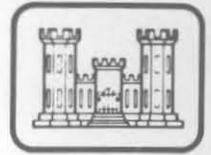
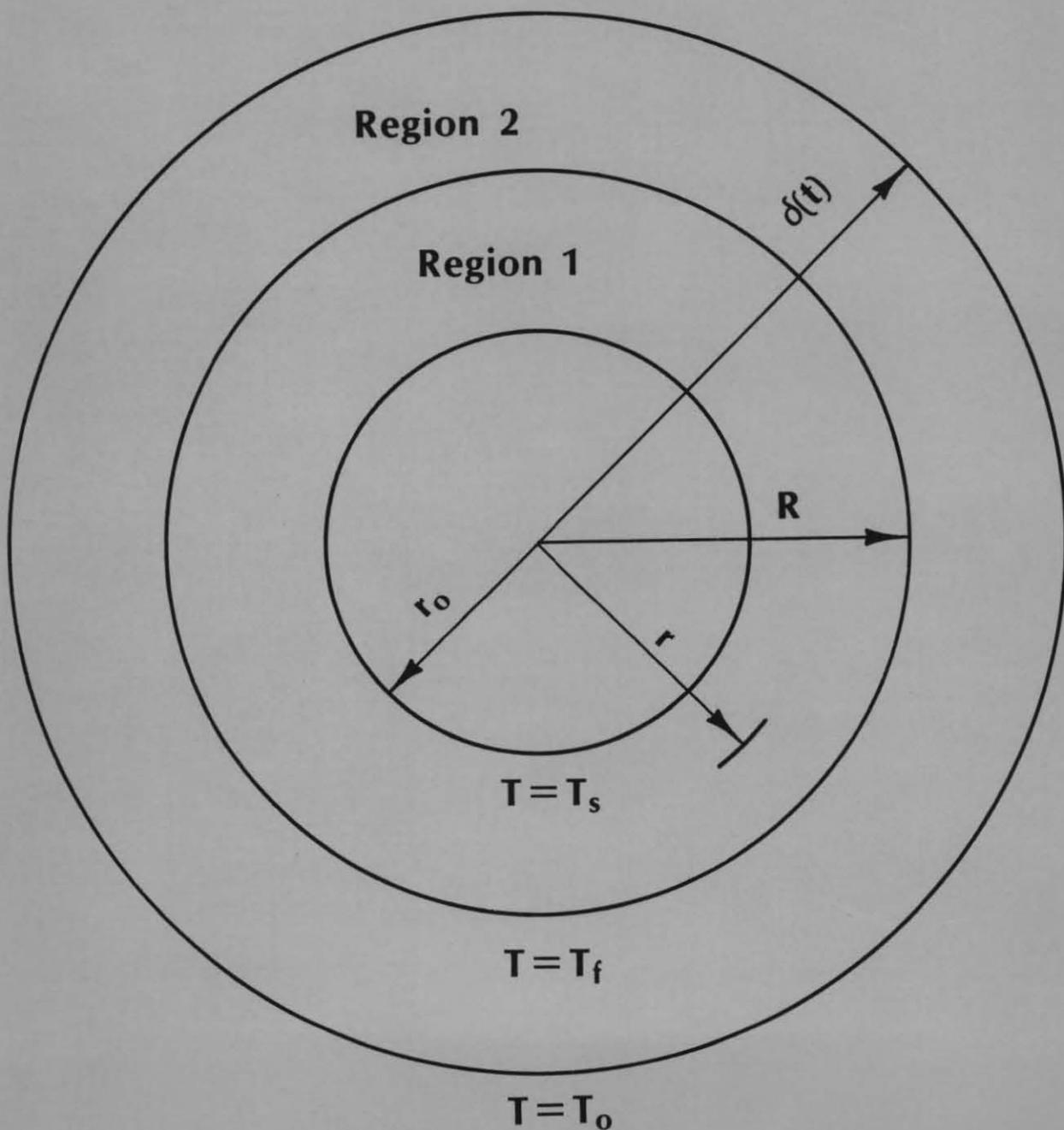


CRREL

REPORT 80-27



Phase change around a circular pipe



CRREL Report 80-27



Phase change around a circular pipe

V.J. Lunardini

December 1980

Prepared for
DIRECTORATE OF MILITARY PROGRAMS
OFFICE OF THE CHIEF OF ENGINEERS
By
UNITED STATES ARMY
CORPS OF ENGINEERS
COLD REGIONS RESEARCH AND ENGINEERING LABORATORY
HANOVER, NEW HAMPSHIRE, U.S.A.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER CRREL Report 80-27	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) PHASE CHANGE AROUND A CIRCULAR PIPE		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) V.J. Lunardini		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS U.S. Army Cold Regions Research and Engineering Laboratory Hanover, New Hampshire 03755		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS DA Project 4A762730AT42, Task D, Work Unit 17, DA Project 4A161101A91D, Work Unit 315
11. CONTROLLING OFFICE NAME AND ADDRESS Directorate of Military Programs Office of the Chief of Engineers Washington, D.C. 20314		12. REPORT DATE December 1980
		13. NUMBER OF PAGES 26
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Cold regions Cylindrical bodies Heat transfer Permafrost Pipes		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) No general, analytical solution exists for phase change around a cylinder, thus, approximate methods have been evaluated. The heat balance integral technique applied to the cylinder gave excellent results when compared to published numerical solutions. Graphical solutions are given for phase change about a cylinder for ranges of the Stefan number, superheat parameter, and property value ratios for typical soils. An approximate, general solution has been derived which is reasonably accurate and can be used for any values of the above-mentioned parameters. The effective thermal diffusivity method has been shown to be useful for practical problems of phase change.		

PREFACE

This report was prepared by Dr. Virgil Lunardini, Mechanical Engineer, Applied Research Branch, Experimental Engineering Division, U.S. Army Cold Regions Research and Engineering Laboratory.

The study was conducted under DA Project 4A762730AT42, *Design, Construction and Operations Technology for Cold Regions*, Task D(04), *Cold Regions Design and Construction*, Work Unit 17, *Heat Distribution Systems in Cold Regions*, and DA Project 4A161101A91D, In-House Laboratory Independent Research, Work Unit 315, *Frost Heave of Chilled Gas Pipelines*.

The report was reviewed by D. Haynes and J. Stubstad of CRREL, Dr. Y. Lee, University of Ottawa, and Dr. R. Reid, University of Tennessee. Appreciation is expressed for the comments of these reviewers.

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PHASE CHANGE AROUND A CIRCULAR PIPE

V.J. Lunardini

INTRODUCTION

The question of freezing and thawing of soils is important for engineering design in permafrost regions. For example, the support of buried pipes, utilidor, and foundations often relies on the mechanical strength of the permafrost and it is necessary to know the extent of thaw endured in the permafrost by heated structures. Thus solutions are sought to conduction problems with moving boundaries for analysis of these questions and synthesis of solutions. Interest in these types of problems has also been stimulated by the requirements of latent heat storage in solar energy systems.

