A LOGNORMAL SIZE DISTRIBUTION MODEL FOR ESTIMATING STABILITY OF BEACH FILL MATERIAL

TECHNICAL MEMORANDUM NO. 16

DEPARTMENT OF THE ARMY
CORPS OF ENGINEERS
A LOGNORMAL SIZE DISTRIBUTION MODEL FOR ESTIMATING STABILITY OF BEACH FILL MATERIAL

by
W. C. Krumbein
and
W. R. James

TECHNICAL MEMORANDUM NO. 16
U.S. Army Coastal Engineering Research Center

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FOREWORD

In specifying fill material to be used in beach widening or nourishment projects, designers usually begin with a criterion that the grain size distribution characteristics of the native sand existing in the beach zone prior to the fill should be closely matched by those of the proposed fill material, and borrow sources are sought accordingly. A previous technical memorandum (No. 102) of the former Beach Erosion Board presented a method for specifying sand for beach fills to permit selection of that material most nearly compatible with natural materials native to a beach. However, ideally matched borrow sources within practicable transport distance of the beach fill site are becoming difficult to find, and so sources with sands of various other grain size distribution characteristics must be investigated and used. The basic problem then becomes one of determining what amount of fill material having different size distribution characteristics is required to be equally as effective as the design quantity of sand having size characteristics of the native beach material.

This report presents an analytical approach employing several simplifying assumptions wherein a mathematical solution to the problem is offered for those cases where material from the borrow source is not as well-sorted as the native beach material. For those cases where the fill material is better sorted than the native beach material, it is shown that there is no mathematical solution and it is further suggested that the determination of equivalent volumes must be based on empirical procedures considering slope adjustments. While it is recognized that procedures discussed in this report embody considerable simplification of complex shore processes, numerical results which can be obtained by the application of these procedures do offer a rational basis for estimating the most probable costs to be associated with a beach fill project.

This report was prepared by Dr. W. C. Krumbein, Professor of Geology at Northwestern University, and Mr. W. R. James, a graduate student in geology at that institution, in pursuance of Contract DA-49-055-CIV-ENG-64-15 with Dr. W. C. Krumbein. This contract provides in part for Dr. Krumbein's consulting advice and service on particular problems involved with the study of beach phenomena.

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>LIST OF FIGURES</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>1</td>
</tr>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2. Selection of a Model</td>
<td>2</td>
</tr>
<tr>
<td>3. Empirical Approach</td>
<td>3</td>
</tr>
<tr>
<td>4. Analytical Approach</td>
<td>6</td>
</tr>
<tr>
<td>5. Discussion of Example Used</td>
<td>8</td>
</tr>
<tr>
<td>6. Theoretical Development</td>
<td>9</td>
</tr>
<tr>
<td>7. Discussion of Table 3</td>
<td>13</td>
</tr>
<tr>
<td>8. Effects of Skewness on Critical Ratio</td>
<td>15</td>
</tr>
<tr>
<td>9. Generalizations from the Simple Model</td>
<td>15</td>
</tr>
<tr>
<td>10. Summary remarks</td>
<td>17</td>
</tr>
</tbody>
</table>

## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Typical example, Virginia Beach, Virginia</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>Observed curves and fitted phi-normal curves for Virginia Beach, Virginia, 1951 data</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>$\Phi_{crit}$ ratio illustrated by data for typical situation given in Table 1 (Case, Table 3)</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>Relations between native sand and borrow material</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>(Cases II, III and IV, Table 3)</td>
<td></td>
</tr>
</tbody>
</table>
A LOGNORMAL SIZE DISTRIBUTION MODEL FOR ESTIMATING STABILITY OF BEACH FILL MATERIAL

by

W. C. Krumbein and W. R. James
Northwestern University
Evanston, Illinois

ABSTRACT

This report discusses the problem of estimating the "extra amount" of beach fill to be used when the available borrow material characteristics are finer than the native sand composing the beach area. It is shown that an estimate of the extra amount of fill needed can be found if the native or design sand is better sorted than the available fill, regardless of whether the mean grain size of the borrow material is finer or coarser than the native sand. However, if the available fill is better sorted than the native, or design sand (again regardless of the relative mean grain sizes), there is no direct mathematical solution to the problem. In these latter cases, until further guiding criteria can be developed, past experience and other empirical data are required in reaching a practical decision regarding the amount of the fill that will be stable when emplaced on a given beach.

The mathematical theory underlying the method of analysis is based on a simple model that assumes essential lognormalcy of the particle size distributions. When this condition is satisfied, several generalizations can be made, which are summarized in a table (Table 3) and illustrated by actual or hypothetical examples. In essence, the theory defines a "critical ratio" of the amount of borrow material needed to produce a size distribution in the stabilized fill that is the same as that of the native sand. When the ratio has a maximum the problem can be solved, and this depends upon the degree of sorting being better in the native sand than in the borrow material.

