having a wide band frequency spectrum, the width of which is based on the frequencies characteristic of tsunamis and consistent with the long wave equation approximations. Then the Fourier procedure outlined above is used to estimate \( R(f) \) for the range of frequencies involved in the input \( X(t) \).
CHAPTER III

SPECIFIC ISLAND MODELS AND RESULTS

Simple Cylindrical Island - Model I

The first island considered is a simple cylinder on a flat bed, Fig. 3.1. The centered difference, finite difference analogues, equations (2.5) through (2.10) are employed to obtain a numerical solution for \( \zeta \) at \( r = n\Delta r \), \( \theta = (m-2)\Delta \theta \) and \( t = k\Delta t \), where \( n, m \) and \( k \) are integers. Since the depth \( h \) is constant for all \( r \geq r_o \), (2.5) and (2.6) are simplified and \( h \) will be factorable. Thus for Model I, (2.5) becomes

\[
Z^{k+1}(n,m) - 2 Z^k(n,m) + Z^{k-1}(n,m) = \gamma h \left[ A \{ Z^k(n+1,m) - Z^k(n,m) \} ight.
\]

\[
- B \{ Z^k(n,m) - Z^k(n-1,m) \} + W],
\]

where \( A \) and \( B \) are now
Figure 3.1 Schematic of simple cylinder island, Model I

Table 3.1 Parameters of Model I

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0$</td>
<td>19 km</td>
<td>(inner boundary radius)</td>
</tr>
<tr>
<td>$h$</td>
<td>4 km</td>
<td>(depth of water)</td>
</tr>
<tr>
<td>$\Delta r$</td>
<td>1 km</td>
<td>(radial increment)</td>
</tr>
<tr>
<td>$\Delta \theta$</td>
<td>$\pi/16$</td>
<td>(angular increment)</td>
</tr>
<tr>
<td>NN</td>
<td>100</td>
<td>(number of radial increments)</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>4 sec</td>
<td>(time increment)</td>
</tr>
</tbody>
</table>
\[
A = (1 + \frac{1}{2} n), \\
B = (1 - \frac{1}{2} n),
\]

and \( \gamma \) and \( W \) remain unchanged. The model parameters are listed in Table 3.1. Both the monochromatic (a sine wave for a broad range of selected frequencies) and the wide band stationary time sequences are used as input to the wave equation model.

Since there is no topography associated with Model I and the forcing function acts only at the cylinder wall, there is no possibility of trapped waves around the island (Longuet-Higgins, 1967). The response is influenced only by scattering at the cylinder and diffraction around it. Therefore, the amplitude response, \( |R(f)| \), at any point on the inner boundary is a monotonic function of frequency with no sharp resonant peaks. The time increment, \( \Delta t \), for both the input and output function is taken as 4 seconds. This increment was determined from the condition for numerical stability described by Vastano and Reid (1967). The analytical solution and the
numerical method of solution of the wave equation with the associated boundary conditions are also contained in this reference.

**Monochromatic input.** The first input $X(t)$ employed to estimate $R(f)$ is monochromatic. The wave program is run for six different wave periods; 2, 4, 6, 8, 10 and 12 minutes. The non-dimensional frequency indicated in this table is that employed by Vastano and Reid (1967) in the analytic solution for the response and is defined by $\tau = 2\pi f r_0 / c$, where $f$ is the frequency, $r_0$ the inner boundary radius, $h$ the water depth and $c = (gh)^{1/2}$, the Lagrangian wave speed. The other numerical results, designated "Numerical (sequence)," contained in the table are discussed below. The relative amplitude $R_1$ and the relative phase $\phi$ are evaluated by direct comparison of the output $Y(t)$ (on both the lee and wave sides) with the input $X(t)$. The phase is measured relative to the incident wave at the 90 degree azimuth position far from the cylinder.

The transfer function is determined for two points on the cylinder for the given direction of wave input. The two positions considered are the "lee side" (180 degree azimuth) and the "wave side" (0 degree azimuth)
on a diameter parallel to the direction of wave propagation.

Table 3.2 shows reasonably good agreement between the numerical and analytic results for both the amplitude $R_1$ and the phase $\varphi$ at the wave side position for nearly every period. However, there is a significant departure of the numerical from the analytic results on the lee side for the 2 minute period wave. Apparently the angular resolution, $\Delta \theta$, is insufficient for the proper rendition of the diffracted waves into the shadow zone for period of two minutes or less.

Recovery of the input - the inverse problem. An attempt to recover the incident wave train from the output is made using the lee side response, $Y(t)$, for the cylinder to estimate the kernel function $G(t)$ for discrete steps, $\Delta t$, based upon a numerical quadrature version of (2.22). In estimating $G(t)$ it is found that the high frequency values of $R(f)^{-1}$ dominated, so a window function is applied in the frequency domain such that $R(f)^{-1}$ is essentially reduced to zero at $f_N$, the maximum frequency. The window is Gaussian with a standard deviation of $300/\pi$. 
Table 3.2 Comparison of amplitudes and phases for numerical and analytic results, Model I