For a homogeneous, semi-infinite slab, initially at a temperature different than the fusion value, there exists a general solution for a step change of surface temperature (Neumann 1860). Recently, Lunardini and Varotta (1980) have given an accurate, but approximate, solution to this problem in convenient form by using the heat balance integral technique. This solution is reproduced here since it will be used later for the cylindrical problem. The phase change depth, for the semi-infinite medium, is given by

$$X = 2\gamma\sqrt{\alpha_1 t} \quad (1)$$

$$\gamma^2 = \frac{-b_1 - \sqrt{b_1^2 - 4aS_T^2}}{2a} \quad (2)$$

where

$$a = \left(S_T + 2 + \frac{2k_{21}\phi S_T}{\alpha_{21}}\right) (S_T + 2)$$

$$b_1 = -2S_T \left(S_T + 2 + \frac{k_{21}\phi S_T}{\alpha_{21}}\right) - \frac{4}{3} \frac{(k_{21}\phi S_T)^2}{\alpha_{21}}$$

$$S_T = c_1(T_s - T_f)/\ell$$

$$\phi = (T_f - T_0)/(T_s - T_f).$$

The superheat parameter ϕ is a measure of the amount the initial temperature differs from the fusion temperature. S_T and ϕ are for the case of thawing. Freezing relations are given in the nomenclature. The Stefan number S_T is the ratio of the sensible to latent heat, while ϕ is the superheat or subcooling parameter.

In contrast to the Neumann problem, there are no general, exact solutions for phase change in cylindrical coordinates. Since this is an important geometry for practical applications, a significant effort has been expended upon analytical solutions limited to certain domains or approximate solutions. The geometry of the system (Fig. 1) is that of a cylinder surrounded by an infinite, homogeneous medium. A review will be made of several simple cases followed by new solutions of the general problem.

ZERO SUPERHEAT, $\phi = 0$

A number of solutions are available for the case when the temperature of the medium is initially uniform at the fusion temperatures. This condition greatly simplifies the problem since only one phase need be considered; that which is changing phase.

The energy condition at the phase change interface is given by the same relation as in the cartesian case

$$-k_1 \frac{\partial T_1}{\partial r} + k_2 \frac{\partial T_2}{\partial r} = \pm \rho \ell \frac{dR}{dt} \quad (3)$$

where the upper sign is for melting and the lower is for freezing. This assumes that the interface motion is in the positive direction of the coordinate r , and that the density is constant.

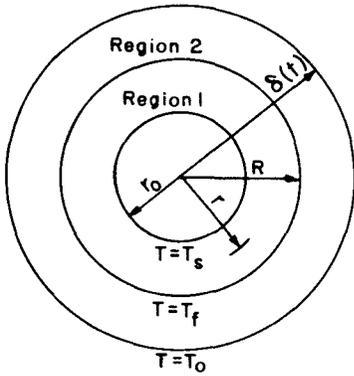


Figure 1. Geometry of cylindrical system.

Constant phase change rate

A simple system which has an exact solution requires that the temperature gradient at the phase change interface be a constant. This case has been discussed by Kreith and Romie (1955) for inward solidification. The problem for outward solidification can be written as follows:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (4)$$

$$T(R, t) = T_f \quad (4a)$$

$$k \frac{\partial T(R, t)}{\partial r} = \rho \ell \frac{dR}{dt} \quad (4b)$$

$$T(r_0, 0) = T_f \quad (4c)$$

The temperature gradient at the phase change interface is specified as

$$\frac{\partial T(R, t)}{\partial r} = G \quad (4d)$$

It is not necessary to solve eq 4 in order to find the location of the freezing front. From eq 4b and 4d

$$\frac{dR}{dt} = \frac{k}{\rho \ell} G \quad (5)$$

Thus the rate of movement of the phase change interface is a constant. The location is

$$R = \frac{kG}{\rho \ell} t + r_0$$

or

$$\beta = 1 + \frac{Gr_0}{(T_f - T_s)} \tau \quad (6)$$

where

$$\tau = \frac{\alpha_1 S_T t}{r_0^2}$$

$$\beta = \frac{R}{r_0}$$

Kreith and Romie (1955) used an iterative-type series solution for the temperature of the solid. This will not be given here since the problem is not too practical due to the necessity of imposing a complicated transient temperature at the cylinder surface.

Zero sensible heat, $S_T = 0$

If the sensible heat of the material is also neglected, then particularly simple solutions are available. Since the Stefan number is a measure of the sensible heat, this situation is equivalent to assuming that the Stefan number is zero. Carslaw and Jaeger (1959) presented a thaw solution for the case using the quasi-steady approximation when the surrounding medium is at the phase change temperature T_f . The quasi-steady method is accurate if $S_T \ll 1.0$. The problem is as follows:

$$\frac{d}{dr} \left(r \frac{dT}{dr} \right) = 0 \quad (7)$$

$$T = T_f; \quad r = R \quad (7a)$$

$$T = T_s; \quad r = r_0 \quad (7b)$$

Equation 3, the interface balance, is now

$$-\left(k \frac{\partial T}{\partial r} \right)_i + \left(k \frac{\partial T}{\partial r} \right)_f = \rho \ell \frac{dR}{dt} \quad (7c)$$

The solution for the phase change location is straightforward and is

$$\frac{1}{2} \beta^2 \ln \beta - \frac{\beta^2}{4} + \frac{1}{4} = \tau \quad (8)$$

Equation 8 will overestimate the thaw beneath the pipe since all of the energy transfer from the pipe will go into thawing. A correction can be made using an effective latent heat in place of ℓ . One example of an effective latent heat is

$$\ell_e = \ell \left(1 + C_{21} \phi S_T + \frac{S_T}{2} \right)$$

The surface heat transfer is

$$q = -k_1 2\pi r_o \left(\frac{\partial T_1}{\partial r} \right)_{r_o}$$

The heat transfer at the cylinder surface can then be written

$$q^* = \frac{qc_1}{2\pi k_1 \ell} = \frac{S_T}{\ln \beta}$$

Finite sensible heat

The heat balance integral can also be conveniently used for the no superheat problem and may also include large values of the Stefan number. The equations are

$$\frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = \frac{r}{\alpha} \frac{\partial T}{\partial t} \quad (9)$$

$$T(R, t) = T_f \quad (9a)$$

$$T(r_o, t) = T_s \quad (9b)$$

$$T(r, 0) = T_f \quad (9c)$$

$$-k \frac{\partial T(R, t)}{\partial r} = \rho \ell \frac{dR}{dt} \quad (9d)$$

Integration of eq 9 once, over the space coordinate, yields the heat balance integral equation

$$\alpha \left[r \frac{\partial T(R, t)}{\partial r} - r_o \frac{\partial T(r_o, t)}{\partial r} \right] = \frac{d\theta}{dt} - RT_f \frac{dR}{dt} \quad (10)$$

where the integrated temperature is

$$\theta = \int_{r_o}^R r T dr \quad (11)$$

The solution method consists of guessing an approximation for the temperature which will satisfy all of the conditions in terms of an unknown parameter such as $R(t)$. Lardner and Pohle (1961) noted that a logarithmic temperature approximation is more appropriate than a polynomial in r since the area is varying with r . They suggest that $T = f(r) \ln r$ be used as an approximation. The temperature is thus assumed to be a logarithmic relation, satisfying eq 9a and 9b.

$$T = T_s - (T_s - T_f) \frac{\ln(r/r_o)}{\ln(R/r_o)} \quad (12)$$

Equations 10-12 then yield a differential equation for the phase change location

$$\left[\left(\frac{\beta}{2} - \frac{\beta^2 - 1}{4\beta \ln \beta} \right) S_T + \beta \ln \beta \right] d\beta = d\tau \quad (13)$$

Equation 13 can be integrated to give

$$\frac{S_T}{4} \left[\beta^2 - 1 - 2(\ln \beta) - (\ln \beta)^2 - \dots - \frac{2^n (\ln \beta)^n}{n!} \right] + \frac{\beta^2}{2} \ln \beta - \frac{\beta^2}{4} + \frac{1}{4} = \tau \quad (14)$$

This equation reduces to the zero sensible heat solution (eq 8) when the Stefan number is zero. Equation 14 is comparable in accuracy to numerical solutions, valid for $\phi = 0$, with any value of S_T .