1. Introduction

A problem of increasing concern in the construction or periodic nourishment of beaches is the location of borrow material* (at economic cost and in adequate supply) having similar or approximately similar textural properties as the native sand in the beach area. In recent years rapid development of land marginal to the coastal zone by public and private interests has had a significant influence on the availability, within economic limits, of borrow material. It is apparent that this trend will

* The term "borrow material" refers to the material that is mechanically removed from an area and transported to the beach zone.
become more critical with time. Thus, in many instances the readily available fill material is finer than may be desirable, as in the replenishment of a beach foreshore with material pumped or dredged from lagoonal areas landward of the beach proper, or hauled from inland sand dunes.

The practical necessity of using available fill material, even though it may be finer than desired, raises the question of how much extra fill should be placed on the foreshore, in order that beach profile adjustment, under natural processes, will have residual dimensions that will comply with design specifications. The authors of this report recently discussed the above general subject with staff members of the Coastal Engineering Research Center of the U. S. Army Corps of Engineers and it was agreed that the problem of borrow material in terms of native beach characteristics could be explored from an analytical standpoint. Such a study may provide further guidance on the development of improved criteria for specifying the characteristics of beach fill material for specific cases. Thus, the relationships developed in this report are presented only as analytical procedures involving several simplifying assumptions. Practical aspects are introduced or discussed to the extent possible, but such factors as the relationship of slope adjustment of the borrow material under prevailing or dominant littoral forces, and the normal alongshore transport of material into the beach area which would be mixed with the fill material, are not considered at this time. Obviously these and other factors must eventually be incorporated before complete guidelines and design criteria for borrow material specifications can be established.

Certain size gradation data for material composing the beach zone and the inland borrow zone were available for Virginia Beach, Virginia, and these data have been utilized to illustrate the present analytical development. It is emphasized that no general conclusions are established, and the results developed through use of the Virginia Beach data are for illustration only, until other factors, mentioned in the preceding paragraph, can be considered.

2. Selection of a Model

In approaching the problem analytically, it is apparent that we are concerned with the following question: how much of a given sand fill placed on a particular beach will remain as a stable addition to the beach, and how much will be winnowed out or otherwise removed by the natural shore processes? Another way of stating the problem is this: if the available fill material does not conform to the characteristics of the native sand, how much extra fill should be placed on the beach to provide the amount of stable sand specified in the design?

We shall approach the problem in the second of these ways, under a set of assumptions that permit selecting a mathematical model as a first approximation to the problem. The assumptions are as follows:

1) The native sand on the beach, which is used as the standard of reference, is in essential equilibrium with local shore processes.
2) That portion of the available fill material which corresponds with the native grain size distribution will be the part of the fill that remains.

3) The problem revolves only around the grain size distributions of the native sand and fill material, without the need for considering other attributes of the sand.

4) The grain sizes in both the native sand and the available fill material are essentially lognormally distributed.

These assumptions greatly simplify the problem, but they are not extreme. That is, the first assumption sets up the native sand as a standard of comparison for the fill material, even though in some practical situations (such as on eroding beaches) the native sand may not be completely stable. In such instances a "design sand" is specified, based on the past history of the beach that is to be replenished. We shall use the term "native sand" in this inclusive sense.

The second assumption states that the fill material will tend to approach the particle size distribution of the native sand by selective sorting and other winnowing action of the operative shore agents. The third assumption, that the particle size distributions of the sands are the controlling factors, is not extreme. This point is developed in detail in an earlier report on beach fill (Technical Memorandum No. 102, Beach Erosion Board, 1957, p. 5), where it is pointed out that other characteristics of beaches, such as foreshore slope, beach firmness, etc., are in large part determined by the particle size distribution.

The last assumption, regarding the nature of the particle size distributions, greatly simplifies the mathematical treatment, and permits examination of some restraints that may be present in this relatively simple model. Before developing the theoretical implications of the model, we shall illustrate the problem graphically with some actual data.

3. Empirical Approach

The two histograms of Figure 1 represent a set of data comparing the particle size distributions of the borrow material and native sand for a typical situation such as at Virginia Beach, Virginia, as of the time of fill operations in early 1951. The histograms are shown in phi-classes (see Technical Memorandum No. 102, Beach Erosion Board, 1957, p. 9 for relations among sieve mesh, diameters in mm., and phi values), in order to have equal-sized logarithmic groupings. It may be noted that the borrow material has a much larger content of fine grains smaller than 0.062 mm. (phi value = 4) than the native sand. Moreover, the native sand has considerably more material larger than 0.25 mm. (phi value = 2) than the borrow material.