<table>
<thead>
<tr>
<th>Period (T in mins)</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Freq. (τ non-dim)</td>
<td>5</td>
<td>2.5</td>
<td>1.67</td>
<td>1.25</td>
<td>1.0</td>
<td>0.83</td>
</tr>
<tr>
<td>Numerical (mono)</td>
<td>1.93</td>
<td>1.89</td>
<td>1.84</td>
<td>1.68</td>
<td>1.71</td>
<td>1.70</td>
</tr>
<tr>
<td>(1) Analytic</td>
<td>1.97</td>
<td>1.89</td>
<td>1.83</td>
<td>1.76</td>
<td>1.71</td>
<td>1.68</td>
</tr>
<tr>
<td>Numerical (seq)</td>
<td>1.93</td>
<td>1.90</td>
<td>1.83</td>
<td>1.69</td>
<td>1.70</td>
<td>1.70</td>
</tr>
<tr>
<td>-------------------</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
</tr>
<tr>
<td>NUMERICAL (MONO)</td>
<td>0.38</td>
<td>0.65</td>
<td>0.76</td>
<td>0.85</td>
<td>0.88</td>
<td>0.92</td>
</tr>
<tr>
<td>(2) Analytic</td>
<td>0.48</td>
<td>0.67</td>
<td>0.78</td>
<td>0.85</td>
<td>0.89</td>
<td>0.93</td>
</tr>
<tr>
<td>Numerical (seq)</td>
<td>0.37</td>
<td>0.64</td>
<td>0.77</td>
<td>0.84</td>
<td>0.88</td>
<td>0.91</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Period (T in mins)</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Freq. (τ non-dim)</td>
<td>5</td>
<td>2.5</td>
<td>1.67</td>
<td>1.25</td>
<td>1.0</td>
<td>0.83</td>
</tr>
<tr>
<td>Numerical (mono)</td>
<td>-298</td>
<td>-150</td>
<td>-108</td>
<td>-87</td>
<td>-72</td>
<td>-62</td>
</tr>
<tr>
<td>(1) Analytic</td>
<td>-291</td>
<td>-152</td>
<td>-107</td>
<td>-86</td>
<td>-69</td>
<td>-65</td>
</tr>
<tr>
<td>Numerical (seq)</td>
<td>-292</td>
<td>-152</td>
<td>-107</td>
<td>-84</td>
<td>-69</td>
<td>-64</td>
</tr>
<tr>
<td>-------------------</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
</tr>
<tr>
<td>NUMERICAL (MONO)</td>
<td>516</td>
<td>264</td>
<td>184</td>
<td>135</td>
<td>113</td>
<td>96</td>
</tr>
<tr>
<td>(2) Analytic</td>
<td>505</td>
<td>265</td>
<td>184</td>
<td>142</td>
<td>114</td>
<td>95</td>
</tr>
<tr>
<td>Numerical (seq)</td>
<td>557</td>
<td>278</td>
<td>182</td>
<td>139</td>
<td>114</td>
<td>99</td>
</tr>
</tbody>
</table>

(1) "wave side" of island (0° azimuth)

(2) "lee side" of island (180° azimuth)
Using the estimated kernel function \( G(t) \), Fig. 3.2, and the output \( Y_a(t) \) in (2.14) for each of the six cases, six separate inputs \( X_a(t) \) are recovered. The input sequences \( X(t) \) and \( X_a(t) \) are then compared and a standard error of estimate calculated, Table 3.3. These can be compared with the unit amplitude of the actual input sequence \( X(t) \).

It is not surprising that the highest standard error occurs for \( T = 2 \) minutes, since as already noted, it is at this period that the numerical results departed significantly from the analytical. The standard error for the other five periods is of the order of 13 per cent of the input amplitude.

Part of the error of estimate is attributable to the window function applied to \( R(f)^{-1} \) in estimating \( G(t) \). This tends to give estimates of \( X_a(t) \) which are of smaller amplitude. In principle a correction can be applied for this systematic error and would considerably improve the estimate. Improving the angular resolution, (i.e., using a smaller \( \Delta \theta \)), would also further reduce the standard error of estimate for the input, especially for the 2 minute period. Indeed, as will be discussed later, this was found necessary in Model II.
Figure 3.2 Plot of kernel function $G(t)\Delta t$ versus $t/\Delta t$ for Model I.
Table 3.3  A measure of reproduction of input from the lee side output for the cylindrical island, Model I

<table>
<thead>
<tr>
<th>Period (T, mins)</th>
<th>Standard Error of Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.313</td>
</tr>
<tr>
<td>4</td>
<td>0.126</td>
</tr>
<tr>
<td>6</td>
<td>0.124</td>
</tr>
<tr>
<td>8</td>
<td>0.129</td>
</tr>
<tr>
<td>10</td>
<td>0.154</td>
</tr>
<tr>
<td>12</td>
<td>0.145</td>
</tr>
</tbody>
</table>
Generalized time sequence input. In order to specify a second input $X$ to estimate $R(f)$ for Model I, a frequency band-limited, generalized time sequence is defined by the Fourier series

$$H_i(k) = \begin{cases} 
0, & k \leq KS, \\
\sum_{j=0}^{M} A(j) \cos \left(2\pi j \left(\frac{k-KS}{NT}\right)\right), & k > KS,
\end{cases} \quad (3.1)$$

where $NT$ is the maximum number of time intervals, $M$ the maximum number of frequency intervals, $k = t/\Delta t$, $j = f/\Delta f$, $j$, $k$ are integers and $KS$ is a constant given by (2.11). The second input $X$ is, therefore, taken as (3.1) at a specified point. The most important factor to be considered in defining $H$ in this way, (3.1), is that the frequency spectrum is known and can be band limited.