FINITE SUPERHEAT

The more practical problems, in which the initial temperature is not at the fusion value, present significantly more difficult analyses. Several methods have been utilized. All of the problems assume that the pipe is buried at an infinite depth because a finite burial depth will effect very severe restraints on the solution methods.

Quasi-steady solution

A simple solution can again be obtained with the quasi-steady state approximation. Khakimov (1957) investigated this problem and introduced the concept of a thermal layer of influence around the buried pipe. Consider the case of thawing of a medium initially frozen at $T = T_o$. A hot cylinder with a surface temperature T_s is inserted in the medium at time zero. The thaw effect (temperature change) is assumed to extend, at any time t , to a finite distance δ . That is, the temperature of the medium will be T_o at some location sufficiently far from the hot cylinder (see Fig. 1). This concept of a temperature penetration is also used in the heat balance integral method and in boundary layer theory.

From experimental evidence, Khakimov (1957) assumed that the ratio $b = (\delta/R)$ was a constant equal to 4.5. The assumption that b is a constant, for a given S_T and ϕ , is correct for the Neumann problem (Lunardini and Varotta 1980), but is invalid for a cylindrical system. Nevertheless it does simplify the equations and yields reasonable approximations.

The temperature will be the solution of

$$\frac{d}{dr} \left(r \frac{du}{dr} \right) = 0 \quad (15)$$

for each region with the boundary conditions

$$u_1 = u_2 = u_f; \quad r = R \quad (15a)$$

$$u_1 = u_s; \quad r = r_o \quad (15b)$$

$$u_2 = 0; \quad r = \delta \quad (15c)$$

where

$$u = T - T_o.$$

The solutions for the temperatures are

$$u_1 = \frac{(T_f - T_s)}{\ln(R/r_o)} \ln(r/r_o) + u_s \quad (16)$$

$$u_2 = \frac{u_f}{\ln(R/\delta)} \ln(r/\delta). \quad (17)$$

The total energy added to the thawing medium will be the latent heat needed to thaw the layer between r_o and R and the sensible heat used to increase the temperature of all the layers up to $r = \delta$. Thus,

$$Q = \pi(R^2 - r_o^2)L + 2\pi C_1 \int_{r_o}^R r u_1 dr + 2\pi C_2 \int_{R}^{\delta} r u_2 dr. \quad (18)$$

Carrying out the integration yields

$$Q = \pi(R^2 - r_o^2)L + 2\pi C_1 \left\{ (T_f - T_s) \left[\frac{R^2}{2} - \frac{(R^2 - r_o^2)}{4 \ln(R/r_o)} \right] + \frac{u_s}{2} (R^2 - r_o^2) \right\} - \pi R^2 p \quad (19)$$

where L = volumetric latent heat

C_1, C_2 = volumetric specific heat of thawed and frozen layers

$$p = C_2 u_f \left(\frac{1 - b^2}{2 \ln b} + 1 \right).$$

During a small time increment the change in the energy absorbed by the system must equal the energy added to the system at the cylinder surface.

$$q = -k_1 2\pi r_o \left(\frac{du_1}{dr} \right)_{r=r_o} = \frac{dQ}{dt}. \quad (20)$$

Combining eq 19 and 20 yields

$$\frac{k_1}{r_o^2 C_1} \int_0^t dt = \int_1^{\beta} m x \ln x + \frac{x}{2} - \frac{1}{4} \frac{x^2 - 1}{x \ln x} dx \quad (21)$$

where

$$m = \frac{L - p + u_f C_1}{C_1 (T_s - T_f)}.$$

The solution to this equation is

$$4\tau = 2m S_T \beta^2 \ln \beta + S_T (1 - m) (\beta^2 - 1) - S_T \psi \quad (22)$$

where

$$m = \frac{1}{S_T} + \phi \left[1 - k_{21} \alpha_{12} \left(\frac{1 - b^2}{2 \ln b} + 1 \right) \right]$$

$$\psi = \int_1^{\beta} \frac{x^2 - 1}{x \ln x} dx.$$

The universal function ψ has been numerically evaluated and is plotted as Figure 2, allowing the thaw for any soil to be approximated. The method is simple but limited in application to first estimates of thaw depth. When ϕ is small the method underpredicts the phase change depth and the accuracy decreases with time since b is not constant.

Heat balance integral solution

The heat balance integral method may be applied to the problem of finite superheat but the labor is increased considerably. The following analysis will allow simple numerical techniques to be applied to systems with typical soil parameters. The problem may be stated for a melting system as follows:

$$\frac{\partial}{\partial r} \left(r \frac{\partial T_1}{\partial r} \right) = \frac{r}{\alpha_1} \frac{\partial T_1}{\partial t} \quad (23)$$

$$T_1(R, t) = T_f \quad (23a)$$

$$T_1(r_o, t) = T_s \quad (23b)$$

$$T_1(r, 0) = T_o \quad (23c)$$

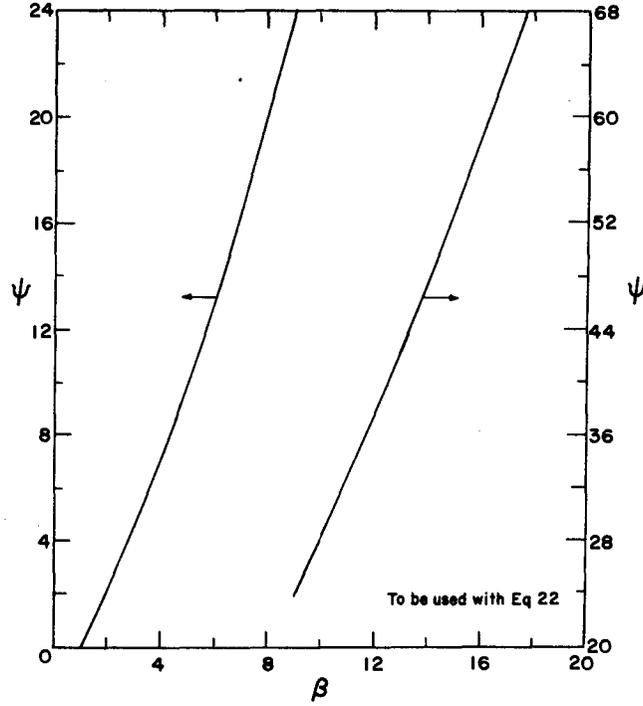


Figure 2. Quasi-steady phase change parameter.