Intuitively it would appear that in order to convert the histogram of the borrow material into the same form as the native material, much of the
FIGURE I, TYPICAL EXAMPLE, VIRGINIA BEACH, VIRGINIA, 1951
fine material must be winnowed out so that sufficient coarse material remains to produce the relatively high percentages of sand in phi-interval 0 to 3 of the native sand. The critical element here is the size class that shows the greatest discrepancy in the two histograms expressed as the ratio of the weight percent in the native sand to the weight percent in the same size class of the borrow material. This class of maximum ratio is not necessarily the modal class, but it can be located by listing the successive ratios for each class, as shown in Table 1.

**TABLE 1**

Native Beach Sand and Borrow Material
Virginia Beach, Va., Spring, 1951

<table>
<thead>
<tr>
<th>Size Grade in mm.</th>
<th>Size Grade in phi units</th>
<th>Native Sand wt. percent</th>
<th>Borrow Material wt. percent</th>
<th>Ratio of wt. percent in Native to wt. percent in Borrow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.032</td>
<td>5</td>
<td>0</td>
<td>13</td>
<td>0.00</td>
</tr>
<tr>
<td>0.032 - 0.062</td>
<td>5 - 4</td>
<td>0</td>
<td>12</td>
<td>0.00</td>
</tr>
<tr>
<td>0.062 - 0.125</td>
<td>4 - 3</td>
<td>5</td>
<td>13</td>
<td>0.38</td>
</tr>
<tr>
<td>0.125 - 0.250</td>
<td>3 - 2</td>
<td>21</td>
<td>24</td>
<td>0.92</td>
</tr>
<tr>
<td>0.25 - 0.50</td>
<td>2 - 1</td>
<td>37</td>
<td>25</td>
<td>1.48</td>
</tr>
<tr>
<td>0.50 - 1.0</td>
<td>1 - 0</td>
<td>31</td>
<td>9</td>
<td>3.44</td>
</tr>
<tr>
<td>1.0 - 2.0</td>
<td>0 - 1</td>
<td>4</td>
<td>2</td>
<td>2.00</td>
</tr>
<tr>
<td>2.0 - 4.0</td>
<td>-1 - 2</td>
<td>1</td>
<td>1</td>
<td>1.00</td>
</tr>
<tr>
<td>4.0 - 8.0</td>
<td>-2 - 3</td>
<td>1</td>
<td>1</td>
<td>1.00</td>
</tr>
<tr>
<td><strong>TOTAL</strong></td>
<td></td>
<td><strong>100</strong></td>
<td><strong>100</strong></td>
<td></td>
</tr>
</tbody>
</table>

Table 1 lists the weight percentage of material in the several size classes of Figure 1, and shows the ratio of the percentages for each class in the last column of the table. As may be seen, the maximum ratio, 3.44, falls in the size class 0.50 to 1.00 mm. (phi values 0 to 1). This maximum ratio is an estimate of the number of tons of borrow material that must be placed on the beach to retain one ton in equilibrium with shore processes. If we assume that the specific gravity of the borrow material is the same as that of the native sand, this means that it is necessary to use 3.44 cubic yards of borrow material for each cubic yard of stable beach material specified in the design.

The exact value of the maximum ratio, computed as in Table 1, will vary according to the class-intervals used, and also as to whether the classes are expressed in arithmetic or logarithmic units. This is illustrated in Table 2 for the same sand as in Table 1. The maximum ratio here is 3.20, which is in fair agreement with the ratio 3.44 of Table 1. The difference
in the individual class ratios is attributable in part to the use of classes defined directly by sieve mesh in Table 2, which yields unequal-sized classes in millimeter terms.

<table>
<thead>
<tr>
<th>Sieve Mesh Classes</th>
<th>Size Grade in mm.</th>
<th>Native Sand wt. percent</th>
<th>Borrow Material wt. percent</th>
<th>Ratio of wt. percent in Native to wt. percent in Borrow</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt; 140</td>
<td>&lt; 0.105</td>
<td>0</td>
<td>35</td>
<td>0.00</td>
</tr>
<tr>
<td>140 - 100</td>
<td>0.105 - 0.149</td>
<td>5</td>
<td>9</td>
<td>0.56</td>
</tr>
<tr>
<td>100 - 60</td>
<td>0.149 - 0.250</td>
<td>22</td>
<td>17</td>
<td>1.29</td>
</tr>
<tr>
<td>60 - 40</td>
<td>0.25 - 0.42</td>
<td>29.5</td>
<td>23</td>
<td>1.28</td>
</tr>
<tr>
<td>40 - 20</td>
<td>0.42 - 0.84</td>
<td>31</td>
<td>11</td>
<td>2.82</td>
</tr>
<tr>
<td>20 - 16</td>
<td>0.84 - 1.19</td>
<td>8</td>
<td>2.5</td>
<td>3.20</td>
</tr>
<tr>
<td>16 - 8</td>
<td>1.19 - 2.38</td>
<td>3</td>
<td>1.5</td>
<td>2.00</td>
</tr>
<tr>
<td>&lt; 8</td>
<td>&gt; 2.38</td>
<td>1.5</td>
<td>1</td>
<td>1.50</td>
</tr>
<tr>
<td>TOTAL</td>
<td></td>
<td>100.0</td>
<td>100.0</td>
<td></td>
</tr>
</tbody>
</table>