Based upon the comparison of the numerical and analytic results for $R_1$ and $\phi$ due to the monochromatic inputs as discussed earlier (where there was a significant departure for a period of 2 minutes), the minimum period is limited for this analysis to
100 seconds, so the maximum frequency of the input

(3.1) will be

\[
\frac{M}{\Delta t \text{N}T} = \frac{1}{100} \text{ hertz,}
\] (3.2)

or

\[
M = \Delta t \text{N}T/100 \] (3.3)

where NT and M equal 3000 and 120 respectively.

The coefficients A(j) are calculated using the relation

\[
A(j) = e^{-\beta j^2} \] (3.4)

where \( \beta = 9/2M^2 \), i.e., with a standard deviation of M/3.

The relative amplitude \( R_1 \) and the relative phase \( \varphi \) are evaluated from the transforms of \( X(t) \) and \( Y(t) \) using (2.20) and (2.21) respectively, for both the lee and wave sides. The numerical version, \( J(j) \), of the Fourier integral (2.16) is the Fourier series approximation.
\[ j(j) = CF(j) + iSF(j), \quad (3.5) \]

where \( CF(j) \) is the cosine transform of (3.5) given by

\[
CF(j) = \frac{1}{2}(X(0) + X(NT)) \\
+ \sum_{k=1}^{NT-1} X(k) \cos \frac{2\pi jk}{NT}, \quad (3.6)
\]

and where \( SF(j) \) is the sine transform of (3.5) given by

\[
SF(j) = \sum_{k=1}^{NT-1} X(k) \sin \frac{2\pi jk}{NT}. \quad (3.7)
\]

To save computing time, tables of sines and cosines with the argument \( 2\pi jk/NT \) \((jk = 0, 1, 2, \ldots, NT)\) are generated and an algorithm employed to selected the proper value from the table for the summing of (3.6) and (3.7)
The results are also presented in Table 3.2 under the heading Numerical (sequence) and show generally the same good agreement with the analytic results as did the monochromatic. Since the results due to these two input sequences are so much the same no attempt is made to recover the incident wave \( X_2(t) \) for the generalized time sequence input.

Cylindrical Island with Step-Sill - Model II

The second island model considered is a modification of the first, i.e., a cylindrical island with a step-sill, Fig. 3.3. The grid characteristics are shown in Table 3.4. The radii are selected in a 1:3 ratio and the depths in a 1:9 ratio, to simplify the computation of the analytical solution to which the numerical results are compared (see Appendix A).

With the addition of a shelf to the island model, trapping of wave energy around the island is made possible and as a result, the amplitude response \( |R(f)| \) is an extremely complicated function of frequency, with many irregular sharp resonant peaks.

The transfer function is determined at the same two points on the island inner boundary and for the same direction of wave input as before. The time
Figure 3.3 Schematic of cylindrical island with a step-sill, Model II.

Table 3.4 Parameters of Model II

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>10 km</td>
<td>(inner boundary radius)</td>
</tr>
<tr>
<td>b</td>
<td>30 km</td>
<td>(step-sill radius)</td>
</tr>
<tr>
<td>$h_1$</td>
<td>$h_2/9$</td>
<td>(depth water, Region 1)</td>
</tr>
<tr>
<td>$h_2$</td>
<td>4 km</td>
<td>(depth water, Region 2)</td>
</tr>
<tr>
<td>$\Delta r$</td>
<td>1 km</td>
<td>(radial increment)</td>
</tr>
<tr>
<td>$\Delta \theta$</td>
<td>$\pi/16$, $\pi/50$</td>
<td>(angular increments)</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>4 sec</td>
<td>(time increment)</td>
</tr>
<tr>
<td>NN</td>
<td>100</td>
<td>(number radial increments)</td>
</tr>
</tbody>
</table>
resolution, \( \Delta t \), for this model is also 4 seconds.

**Model equations.** The wave equation, inner \((r = a)\) and outer boundary conditions are given by (2.1) through (2.4), but now must conform to two depth regimes; Region I \((i = 1)\) with a depth of \( h_1 = h_2/9 \) and Region II \((i = 2)\) with a depth \( h_2 = 4000 \) m. There must also be included a boundary condition at the step-sill \((r = b)\). For Model II, therefore, the wave equation is

\[
\frac{\partial^2 \zeta_i}{\partial t^2} = gh_i \left[ r^{-1} \frac{\partial}{\partial r} \left( r \frac{\partial \zeta_i}{\partial r} \right) + r^{-2} \frac{\partial^2 \zeta_i}{\partial \theta^2} \right], \quad (3.8)
\]

with boundary conditions

\[
\frac{\partial \zeta_i}{\partial r} = 0 \quad \text{at} \quad r = a, \quad (3.9)
\]

\[
\begin{align*}
\zeta_1 &= \zeta_2, \\
h_1 \frac{\partial \zeta_1}{\partial r} &= h_2 \frac{\partial \zeta_2}{\partial r} \quad \text{at} \quad r = b, \quad (3.10)
\end{align*}
\]

and
\[
\frac{\partial}{\partial t} (r^{1/2} \zeta_s) + c_s \frac{\partial}{\partial r} (r^{1/2} \zeta_s) = 0,
\]

as \( r \) approaches infinity, where

\[
\zeta_s = \zeta_2 - \eta \left(t + \frac{x}{c_2}\right),
\]

\[
x = r \cos \theta,
\]

and \( \zeta_s \) is the scattered portion of \( \zeta \), \( \zeta_2 \), is \( \zeta \) in Region II, \( \eta \) is the incident portion and 

\[
c_s = \left(gh_2\right)^{1/2}. 
\]