$$-k_1 \frac{\partial T_1(R, t)}{\partial r} + k_2 \frac{\partial T_2(R, t)}{\partial r} = \rho_1 l \frac{dR}{dt} \quad (23d)$$

$$\frac{\partial}{\partial r} \left(r \frac{\partial T_2}{\partial r} \right) = \frac{r}{\alpha_2} \frac{\partial T_2}{\partial t} \quad (24)$$

$$T_2(R, t) = T_f \quad (24a)$$

$$T_2(\delta, t) = T_o \quad (24b)$$

$$\frac{\partial T_2(\delta, t)}{\partial r} = 0. \quad (24c)$$

The integrated temperatures are defined as usual.

$$\theta_1 = \int_{r_o}^R r T_1 dr. \quad (25)$$

$$\theta_2 = \int_R^{\delta} r T_2 dr. \quad (26)$$

The heat balance integral equations are

$$\alpha_1 \left[R \frac{\partial T_1(R, t)}{\partial r} - r_o \frac{\partial T_1(r_o, t)}{\partial r} \right] = \frac{d\theta_1}{dt} - R T_f \frac{dR}{dt} \quad (27)$$

$$-\alpha_2 R \frac{\partial T_2(R, t)}{\partial r} = \frac{d\theta_2}{dt} - T_o \delta \frac{d\delta}{dt} + T_f R \frac{dR}{dt} \quad (28)$$

An approximation for T_1 that satisfies eq 23a–23c is

$$T_1 = T_s - \frac{(T_s - T_f) \ln(r/r_o)}{\ln(R/r_o)}. \quad (29)$$

A similar approximation for T_2 that satisfies eq 24a–24c is

$$T_2 = T_o + (T_f - T_o) \frac{(\delta - r) \ln(r/\delta)}{(\delta - R) \ln(R/\delta)} \quad (30)$$

Equations 25, 26, 29, and 30 applied to eq 27 and 28 now lead to the following coupled system of equations.

$$\alpha_{12} S_T \frac{dF}{dr} = \frac{1}{\Omega - 1} + \frac{1}{\ln \Omega} \quad (31)$$

$$\left[S_T \left(\frac{\beta}{2 \ln \beta} - \frac{\beta^2 - 1}{4\beta (\ln \beta)^2} \right) + \beta \right] \frac{d\beta}{dr} = \frac{1}{\ln \beta} -$$

$$k_{21} \phi \left(\frac{1}{\Omega - 1} + \frac{1}{\ln \Omega} \right) \quad (32)$$

Table 1. Frozen and thawed properties ratios used with Figures 6 through 22.

x_R	Thaw		Freeze	
	$k_{12} = k_t/k_f$	$\alpha_{12} = \alpha_t/\alpha_f$	$k_{12} = k_f/k_t$	$\alpha_{12} = \alpha_f/\alpha_t$
0	1	1	1	1
0.1	0.873	0.792	1.1455	1.2626
0.2	0.7621	0.6326	1.3122	1.5808
0.3	0.6653	0.5091	1.5031	1.9642
0.4	0.5808	0.4121	1.7218	2.4266
0.5	0.5070	0.3354	1.9724	2.9815
0.6	0.4426	0.2742	2.2593	3.6470
0.7	0.3864	0.2252	2.5880	4.4405
0.8	0.3373	0.1855	2.9647	5.3908
0.9	0.2945	0.1533	3.3956	6.5231
1.0	0.2571	0.1271	3.8895	7.8685

$$F(\beta, \Omega) = \frac{\beta^2}{2} + \frac{\beta^2}{(1-\Omega) \ln \Omega} \left[-\frac{\Omega}{4} (\Omega^2 - 1) + \left(\frac{\Omega}{2} - \frac{1}{3} \right) \ln \Omega + \frac{1}{9} (\Omega^3 - 1) \right] \quad (33)$$

where $\Omega = \delta/R$. Equations 31-33 were solved numerically with a fourth order, Runge-Kutta, predictor-corrector technique. Since the problem, as specified, is initially singular at the origin, the Neumann solution was used to start the calculation. Sparrow et al. (1978) solved this problem numerically, with $\alpha_{12} = k_{12} = 1$

for a range of S_T and ϕ , also using the Neumann solution as a start. The results of the much simpler method presented here are within 5% of the values reported by Sparrow et al. The comparisons are shown in Figures 3-5. The calculations have been generalized for a range of α_{12} , k_{12} which are pertinent for soil systems, and are presented in Figures 6-22. The property values associated with each value of x_R are given in Table 1. These property ratios are obtained using the method of Lunardini and Varotta (1980). While the curves have been developed particularly for soils, they are valid for any medium with the same properties.

Tien and Churchill (1965) also numerically evaluated freezing outside a cylinder. Their calculations are more extensive than Sparrow et al. (1978) and the numerical technique was entirely different but the results are essentially the same.

Approximate methods

The numerical solutions, while very good, are often not as convenient as analytical solutions. Thus further work will be carried out to obtain approximate solutions which yield results with acceptable accuracy. The following analyses deal with approximate methods.

Coordinate transformation

A method suggested by Lin (1971) uses a coordinate transformation to reduce a problem with a variable phase change area, such as a cylinder, to one with a constant

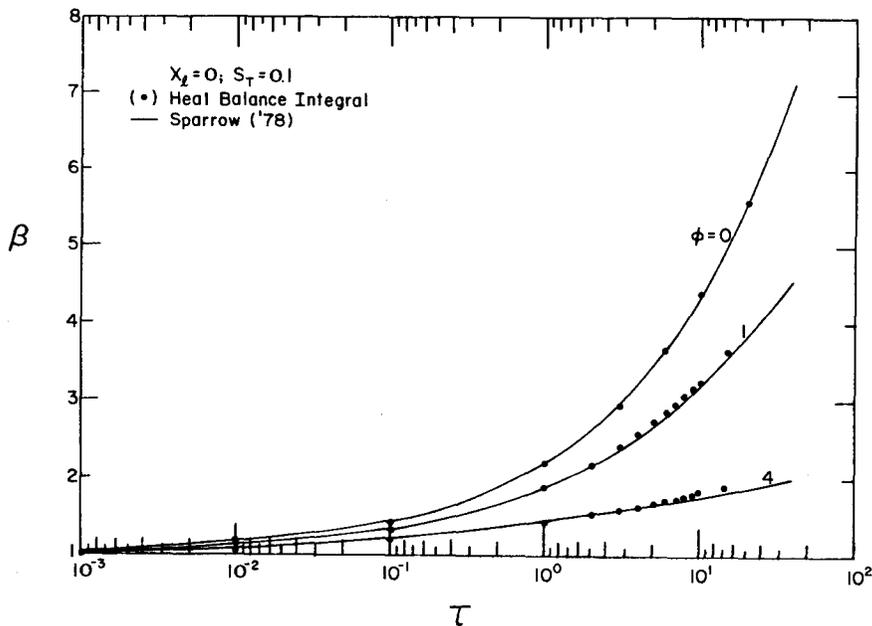


Figure 3. Accuracy of heat balance integral solution, phase change $S_T = 0.1$, $\alpha_{12} = k_{12} = 1.0$.

phase change area—the semi-infinite solid. Since the relations for the semi-infinite solid are well known (see eq 1 and 2), this is a useful procedure but, as will be seen, is limited in accuracy.