4. Analytical Approach

When the particle size distributions of the beach sand and fill material are known, or can be assumed, the foregoing analysis can be expressed mathematically. If the native sand and available borrow material are both essentially lognormally distributed, and if the native sand is better sorted than the borrow material, the maximum ratio can be had directly from their mean values and standard deviations. The theory is developed in Section 6; we use the formulas here to illustrate the method for the typical situation shown in Table 1 and Figure 2.

Figure 2 shows the cumulative frequency curves for the native sand and borrow material of Table 1 and Figure 1. The observed curves are drawn on arithmetic probability paper, using the phi scale values in Table 1. The straight lines through the observed data are obtained by the following procedure: Read the phi values of the points where the two observed curves cross the 84 and 16 percentile values on the vertical frequency scale. The results are:

<table>
<thead>
<tr>
<th></th>
<th>$\phi$ 84</th>
<th>$\phi$ 16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Native sand</td>
<td>2.41</td>
<td>0.59</td>
</tr>
<tr>
<td>Borrow material</td>
<td>4.72</td>
<td>1.20</td>
</tr>
</tbody>
</table>
FIGURE 2. OBSERVED CURVES AND FITTED PHI-NORMAL CURVES, FOR VIRGINIA BEACH, VIRGINIA, 1951 DATA
The phi mean is obtained by the relation \( M_\phi = (\phi_{84} + \phi_{16})/2 \), and the phi standard deviation is had by the relation \( s_\phi = (\phi_{84} - \phi_{16})/2 \). (The reader is referred to Technical Memorandum 102, pp. 13-15 for discussion of these graphic methods.) The results are:

<table>
<thead>
<tr>
<th></th>
<th>Native Sand</th>
<th>Borrow Material</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phi mean ( M_\phi )</td>
<td>1.50</td>
<td>2.96</td>
</tr>
<tr>
<td>Phi standard deviation ( s_\phi )</td>
<td>0.91</td>
<td>1.76</td>
</tr>
<tr>
<td>Phi variance ( s_\phi^2 )</td>
<td>0.83</td>
<td>3.10</td>
</tr>
</tbody>
</table>

The equation for computing the Phi Ratio, \( R_{\phi_{crit}} \) (see theory in later section), is as follows:

\[
R_{\phi_{crit}} = \left(\frac{s_{\phi_b}}{s_{\phi_n}}\right) e^{-(M_{\phi_b} - M_{\phi_n})^2/2(s_{\phi_n}^2 - s_{\phi_b}^2)}
\]  

(Equation 1)

By inserting the corresponding values we obtain:

\[
R_{\phi_{crit}} = 193 e^{-(2.13)/2(-2.27)} = 193 e^{0.47}
\]

Note that the exponent has become positive. The value \( e^{0.47} \) can be found in standard tables of the exponential function, and its value is 1.60. Hence,

\[
R_{\phi_{crit}} = 1.93 \times 1.60 = 3.09
\]

Thus, the ratio at the critical phi value (i.e. the maximum ratio as used in Tables 1 and 2) is 3.09, as against 3.44 and 3.20 obtained by the approximate methods used in the tables. These results are in fair agreement, and of the three, the best choice is 3.09. This is because the analytical method is independent of the size classes used in the analysis; in effect, the maximum ratio is found analytically in terms of the continuous frequency distributions as a whole, rather than by class increments.

5. Discussion of the Example Used

The above example, utilizing the Virginia Beach data, represents only one of several possible situations that may arise in the relations between native sand and borrow material. It is that case in which the mean grain size in mm. is larger in the native sand than in the borrow material, and in which the degree of sorting of the native sand is better than that in the borrow material. To avoid confusion here, because of the logarithmic measures used in the computations, it is to be recalled that the phi value increases as the sand becomes finer, and that the phi standard deviation
becomes larger as the sand becomes less well sorted. Thus, the diameter equivalent in mm. of the two sands is as follows: For the native sand, \( M_{0n} = 1.50 \), equivalent to 0.35 mm., and for the borrow material \( M_{0b} = 2.96 \), equivalent to 0.13 mm. (See Appendix A of Technical Memorandum No. 102 of the Beach Erosion Board for a conversion table from \( \varnothing \) to mm.)