The finite difference analogues of (3.8), (3.9), (3.11) and (3.12) are given by (2.5) through (2.12). It remains to consider how the interior conditions (3.10) are applied to the finite difference algorithm. Physically these conditions assert that the water level and the radial component of volume transport of fluid must be continuous across the step.

**Finite difference analogue at step.** At \( r = b \) (\( n = N \) say) a difficulty arises since \( Z_1(N + 1, m) \) and \( Z_2(N - 1, m) \) are undefined. However, they can be defined by introducing conditions (3.10) which in finite difference form are
\[ Z_k^k (N, m) = Z_2^k (N, m) , \]

\[ h_1 [ Z_1^k (N + 1, m) - Z_1^k (n - 1, m)] = h_2 [ Z_2^k (N + 1, m) - Z_2^k (N - 1, m)]. \]

Clearly these hold for \( k + 1 \) and \( k - 1 \) as well and hence, the first of these conditions requires that the right hand side of (2.5) for \( i = 1 \) be equal to that for \( i = 2 \). The later relation together with the second relation in (3.13) then yield

\[ Z_1^k (N + 1, m) = h_2 h_1^{-1} Z_2^k (N + l, m) \]

\[ + (h_2 - h_1)(2h_1)^{-1} [ W - 2Z_1^k (N, m)]. \]

\[ Z_2^k (N - 1, m) = h_1 h_2^{-1} Z_1^k (N + l, m) \]

\[ - (h_2 - h_1)(2h_2)^{-1} [ W - 2Z_2^k (N, m)]. \]

Using either of the above relations in (2.5) for corresponding \( i \) then provides a prognostic relation for \( Z \) at the step, which incorporates the step
boundary conditions. The generalized equations (2.5) and (2.6), which provide for a variable depth, could have been used in lieu of (3.14) but the method embodied in (3.14) is chosen because it is felt that it gives more exact results.

Generalized time sequence input with random phase. The raw input used to determine $R(f)$ for Model II is of the same form as (3.1), but with random phase as part of the cosine argument,

$$
H_2(k) = \begin{cases} 
0, & k \leq KS, \\
\sum_{j=0}^{M} A(j) \cos \left( \frac{2\pi j (k-KS)}{NT} + \phi_j \right), & k > KS,
\end{cases}
$$

where $\phi_j$ is generated by a random number generator such that it has uniform probability density for the range $-\pi$ to $\pi$ radians. Indeed, for large $M$ (3.15) has nearly Gaussian statistics. The terms $A(j)$ and $M$ are identical with (3.4) and (3.3) respectively, and as before the input $X$ is $H_2$ at a point.
Since NT is a finite number the sequence $H_2(k)$ is truncated at NT, which in effect modifies the sequence with a "box-car" window, $\chi(k)$,

$$\chi(k) = \begin{cases} 0, & k > NT, \\ 1/2, & k = NT, \\ 1, & k < NT, \end{cases}$$

(3.16)

where $k = t/\Delta t$. The corresponding spectral filter is $\sin \pi f/\pi f$, which has large negative and positive side lobes and as a result will distort the amplitude and phase of the desired spectral response.

Accordingly, input $H_2(k)$ is modified before hand with an exponential window, $E(k)$,

$$E(k) = \exp \left[- \frac{(k - KS)}{Am}\right]$$

(3.17)

such that $E(3000) = .001$, with $Am = 435$. The response $Y(k)$ is estimated at the desired locations by application of the recursion relations (2.5), (2.7), (2.8) and (3.14).

In order to test the adequacy of Model II to estimate $R(f)$, comparisons are made between the analytic (Appendix A) and the numerical solutions
of $R(f)$ versus $f$. The complicated nature of the response $R(f)$ versus $f$ makes the comparison less than satisfactory. A much clearer means of comparison is obtained by recording a time history of water level anomalies at all island inner boundary points, $\Gamma(m,k)$, and evaluating the different wave modes around the island, i.e.,

$$Y(t,k) = \sum_{m=2}^{MM} P_t W_m \Gamma(m,k) \cos \left[ t (m-2)\Delta \theta \right], \quad (3.18)$$

where $\Gamma(m,k)$ are water level anomalies $z^k(NB,m)$ at the inner boundary, NB, for all $m$ and $k$, and

$$W_m = \begin{cases} 1 , & m = 2; \text{MM}, \\ 2 , & m = 3; 4, \ldots, \text{MM} - 1, \end{cases} \quad (3.19)$$

$$P_t = \begin{cases} 2 \text{MM}, & t = 0, \\ \text{MM} , & t \neq 0, \end{cases} \quad (3.2)$$

where $t = 0, 1, 2, \ldots, 15$ (wave mode).
The spectra of $Y(t,k)$ are then calculated and compared with the individual analytic responses for different $t$. In the first test $\Delta \theta$ was taken as $\pi/16$ corresponding to $MM = 16$. This comparison indicates a large discrepancy between the numerical and analytic spectra as can be seen by a study of Fig. 3.4.