The following transformation will reduce the cylindrical system to the constant area case.

$$y = r_o \ln(r/r_o). \quad (34)$$

The phase change interface, which is the value of y when $r = R$, is related by

$$\eta = r_o \ln(R/r_o). \quad (35)$$

The governing equations for the cylindrical system then transform into the following system which is valid near the phase change interface where $r \approx R$.

$$\frac{\partial^2 T}{\partial y^2} = \frac{-k}{\rho \ell \alpha} \frac{\partial T(\eta, \eta)}{\partial y} \frac{\partial T}{\partial \eta} \quad (36)$$

$$T(y, 0) = T_f \quad (36a)$$

$$T(\eta, \eta) = T_f \quad (36b)$$

$$T(0, \eta) = T_s. \quad (36c)$$

The solutions of this system of equations are universal functions for all cross-sectional areas. However, the solutions are only valid near the phase change interface. The system of eq 36 need not be solved to use the method. The phase change interface rate of movement is given by

$$\frac{d\eta}{dt} = \frac{k}{\rho \ell} \left[\frac{A(r_o)}{A(R)} \right]^2 \frac{\partial T(\eta, \eta)}{\partial y}. \quad (37)$$

For the constant area case $A(r_o) = A(R)$ and

$$\frac{d\eta}{dt} = \frac{k}{\rho \ell} \frac{\partial T(\eta, \eta)}{\partial y} = g(\eta). \quad (38)$$

Thus if the velocity of the phase change interface for the constant area case is given by

$$\frac{dX}{dt} = g(X) \quad (40)$$

then, the phase change interface velocity for the cylindrical system is

$$\frac{dR}{dt} = \frac{r_o}{R} g\left(r_o \ln \frac{R}{r_o}\right). \quad (41)$$

The plane or Neumann solution is given by eq 1 and 2. From these relations

$$g(X) = \frac{2\gamma^2 \alpha_1}{X}. \quad (42)$$

Finally, the velocity of the cylindrical interface is

$$\frac{dR}{dt} = \frac{2\gamma^2 \alpha_1}{R \ln(R/r_o)}. \quad (43)$$

Equation 43 may be integrated to give

$$2\beta^2 \ln \beta - \beta^2 + 1 = \frac{8\gamma^2}{S_T} \tau. \quad (44)$$

Equation 44 may be compared to the quasi-steady solution, eq 8, for the case of no superheat, i.e. $S_T = 0$, $\phi = 0$. In the limit as $\phi \rightarrow 0$, eq 2 is $\gamma^2 = (S_T/2 + S_T)$ and in this case eq 44 is identical to eq 8.

While eq 44 is in an extremely convenient form for cylindrical systems, its accuracy is limited to certain ranges of S_T , ϕ and τ . This can be seen by comparing the phase change interface position β for a cylindrical system, calculated with eq 44, and the values computed numerically by Sparrow et al. (1978), given in Figures 23 and 24.

Equation 44 is accurate if $\tau/S_T < 1.0$, when $\phi = 4$. For smaller values of ϕ the time limit when eq 44 is accurate will increase. Equation 44 is so simple that it may be of value for quick, more or less crude, estimates, especially when $\phi \approx 0.0$. However a more accurate closed form relation for cylinders with superheat or subcooling will be evaluated in the next section.

Effective thermal diffusivity

Churchill and Gupta (1977) have introduced another method which allows the Neumann solution to be used for more complex geometries. The procedure involves replacing the nonlinear phase change problem with its linear analogue which does not include phase change. The thermal diffusivity of the latter problem is then replaced by an effective diffusivity which includes the latent heat. Since many solutions are available for non-phase change problems the method has potential for application to numerous freeze/thaw problems.

The derivation of the effective diffusivity to use is based upon the fact that the solution to the zero latent heat analogue of the Neumann problem (simply transient conduction in the semi-infinite medium) can be forced to agree with the Neumann solution if an effective diffusivity replaces the actual diffusivity. If the location of the freezing surface is important then one

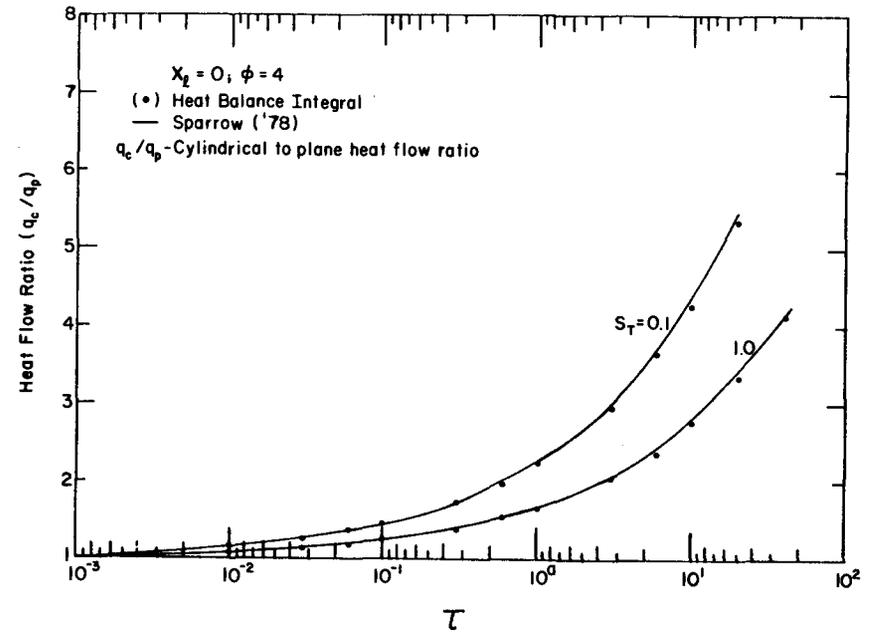
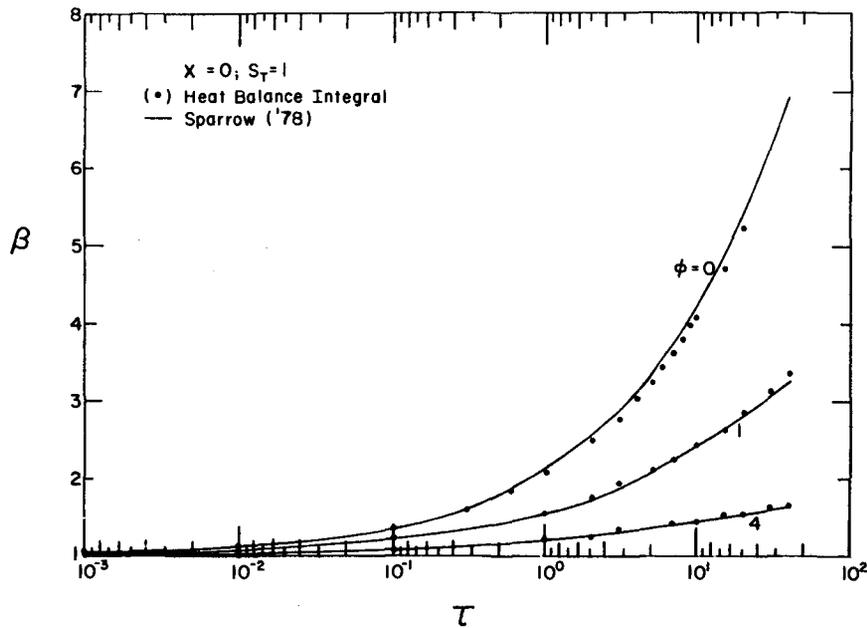


Figure 4. Accuracy of heat balance integral solution, phase change $S_T = 1.0$, $\alpha_{12} = k_{12} = 1.0$.