Relations between native sand and borrow material are summarized in Table 3; in which the typical situation shown in Table 1 and Figure 2 forms the first entry (Case I). In addition, four other situations that could arise are listed in the table. For example, if dune sand is used as borrow material, it usually has a smaller mean diameter than the native sand, and is better sorted, so that this situation gives rise to Case II in Table 3. Under these circumstances, the critical ratio \( R_{\varnothing_{crit}} \) is a minimum instead of a maximum, and theoretically the borrow material can never be made to have the same frequency distribution as the native material, even if the mean size is larger in the borrow material. In practice, of course, there will be some portion of the borrow sand that will remain stable on the beach, but it will not have the particle size characteristics of the native sand.

Case III in Table 3 is the case where the phi means of the native sand and borrow material are the same, and the phi standard deviation of the borrow material is larger than that of the native sand. In this case the critical ratio \( R_{\varnothing_{crit}} \) occurs at the phi mean.

Case IV in Table 3 has the conditions where the phi standard deviations of the native sand and borrow material are the same but with different mean values. Here again it is theoretically not possible for the borrow material to attain the size distribution of the native sand.

The last entry in Table 3 (Case V) is the situation in which the native sand and the borrow material have the same mean grain size and the same degree of sorting. Theoretically, \( \varnothing_{crit} \) is here indeterminate, inasmuch as the critical phi ratio \( R_{\varnothing_{crit}} \) is equal to 1.0 all the way along the \( \varnothing \) axis. This is the case in which no transformation is needed inasmuch as the two size distributions are the same.

Table 3 will be discussed in greater detail in Section 7, after the theory leads to the several cases in the table is described.

6. Theoretical Development

It was stated earlier that when the particle size distributions of the native sand and borrow material are known, the problem becomes purely mathematical. For purposes of discussion it will be assumed that both the native sand and the borrow material are essentially lognormally distributed. That is, when the phi transformation is applied, the distribution is normal in \( \varnothing \). Hence, conventional normal distribution theory may be used directly.
The Critical Phi Ratio, $R_{\phi \text{crit}}$, and Relations Between Native Sand and Borrow Material

<table>
<thead>
<tr>
<th>Case</th>
<th>Relations between Native Sand and Borrow Material</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$M_{\phi n} \neq M_{\phi b}$ $\sigma_{\phi n} &lt; \sigma_{\phi b}$</td>
<td>$R_{\phi \text{crit}}$ is a maximum; i.e., the borrow material can be transformed into the native material.</td>
</tr>
<tr>
<td>II</td>
<td>$M_{\phi n} \neq M_{\phi b}$ $\sigma_{\phi n} &gt; \sigma_{\phi b}$</td>
<td>$R_{\phi \text{crit}}$ is a minimum; i.e., the borrow material cannot be transformed into the native material.</td>
</tr>
<tr>
<td>III</td>
<td>$M_{\phi n} = M_{\phi b}$ $\sigma_{\phi n} &lt; \sigma_{\phi b}$</td>
<td>$R_{\phi \text{crit}}$ is a maximum and occurs at the phi mean; the borrow material can be transformed into the native material.</td>
</tr>
<tr>
<td>IV</td>
<td>$M_{\phi n} \neq M_{\phi b}$ $\sigma_{\phi n} = \sigma_{\phi b}$</td>
<td>$R_{\phi \text{crit}}$ is at an inflection point; i.e., the borrow material cannot be transformed into the native material.</td>
</tr>
<tr>
<td>V</td>
<td>$M_{\phi n} = M_{\phi b}$ $\sigma_{\phi n} = \sigma_{\phi b}$</td>
<td>$R_{\phi \text{crit}} = 1.0$; i.e., the two curves are identical and no transformation is needed.</td>
</tr>
</tbody>
</table>

The equation of the normal phi curve is:

$$Y = f(\phi) = \frac{1}{\sigma_\phi \sqrt{2\pi}} e^{-\frac{(\phi - \mu_\phi)^2}{2\sigma_\phi^2}}$$

(Equation 2)

where $\mu_\phi$ is the population mean and $\sigma_\phi$ is the population standard deviation. When the observational data are based on samples, we use $M_\phi$ for the sample mean and $\sigma_\phi$ for the sample standard deviation, as in previous paragraphs and in Table 3. By inserting the additional subscripts $n$ and $b$, we distinguish between the native sand and the borrow material.
The ratio between the frequency of the native sand distribution at any given phi value \( f(n) \), to the frequency of the borrow material distribution at the same phi value, \( f(n) \), to the frequency of the borrow material distribution of the native sand by the normal phi distribution of the borrow material. This division yields the following phi ratio, \( R_\phi \):

\[
R_\phi = \frac{(s_{\phi n}^2/s_{\phi b}^2)}{(s_{\phi n}^2/s_{\phi b}^2)}
\]

(Equation 3)

By setting the first derivation of this expression equal to zero, the extrema (i.e., the maximum and minimum values of the ratio) can be found as follows:

\[
\phi_{\text{crit}} = \frac{(s_{\phi n}^2 M_{\phi b} - s_{\phi b}^2 M_{\phi n})}{s_{\phi n}^2 - s_{\phi b}^2}
\]

(Equation 4)

and the value of the ratio at the critical phi value, \( R_{\phi \text{crit}} \), is the expression given in Equation 1 in Section 3.