To determine the cause of this discrepancy and to test the adequacy of the step-boundary condition (3.14), a Gaussian pulse is used as an input into the wave program and the results from two tests compared with ray theory, (Appendix B). The parameters used for the tests are given in Table 3.5. A comparison of the two test runs and the analytic results show first that the angular resolution ($\Delta \theta = \pi/16$) used previously in Model I is not nearly good enough for Model II, (Figs. 3.5 and 3.6). Next it showed that increasing the angular resolution by a factor of more than three ($\Delta \theta = \pi/50$) gave reasonable agreement with ray theory, Table 3.6.

The wave equation (3.8) is again integrated numerically with the new angular resolution, and the amplitude and spectra of $Y(t,k)$ re-calculated as above. The comparison between these results and
Figure 3.4 Comparison of numerical and analytic frequency spectra for wave mode zero ($t = 0$) and $\Delta \theta = \pi/16$, Model II.
Table 3.5 Test parameters for Gaussian input, Model II

\[ \sigma = \text{standard deviation Gaussian pulse} \]
\[ \Delta t = 4 \text{ sec} \]
\[ X(t) = \exp[-(t/\sigma)^2] \]

3\(\sigma\) taken as truncation point of Gaussian tails

<table>
<thead>
<tr>
<th>Test 1</th>
<th>Test 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta \theta = \pi/16)</td>
<td>(\Delta \theta = \pi/50)</td>
</tr>
<tr>
<td>(\sigma = 4 \Delta t)</td>
<td>(\sigma = 8 \Delta t)</td>
</tr>
</tbody>
</table>

Both tests were evaluated at \((n,m) = (10,2), (20,2), (30,2)\) and test 2 at \((50,2)\), where \(r = n\Delta r\), \(\theta = (m-2)\Delta \theta\), and \(m = 2\) at \(g = 0^\circ\). The inner boundary is at \(n = 10\), the step boundary at \(n = 30\).
\[ (n,m) = (30,2) \]

\[ \Delta t = 4 \text{ secs} \]

\[ t = k \Delta t \]

Figure 3.5 Comparison of results of Gaussian pulse input tests (Model II), at the step-sill.
Figure 3.6 Comparison of results of Gaussian pulse input tests (Model II), on the sill.

\[(n, m) = (20, 2)\]

\[\Delta t = 4 \text{ secs}\]

\[t = k \Delta t\]
Table 3.6 A comparison of numerical and ray theory arrival times and wave heights for plane Gaussian pulse input, Model II

<table>
<thead>
<tr>
<th>(n,m)</th>
<th>Arrivals</th>
<th>t/\Delta t</th>
<th>Heights</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Num.</td>
<td>Ray</td>
<td>Num.</td>
</tr>
<tr>
<td>50,2</td>
<td>1st Arr.</td>
<td>88</td>
<td>88</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>1st Refl.</td>
<td>138</td>
<td>139</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>tail</td>
<td>166</td>
<td></td>
<td>-0.06</td>
</tr>
<tr>
<td></td>
<td>2nd Refl.</td>
<td>290</td>
<td>290</td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td>tail</td>
<td>312</td>
<td></td>
<td>-0.19</td>
</tr>
<tr>
<td>30,2</td>
<td>1st Arr.</td>
<td>112</td>
<td>113</td>
<td>1.44</td>
</tr>
<tr>
<td></td>
<td>tail</td>
<td>141</td>
<td></td>
<td>-0.10</td>
</tr>
<tr>
<td></td>
<td>1st Refl.</td>
<td>265</td>
<td>265</td>
<td>0.27</td>
</tr>
<tr>
<td></td>
<td>tail</td>
<td>286</td>
<td></td>
<td>-0.26</td>
</tr>
<tr>
<td>20,2</td>
<td>1st Arr.</td>
<td>151</td>
<td>151</td>
<td>1.60</td>
</tr>
<tr>
<td></td>
<td>tail</td>
<td>174</td>
<td></td>
<td>-0.12</td>
</tr>
<tr>
<td></td>
<td>1st Refl.</td>
<td>227</td>
<td>227</td>
<td>0.82</td>
</tr>
<tr>
<td></td>
<td>tail</td>
<td>248</td>
<td></td>
<td>-0.56</td>
</tr>
<tr>
<td></td>
<td>2nd Refl.</td>
<td>304</td>
<td>303</td>
<td>-0.62</td>
</tr>
<tr>
<td></td>
<td>tail</td>
<td>326</td>
<td></td>
<td>0.11</td>
</tr>
<tr>
<td></td>
<td>3rd Refl.</td>
<td>379</td>
<td>379</td>
<td>-0.23</td>
</tr>
<tr>
<td>10,2</td>
<td>1st Arr.</td>
<td>189</td>
<td>189</td>
<td>3.24</td>
</tr>
<tr>
<td></td>
<td>tail</td>
<td>209</td>
<td></td>
<td>-0.99</td>
</tr>
<tr>
<td></td>
<td>1st Refl.</td>
<td>342</td>
<td>341</td>
<td>-1.07</td>
</tr>
<tr>
<td></td>
<td>tail</td>
<td>365</td>
<td></td>
<td>0.74</td>
</tr>
</tbody>
</table>

(1) Loss of height probably due to creation of tail and diffraction.
(2) Compounding of loss by tails formed at shore and at step, plus diffraction.
(3) Slight loss due to tail formation at step.
(4) Triple compounding of tail loss and diffraction.
the analytic are presented in Figs. (3.7) through (3.11). The shift in frequency of the spectral peaks as \( f \) increases is probably due to the limited radial and angular grid resolution. The lack of resolution apparent for the first few sharp peaks of the high wave modes is apparently due to the finite record length.