Figure 5. Heat flow rate comparisons $\phi = 4.0$, $\alpha_{12} = k_{12} = 1.0$.

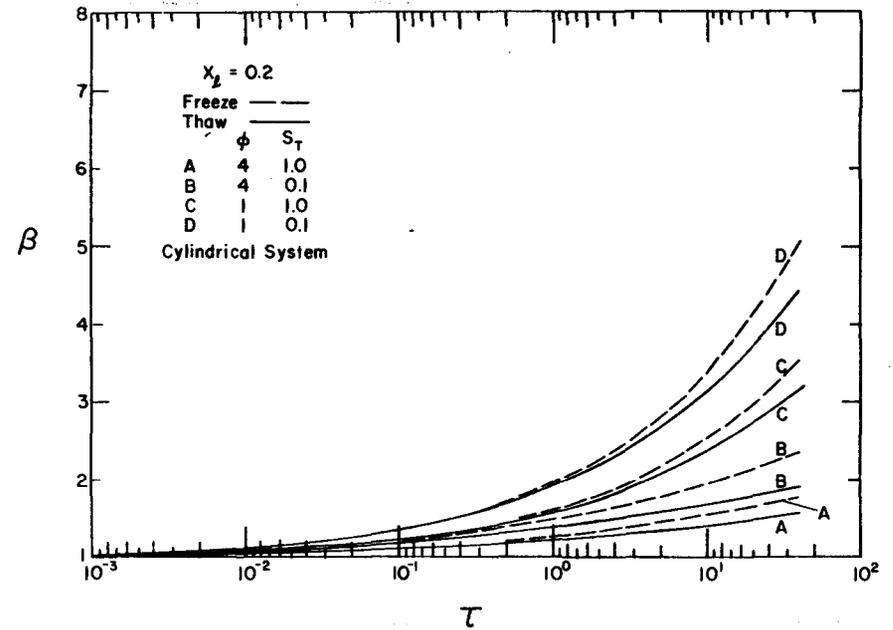
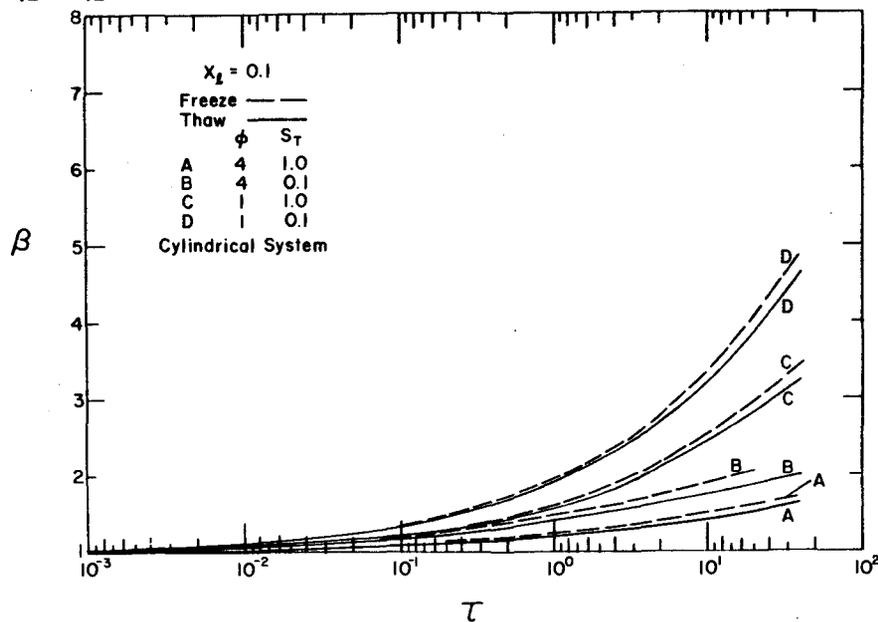


Figure 6. Phase change position vs time $x_q = 0.1$.

Figure 7. Phase change position vs time $x_q = 0.2$.

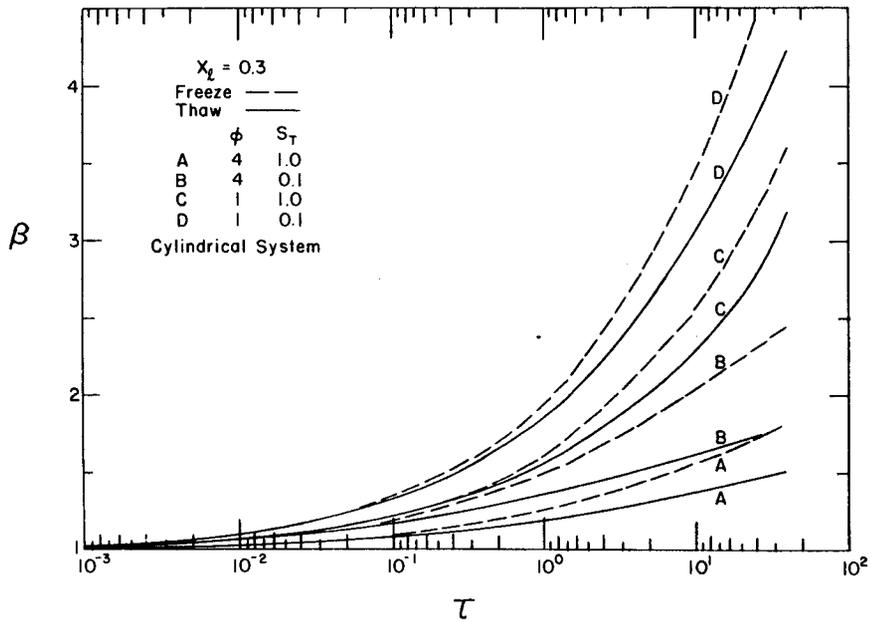


Figure 8. Phase change position vs time $x_q = 0.3$.