We may illustrate the use of Equation 4 with the data of the typical example given in Table 1. Figure 3 shows the two distributions as normal curves. The physical meaning of \( \phi_{\text{crit}} \) is that it represents the phi value at which the ratio \( R_{\phi \text{crit}} \) is a maximum in our example. By inserting the values of the phi mean and phi variance of the two samples, we obtain:

\[
\phi_{\text{crit}} = \frac{[(0.83 \times 2.96) - 3.10 \times 1.50] / (0.83 - 3.10)}{(-2.19)/(-2.27)} = 0.96
\]

The ordinate at \( \phi = 0.96 \) in Figure 3 is thus the specific diameter at which the maximum ratio occurs. It will be recalled that the analysis in Table 1 showed the maximum to be in the phi class 0 - 1, and this present analysis indicates that it is very close to \( \phi = 1 \). What this illustrates is that the method of analysis finds that location along the phi axis at which the height of the same ordinate from the base to its intersection with the lower curve, is a maximum. By visualizing other ordinates along the phi axis, it is apparent that this maximum occurs in the area where both curves are rising toward their maxima, and that on either side of \( \phi_{\text{crit}} \) the ratio between the ordinate heights diminishes away from its greatest value.

If we now compute the heights of the ordinates for the native sand and the borrow material at \( \phi = 0.96 \), the ratio of these heights should be \( R_{\phi \text{crit}} = 3.09 \). Ordinate heights for normal curves are readily obtained from standard tables of normal curve ordinates by computing \( z = (\phi - M_\phi) / s_\phi \). Thus, \( z_{\phi n} = (0.96 - 1.50)/0.91 = -0.593 \); and \( z_{\phi b} = (0.96 - 2.96)/1.76 = -1.136 \). (The negative values here merely mean that the ordinates lie to the left of their respective means.) The ordinates at these \( z \)-values, relative to the central highest ordinates of the curves, are \( H_n = 0.841 \) and \( H_b = 0.528 \) taken from standard tables. By multiplying the \( z \) values by the central ordinates
Height of native sand ordinate at $M_{\phi_n}$ is 43.8%. Height of borrow material ordinate at $M_{\phi_B}$ is 22.6%.

**FIGURE 3** $\phi_{\text{crit}}$ RATIO ILLUSTRATED BY DATA FOR TYPICAL SITUATION GIVEN IN TABLE I, (CASE I - TABLE 3)
We obtain \( Y_n = 0.841 \times 43.8 = 36.84 \) and \( Y_b = 0.528 \times 22.6 = 11.93 \). The ratio \( Y_n/Y_b = 36.84/11.93 = 3.09 = R_{\text{crit}} \). Thus, the two computations check.

Finally, by taking the second derivative of the expression in Equation 3 at the critical point \( (\varnothing_{\text{crit}}) \), the several conditions brought out in Table 3 can be derived. That is, it may be determined whether the ratio at the critical point is a maximum (as it is in Case I of Table 3), a minimum (as in Case II, Table 3), whether it occurs at an inflection point of the frequency distributions (as in Case III of Table 3), etc.

7. Discussion of Table 3

Figure 4 illustrates the several cases in Table 3 (except the first one, which is shown in Figure 3, and the last one, where the curves are identical). The top pair of curves are an example from Case II where the native sand is coarser and less well-sorted than the borrow material. This could arise when dune sand is used as borrow material, inasmuch as dune sand tends to be better sorted and finer than beach sand. Thus, although these two curves resemble those in Figure 3, the problem is that the spread of the size distribution for the borrow material is less than that of the native sand, and no amount of winnowing action can make the two coincide. In actual practice, where this situation arises, some of the dune sand will be stable on the beach, but normally only by an adjustment of foreshore slope to a much more gentle angle than is present with the native sand.

The central pair of curves in Figure 4 represent Case III, where the two phi means are the same, and in which the spread of the borrow material is greater than that of the native sand. This is almost an ideal situation (though not as ideal as Case V, where the borrow material exactly coincides with the native sand), inasmuch as winnowing action tends to reduce \( s_{\varnothing_b} \) to about the same order as \( s_{\varnothing_n} \). However, because the borrow material has a wider spread toward coarse material than the native sand, this situation may give rise to stringers or zones of very coarse sand or pebbly material on the restored beach. The finer sizes in the borrow material would presumably present no problem, inasmuch as they would be winnowed out by the shore processes.