In an attempt to increase this resolution of the lower frequency spectral peaks for higher order wave modes, the wave program is re-run using (3.15) as input. Then the resulting response \( Y(t,k) \) and the input \( H(k) \) are modified with a "Hanning" window, which according to Blackman and Tuckey (1958) will keep the side lobes of the spectral filter as small as possible while concentrating the main lobe at \( f = 0 \). This window is given by

\[
D_2 = \begin{cases} 
1/2 \left[ 1 + \cos \pi \left( k - K_S \right) / NT \right], & |k| < NT, \\
0, & |k| > NT,
\end{cases}
\]

(3.21)

where \( t = k\Delta t \), \((k = 0, 1, 2, \ldots, NT)\).

The results indicate that the windowing not only did not improve the resolution as desired, but actually
caused a decrease in the amplitude of the spectral peaks and a general disruption of the smoothness evident in Figs. (3.7) through (3.11).

It is apparent, therefore, that the desired resolution will be obtained only by a many-fold increase in the record length; however, the computer time demanded for such resolution unfortunately transcends that available for this study.

Since R(f) cannot be resolved sufficiently under the present conditions, a good estimate of G(t) is unobtainable by the numerical approach used in Model II.
Figure 3.7 Comparison of numerical and analytic spectra for wave mode zero ($\ell = 0$) and $\Delta \theta = \pi/50$, Model II.
\( t = 1 \)
\( \Delta \theta = \pi / 50 \)
\( \Delta \tau = 0.078 \)

**Figure 3.8** Comparison of numerical and analytic spectra for wave mode one (\( t = 1 \)) and \( \Delta \theta = \pi / 50 \), Model II.
Figure 3.9 Comparison of numerical and analytic spectra for wave mode two ($\ell = 2$) and $\Delta \theta = \pi/50$, Model II.
Figure 3.10 Comparison of numerical and analytic spectra for wave mode three \((t = 3)\) and \(\Delta \theta = \pi/50\), Model II.
Figure 3.11 Comparison of numerical and analytic spectra for wave mode four ($\lambda = 4$) and $\Delta \theta = \pi/50$, Model II.

$\lambda = 4$
$\Delta \theta = \pi/50$
$\Delta \tau = 0.078$

analytic
numerical
CHAPTER IV

SUMMARY AND SUGGESTIONS FOR FURTHER RESEARCH

The purpose of this study was to develop a numerical method for recovering the deep water incident wave field, based on a reliable estimate of the transfer function and recorded marigram data for a given point near or at the island shoreline.

This was done quite adequately for the cylindrical island, Model I. With the addition of the step-sill (Model II) and the inherent complications to the spectral response from the resonant phenomena, however, a good response function $R(f)$ could not be obtained adequately. The results of this study seem to very strongly suggest, however, that given sufficient angular and radial grid resolution in the numerical solution of the wave equation and a sufficiently long record, the desired results could be obtained.

The problem encountered using this type of numerical technique to generate the needed response $Y(t)$ due to $X(t)$, is the large amount of computing time required. For instance, with $\Delta \theta = \pi/50$, $\Delta r = 1$ km, and $\Delta t = 4$ seconds, the numerical integration of (2.1) with the appropriate boundary conditions for 3000 time
55

iterations, takes 68 minutes on the IBM 360/65 computer. It is estimated that for really adequate definition of the response function for Model II, $\Delta \theta$ and $\Delta r$ should be $\pi/100$ and 0.5 km respectively. In order to maintain stable computations this would require $\Delta t = 2$ seconds. Finally the record length ought to be of the order of 5 hours thus requiring 9000 iterations. The machine time required for this would involve almost 14 hours on the IBM 360/65. On the other hand, the spectral analysis portion of the model, particularly with the use of various Fast Fourier Transform algorithms, represents a small part of the machine time.

It may also be true that the very sharp shelf boundary used in Model II made the resolution of $R(f)$ versus $f$ unrealistically complicated. For a more gradually sloping bathymetry the associated $R(f)$ probably would have broader resonant peaks which the present system could resolve.

It is suggested that in further studies, this method be tested on islands with more realistic bathymetry. As a first test it is suggested that the paraboloidal island (Vastano and Reid, 1967) be used, since the spectral response is known to be less complicated than that of Model II. The necessity for
increased resolution for the paraboloidal island will undoubtedly hold true.
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Van Dorn, W.G.  