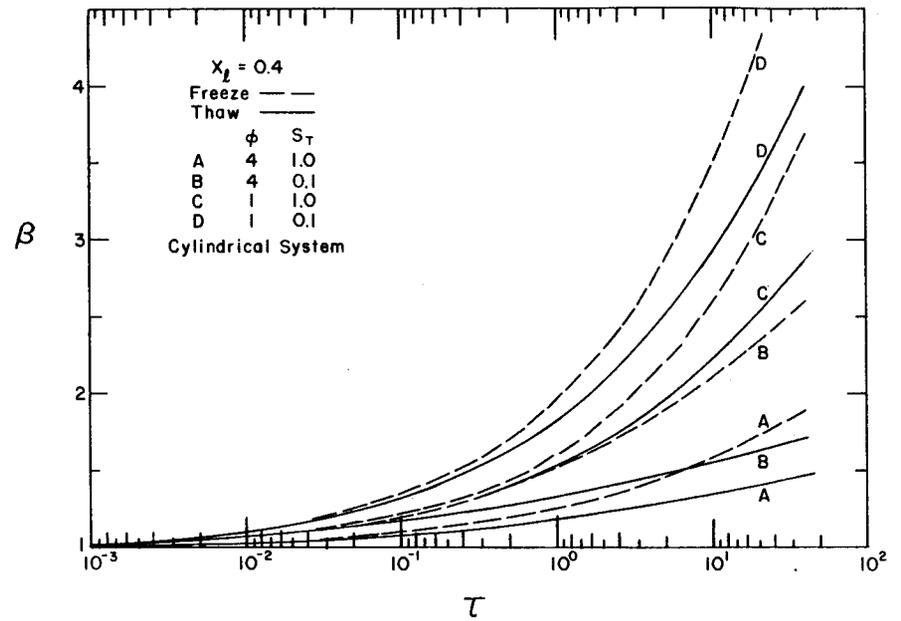


Figure 9. Phase change position vs time $x_q = 0.4$.

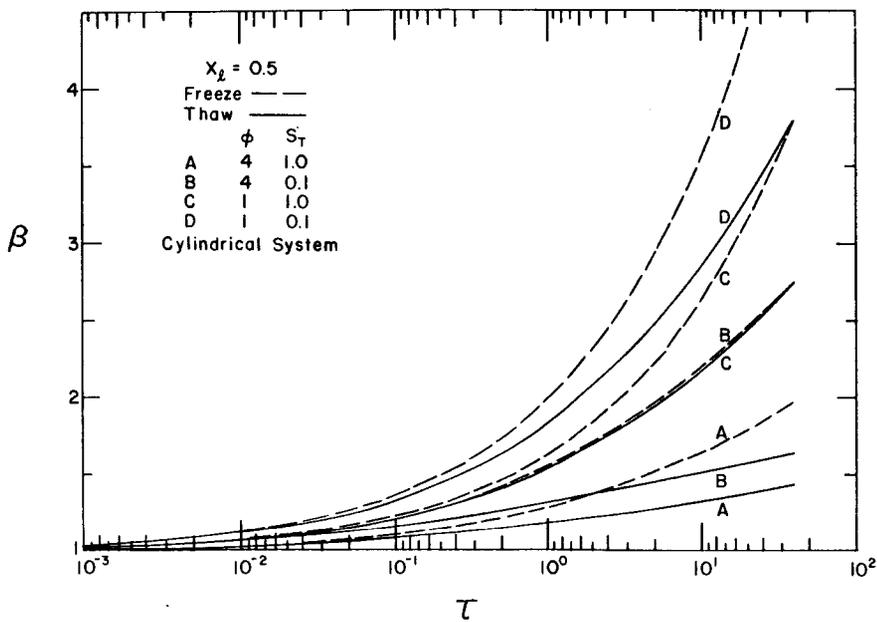


Figure 10. Phase change position vs time $x_q = 0.5$.

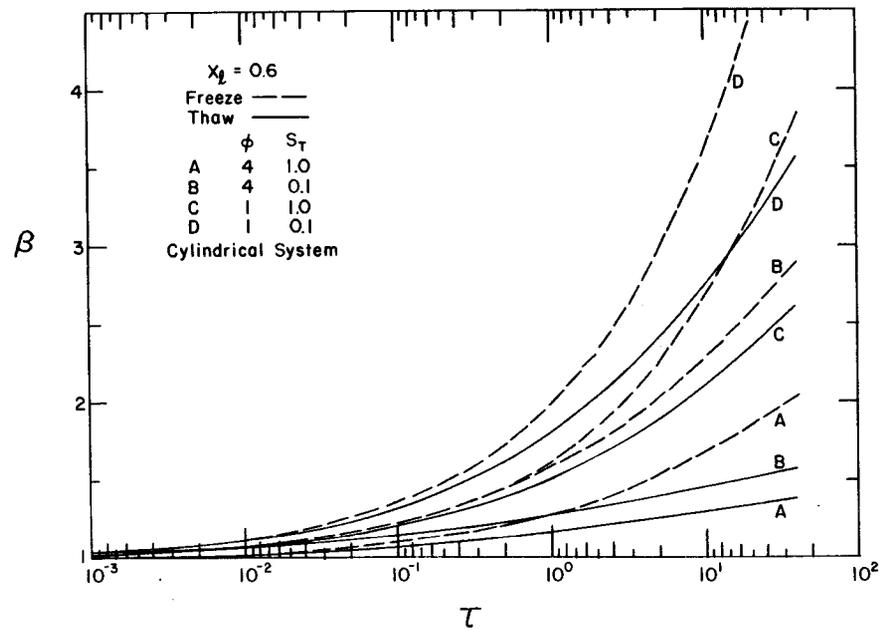


Figure 11. Phase change position vs time $x_q = 0.6$.

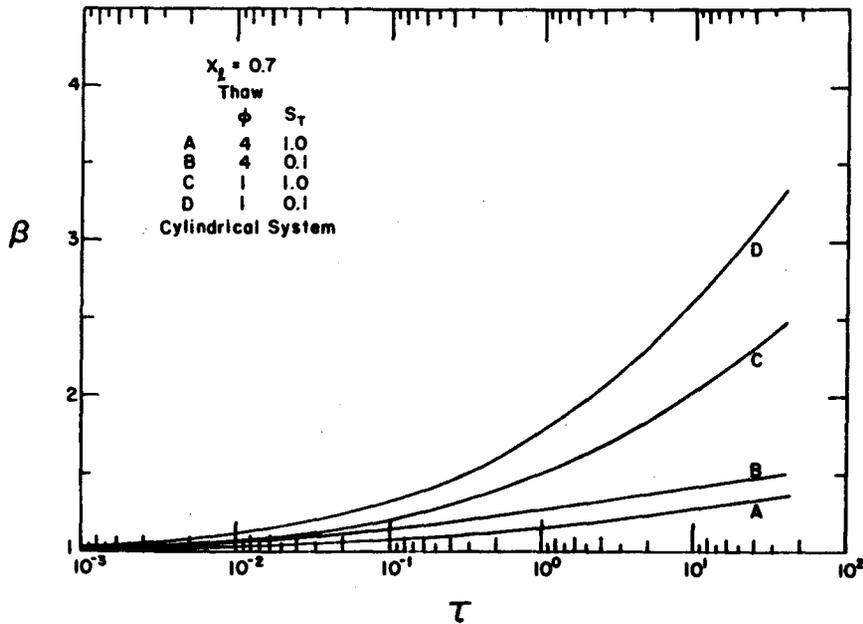


Figure 12. Phase change position vs time $x_2 = 0.7$, thaw.