In Case III the critical phi ratio, \( R_{\varnothing_{\text{crit}}} \), occurs at the phi mean, and is equal to \( s_{\varnothing_b}/s_{\varnothing_n} \). That is, if the typical situation shown in Table 1 and Figure 2 had been one in which native sand and borrow material were of the same grain size (i.e., had the same phi means), the maximum ratio would be simply \( 1.76/0.91 = 1.93 \), and something less than two cubic yards per design cubic yard of fill would be needed, rather than something more than three cubic yards per design cubic yard. This points up the advantage of having the mean grain size of the borrow material approach as nearly as possible to that of the native sand, in fact, there is some advantage in having the fill material somewhat coarser than the native sand.

The bottom pair of curves in Figure 4 are an example of Case IV in Table 3, where the phi standard deviations are the same in the native sand and borrow material, but in which the phi means do not coincide. (It does not
FIGURE 4 RELATIONS BETWEEN NATIVE SAND AND BORROW MATERIAL (CASE II, III, AND IV, TABLE 3)
make any difference here whether the borrow material is coarser or finer, but for illustration we assume that the borrow material has a smaller mean grain size than the native sand.) In this case the borrow material theoretically cannot be transformed into the same size distribution as the native sand, but again in practice it is possible to infer what may happen. There would presumably be a tendency for winnowing the fines from the borrow material, which would have the effect of introducing some asymmetry into the size distribution of the sand that remained stable on the beach. This stability would presumably also involve an adjustment of the foreshore slope to a gentler surface than occurs with the native sand.

8. Effects of Skewness on the Critical Ratio

The mathematical model that underlies the foregoing theoretical development assumes that both the native sand and the borrow material have lognormal grain-size distributions, which are normal when the diameters are expressed in the phi notation. As long as the observed particle size distributions do not depart from a straight line much farther than the examples in Figure 2, the conditions of the sedimentary model are probably satisfied. However, if either or both of the size distribution curves are strongly skewed, the theoretical development becomes much more complex, involving terms as high as fifth degree if both curves are skewed. If the effects of Kurtosis (i.e., unusual peakedness of flatness of the curves) is included, the theoretical solution would be even more complicated. Thus, for curves that are strongly bowed or S-shaped on probability paper, the theoretical model used in this analysis is over-simplified.

9. Generalizations from the Simple Model

The foregoing theoretical development and the summary material in Table 3 and Figure 4 suggest that at least three general relations between native sand and borrow material can give rise to satisfactory fill conditions, on the several assumptions listed in Section 2. For this simple model, the three situations are:

1) Case V of Table 3, where the size distribution of the borrow material has the same phi mean and phi standard deviation as the native sand. Here \( R_{\text{crit}} = 1.0 \), and the amount of fill emplaced is that specified in the design plans.

2) Case III of Table 3, where the borrow material has the same phi mean as the native sand, but the borrow material is less well-sorted than the native sand (i.e., \( s_{\text{b}} > s_{\text{n}} \)). Here the principal adjustment required of the fill material is that \( s_{\text{b}} \) approach \( s_{\text{n}} \) as sorting and winnowing action takes place. Here \( R_{\text{crit}} = s_{\text{b}}/s_{\text{n}} \), which probably lies between 1.5 and 2.0 in most cases. That is, the amount of borrow material to be emplaced may be expected to be no more than twice the specified design amount. (Exceptions can occur here, and the actual test is the value of the ratio \( s_{\text{b}}/s_{\text{n}} \).)

3) Case I of Table 3, which can have two variants:
a. Where $M_{\text{n}} < M_{\text{b}}$, that is, where the native sand has a coarser average grain size than the borrow material, with $s_{\text{n}} < s_{\text{b}}$. This is true in the typical situation shown in Table 1 and Figure 2. $R_{\text{c}}$ may be expected not to exceed the value 5, unless the grain size of the borrow material is much smaller than that of the native sand, and at the same time the phi standard deviation of the borrow material is much greater than that of the native sand. Where there is great disparity between the two materials, as long as they belong to Case III, however, the value of $R_{\text{c}}$ can be computed and used as a guide to the amount of fill to be emplaced per unit volume of design sand.

b. Where $M_{\text{n}} > M_{\text{b}}$, that is, where the native sand is finer in average size than the borrow material, and with the borrow material less well-sorted than the native sand, as in 3a. Here the problem arises that there is less fine-grained material and more coarse material in the fill than in the native sand. This could give rise to some shift toward a coarser average grain size, perhaps with some stringers or zones of coarse material on the replenished beach. Even so, $R_{\text{c}}$ can be computed and used as a guide in specifying the amount of fill needed to satisfy design plans.