Vastano, A.C. and R.O. Reid  
APPENDIX A

THE ANALYTIC SOLUTION TO THE STEP-ISLAND PROBLEM

The schematic diagram of the island (Model II) is given in Figure 3.3. Equations (3.8) through (3.12) will be used along with

\[ \zeta_s = \zeta - F e^{i(\beta x - \omega t)} \]  \hspace{1cm} (A.1)

where

\[ \beta = \omega / (gh_2)^{1/2}, \quad \omega = 2\pi f. \]

In this solution, however, the 'wave side' is at \( \theta = \pi \), the 'lee side' at \( \theta = 0 \). For region 1 \((a < r < b)\), a solution of (3.8) is

\[ \zeta(r, \theta) = \sum_{n=0}^{\infty} [A_n J_n(\alpha r) + B_n Y_n(\alpha r)] \cos n\theta e^{-i\omega t}, \]  \hspace{1cm} (A.2)

where

\[ \alpha = \omega / (gh_1)^{1/2}. \]
For region 2 \((r \geq b)\),

\[
\zeta(r, \theta) = F e^{i(\beta x - \omega t)} + \sum_{n=0}^{\infty} C_n H_n(\beta r) \cos n\theta e^{-i\omega t}. \tag{A.3}
\]

The functions \(J_n\) and \(Y_n\) are Bessel functions of the first and second kind, respectively, and \(H_n\) the Hankel function. Primes denote derivatives.

\[
B_n = - \frac{J_n'(\alpha b)}{Y_n'(\alpha b)} A_n, \tag{A.4}
\]

and

\[
A_n = \frac{2\beta h_2 F i^{n+1} \epsilon_n Y_n'(\alpha a)}{Q\pi \beta b}, \tag{A.5}
\]

where

\[
\epsilon_n = \begin{cases} 
1, & n = 0, \\
2, & n \neq 0,
\end{cases}
\]

and

\[
Q = \alpha h_1 H_n(\beta b)[J_n'(\alpha a)Y_n'(\alpha b) - J_n'(\alpha b)Y_n'(\alpha a)] \\
+ \beta h_2 H_n'(\beta b)[J_n(\alpha b)Y_n'(\alpha a) - J_n'(\alpha a)Y_n(\alpha b)]. \tag{A.6}
\]
At the island $r = a$, (A.2) yields

$$\zeta(a, \theta) = \frac{(2/\pi)^2 h_2 F/b}{\alpha a Q} \sum_{n=0}^{\infty} \frac{e_n^{i(n+1)\cos \theta}}{e^{-i\omega t}}. \quad (A.7)$$

For the special case where $a = b$ (i.e., no shelf), (A.6) reduces to

$$Q = \beta h_2 H_n'(\beta b) \frac{2}{\pi \alpha a},$$

so, (A.7) yields

$$\zeta(a, \theta) = \frac{2F}{\pi} \sum_{n=0}^{\infty} \frac{e_n^{i(n+1)\cos \theta}}{\beta b H_n'(\beta b)} e^{-i\omega t}, \quad (A.8)$$

which checks with the cylindrical island solution (Vastano and Reid, 1967).

Now let $x = wa/c_1$, $y = wb/c_1$, $z = wb/c_2$, then $y = bx/a$, $z = (h_1/h_2)^{1/2}bx/a$, and $c_1 = (gh_1)^{1/2}$ is the Lagrangian wave speed, where $i = 1, 2$. For simplicity consider $(h_1/h_2)^{1/2} = a/b$, then $z = x$. Also let $\lambda = b/a$, so (A.6) becomes
\[ Q = h_n b^{-1} \left\{ \frac{1}{\lambda} H_n(x) \left[ J_n'(x) Y_n'(\lambda x) - J_n'(\lambda x) Y_n'(x) \right] \right. \]
\[ + H_n'(x) \left[ J_n(\lambda x) Y_n'(x) - J_n'(x) Y_n(\lambda x) \right] \}, \quad (A.9) \]

and (A.7) becomes
\[
\zeta(x, \theta) = \left( \frac{2}{\pi} \right)^2 \sum_{n=0}^{\infty} \frac{\epsilon_n i^{n+1} \cos n\theta}{G_n(x)} e^{-i\omega t}, \quad (A.10)\]

where \( G_n(x) = xQ \). The spectral response for wave mode \( n \) is
\[
Z_n(x) = \left( \frac{2}{\pi} \right)^2 \epsilon_n / G_n(x). \quad (A.11)\]

For the purpose of this analysis, let
\[
R(x, \theta) = \zeta(x, \theta) e^{i\omega t}. \quad (A.12)\]

In general \( R(x, \theta) \), \( G_n(x) \) and \( Z_n(x) \) are complex. Equation (A.12) can be written
\[
R(x, \theta) = \sum_{n=0}^{\infty} Z_n(x) i^{n+1} \cos n\theta. \quad (A.13)\]
The function $G_n(x)$ can be expressed in its real and imaginary parts as

$$G_n^r(x) = x^2[\lambda^{-1}J_n(x)JY + J'_n(x)JYP], \quad (A.14)$$

$$G_n^i(x) = x^2[\lambda^{-1}Y_n(x)JY + Y'_n(x)JYP],$$

since $H_n(x) = J_n(x) + iY_n(x)$, where

$$JYP = J'_n(x)Y'_n(x) - J_n(x)Y'_n(x), \quad (A.15)$$

$$JYP = J'_n(x)Y'_n(x) - J_n(x)Y_n(x),$$

and $G_n(x) = G_n^r(x) + iG_n^i(x)$. $Z_n(x)$ becomes

$$Z_n(x) = \left(\frac{2}{\pi}\right)^2 \epsilon n G_n(x) \frac{G_n^r(x) - iG_n^i(x)}{G_n^r(x)^2 + G_n^i(x)^2}, \quad (A.16)$$