Cases 3 a and b differ only in the relative effects of having fill material either coarser or finer than the native sand. As long as $s_{\text{n}} < s_{\text{b}}$, and $M_{\text{n}} > M_{\text{b}}$, Case I holds.

The other two cases in Table 3 (Cases II and IV) present special problems, inasmuch as theoretically in neither instance can the size distribution of the borrow material be transformed to that of the native sand. Except in very extreme cases, some of the fill material will be stable on the beach, but specification of the "extra amount" required is mathematically indeterminate. We can, nevertheless, make some speculations for Cases II and IV.

In Case II there are two variants, first, in which the native sand is coarser than the fill material, and second, in which the native sand is finer than the borrow material. The dune sand illustration of Figure 4 in Section 6 represents the first variant, and here, as stated, the adjustment would be toward a beach with a more gentle slope. If the borrow material is coarser than the native sand in Case II, the adjustment would probably involve some steepening of the foreshore. Because of the relative ease of winnowing the fines from emplaced material as against removing or redistributing coarse material, the implication is that a larger "extra amount" of fill is required when the borrow material is finer instead of coarser than the native sand. It is a matter of judgment only, but the likelihood is that the "extra amount" of fill needed in both these cases may lie somewhere between 5 and 10 times the design volume.

Case IV also has two variants, the same as in Case II. That is, the borrow material may be coarser or finer than the native sand. One of these variants, with finer borrow material, is shown in the bottom curves of Figure 4. As stated, in this variant the net effect presumably is that the fines will be winnowed out, leaving behind a skewed distribution. If the
borrow material is coarser in Case IV, the likelihood would appear to be a somewhat coarser end product, with steepening of the foreshore slope. As in Case II, the critical phi ratio $R_{\text{crit}}$ cannot be directly computed. Qualitative assessment of Case IV suggests that when the borrow material is coarser than the native sand, the "extra amount" of fill may be something less than 5 times design volume, whereas when the borrow material is finer than the native sand, the ratio may be as high as 10 to 1.

10. Summary Remarks

It is to be emphasized that the generalizations of Table 3 and the discussion in Section 9 are based on a mathematical model specified by the assumptions in Section 2. This model represents a considerable simplification of the complex shore processes acting on a beach. Although the emplaced borrow material will tend to assume a particle size distribution that is stable under the prevailing conditions, such adjustment may bring with it a winnowing out of fine materials, a modification of the foreshore and nearshore bottom slopes, and other changes in the gross characteristics of the beach area. Thus, the numerical results of the present analysis are first approximations only, and the generalizations apply only to the extent that the assumptions of Section 2 are satisfied.

It is our opinion, nevertheless, that this simple model does establish the importance of the relative degree of sorting of the native sand and borrow material. That is, it is very likely true that optimum adjustment will occur if the native sand is better sorted than the borrow material, (i.e., that $\sigma_{fn}$ is smaller than $\sigma_{fb}$), regardless of whether the mean grain size of the borrow material is larger or smaller than that of the native sand. If this condition of relative degrees of sorting is satisfied, the "extra amount" of fill required will depend largely on how much finer the available borrow material is than the native sand.

The principal questions that remain unanswered with the simple model used here have to do with available borrow material that is better sorted than the native sand (i.e., that $\sigma_{fb}$ is smaller than $\sigma_{fn}$). Under these circumstances the method of analysis illustrated in Tables 1 and 2 is not directly applicable. For such situations it is necessary to develop empirical procedures regarding slope adjustments of the available material for the coastal area of interest, in order to make practical decisions regarding the "extra amount" of fill needed to meet design specifications.
An analytical approach to the problem of estimating the "extra amount" of beach fill needed when available borrow material is finer than native sand composing the beach area is discussed. A mathematical solution is offered for those cases where borrow material is less well-sorted than native beach material. If fill is better sorted, it is shown that there is no direct mathematical solution and required fill quantities must be based on past experience and empirical procedures. Mathematical theory underlying the method of analysis is based on a simple model assuming lognormalcy of particle size distributions. A "critical ratio" of amount of borrow material needed to produce the size distribution of the native sand is defined such that when the ratio has a maximum, the problem can be solved.
An analytical approach to the problem of estimating the "extra amount" of beach fill needed when available borrow material is finer than native sand composing the beach area is discussed. A mathematical solution is offered for those cases where borrow material is less well-sorted than native beach material. If fill is better sorted, it is shown that there is no direct mathematical solution and required fill quantities must be based on past experience and empirical procedures. Mathematical theory underlying the method of analysis is based on a simple model assuming lognormalcy of particle size distributions. A "critical ratio" of amount of borrow material needed to produce the size distribution of the native sand is defined such that when the ratio has a maximum, the problem can be solved.