and can easily be sorted into its real and imaginary parts. Furthermore,
\[ Z_n(x) i^{n+1} = \begin{cases} -z^1_0, 4, 8, \ldots & z^r_0, 4, 8, \ldots \\ -z^r_1, 5, 9, \ldots & -z^i_1, 5, 9, \ldots \\ z^i_2, 6, 10, \ldots & z^i_2, 6, 10, \ldots \\ z^r_3, 7, 11, \ldots & z^i_3, 7, 11, \ldots \end{cases} \] (A.17)

From this sorting, it is possible to represent (A.13) in polar form, i.e.,

\[ M(x, \theta) = [R^r(x, \theta)^2 + R^i(x, \theta)^2]^{1/2} \]

(A.18)

\[ \varphi(x, \theta) = \tan^{-1} \left[ R^i(x, \theta) / R^r(x, \theta) \right] \]

for all \( \theta \), while (A.18) gives the spectral response for all modes at selected azimuth position, \( \theta \), around the island. Since (A.18) is a very complicated function of \( x \) and therefore difficult to reproduce by the numerical analysis of \( \zeta(n, m) \) at points around the inner boundary, the much smoother spectral response (A.16) was used for each wave mode \( n \), to test the analytic and numerical solutions for the step-sill island model.
APPENDIX B

A TEST OF THE STEP-BOUNDARY CONDITION AND THE WAVE EQUATION NUMERICAL MODEL WITH A GAUSSIAN PULSE

In order to test the adequacy of the step-cylinder wave equation model (Model II) in general, and the step-boundary condition in particular to render a correct response to a given input wave $X(t)$, a plane Gaussian pulse is used as the input and the results compared with those predicted by ray theory. Ray theory does not provide for diffraction, so the test stations selected were all at the zero degree azimuth.

Two tests were run using the parameters indicated in Table 3.5. Both tests indicate that the step boundary condition (3.14) is working properly. In test 1 the first arrival times at all stations were what ray theory would predict. The time of arrival of the first reflections also were rendered quite accurately, but in between these peaks there was a great deal of erroneous oscillation in the water level. Test 2 resulted in the elimination of these oscillations (Figs. 3.5 and 3.6) and rendered all arrivals at almost exactly the point in time where ray theory would predict, Table 3.6. Figures B.1, B.2, 3.5 and 3.6 show
the results at the four locations considered.

The other parameter used as an indicator was the wave amplitude at these stations. For the curved boundary surfaces of the step-island model, the correct determination of the wave amplitudes for the stations in question revolves around the proper determination of the location of the focal-point of the ray on the x-axis and the corresponding refraction factors. Figures B.3 and B.4 show schematically how $x_i$ and $\theta_i$ are found for a representative ray being reflected and transmitted at each of the two boundaries. Table 3.6 is a summary of the amplitudes and arrival times for the numerical and ray theory results. In ray theory no provision is made for tail formation so no values appear in the table under that heading. Furthermore, the decrease in amplitude in the numerical results can be explained by the formation of the tails.

Even though the increased width of the second pulse was undoubtedly a factor in the improved accuracy, the test shows clearly that sufficient grid resolution is of utmost importance.
Figure B.1  Presentation of results of Gaussian pulse input test 2, (Model II), in deep water. Plot is pulse height versus $t/\Delta t$. 

$(n,m) = (50,2)$ 

$\Delta t = 4 \text{ secs}$

$t = k \Delta t$ 

$\Delta \theta = \pi/50$
Figure B.2 Comparison of results of Gaussian pulse test (Model II), at the inner boundary. Plot is pulse height versus $t/\Delta t$. 

$(n,m) = (10,2)$

\[ \Delta t = 4 \text{ sec} \]

$t = \Delta t$
\[ \theta_0 / \approx 1, \text{ so } \theta_1 / \theta_0 = c_1 / c_2 = 1/3 \]

\[ r_1 = 3r_0 \quad \therefore x_1 = 1/2 \ r_1 \]

\[ c_1 = 1/3 \ c_2 \quad \therefore x_2 = 1/2 \ r_1 \]

\[ \theta_2 = 2/3 \ \theta_0 \]

\[ \theta_3 = 5/3 \ \theta_0 \]

\[ \theta_4 = \theta_0 \]

\[ \theta_5 = 8/3 \ \theta_0 \]

\[ x_3 = 5/24 \ r_1 \]

Figure B.3 Schematic of typical ray path used to calculate refraction factors for Model II; inbound ray.
\[ \begin{align*}
\theta_8 &= \frac{21}{9} \theta_0 \\
\theta_7 &= \frac{3}{9} \theta_0 \\
\theta_9 &= \theta_n \\
\theta_9 &= \frac{30}{9} \theta_0 \\
\theta_{10} &= 2 \theta_0 \\
x_4 &= \frac{7}{10} r_1 \\
x_5 &= \frac{3}{18} r_1 
\end{align*} \]

Figure B.4 Schematic of typical ray path used to calculate refraction factors for Model II; outbound ray.