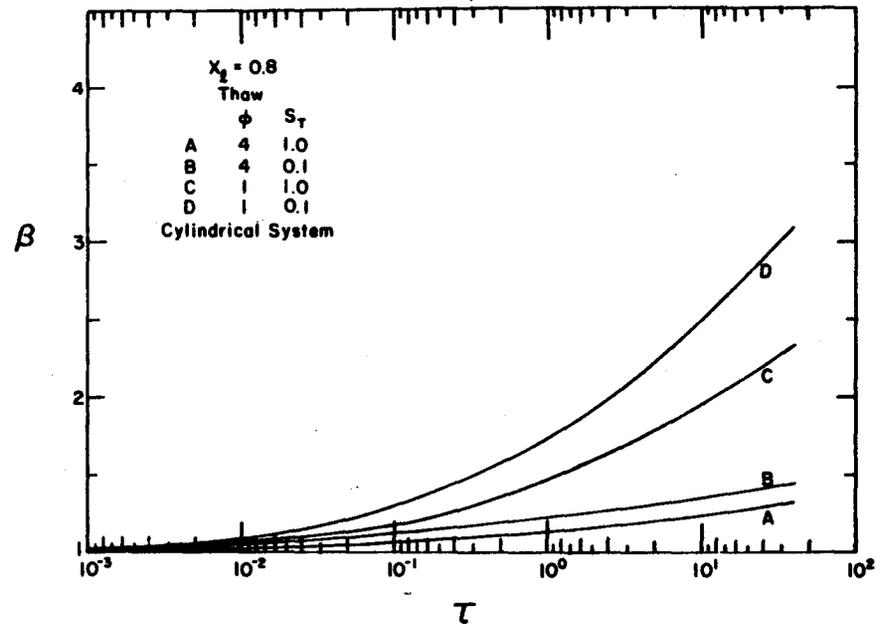


Figure 13. Phase change position vs time $x_2 = 0.8$, thaw.

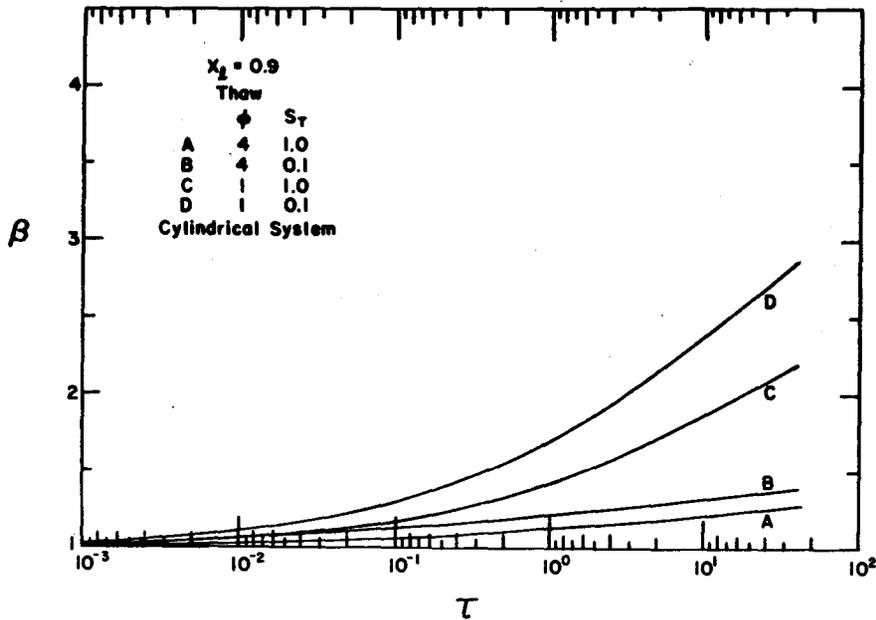


Figure 14. Phase change position vs time $x_2 = 0.9$, thaw.

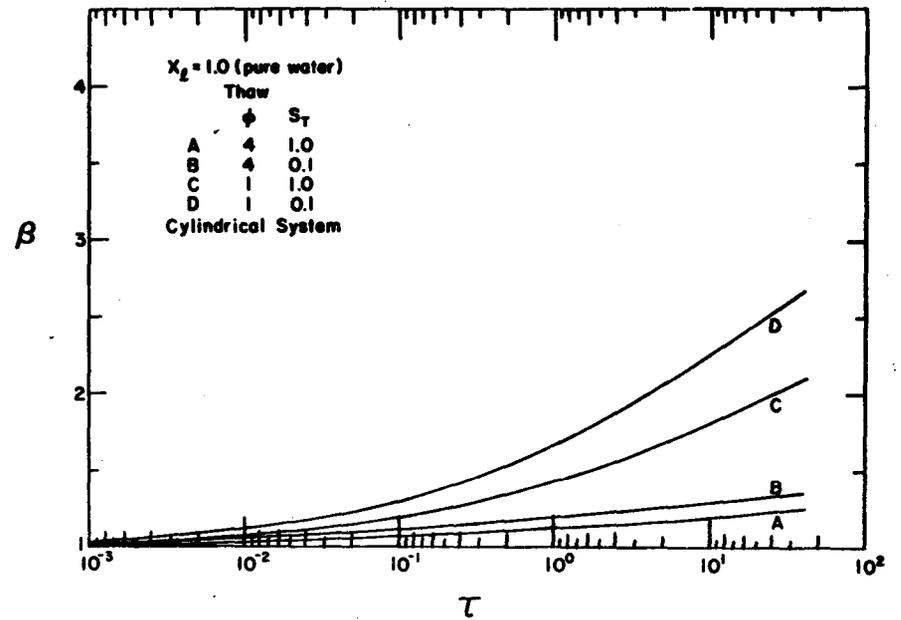


Figure 15. Phase change position vs time $x_2 = 1.0$ (pure water), thaw.

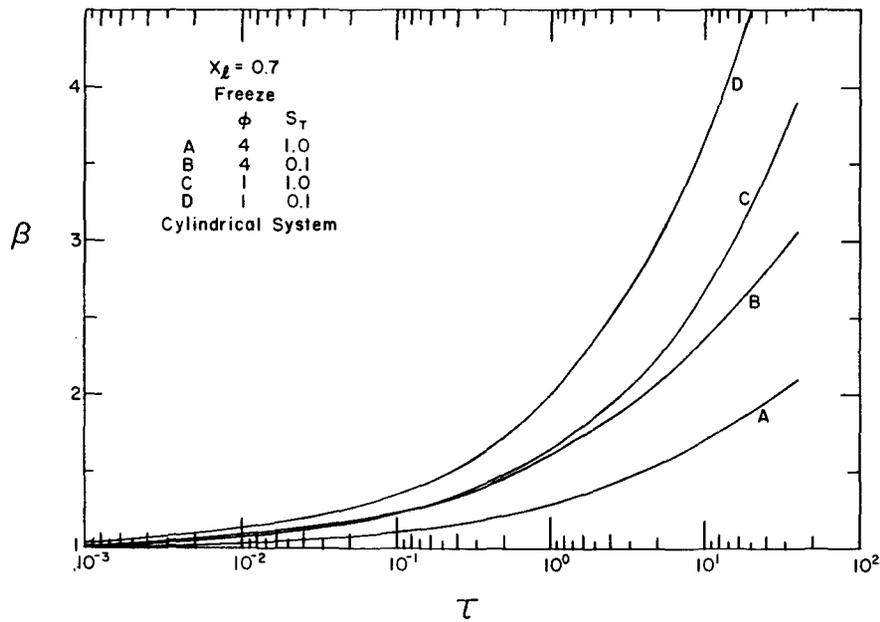


Figure 16. Phase change position vs time $x_0 = 0.7$, freeze.

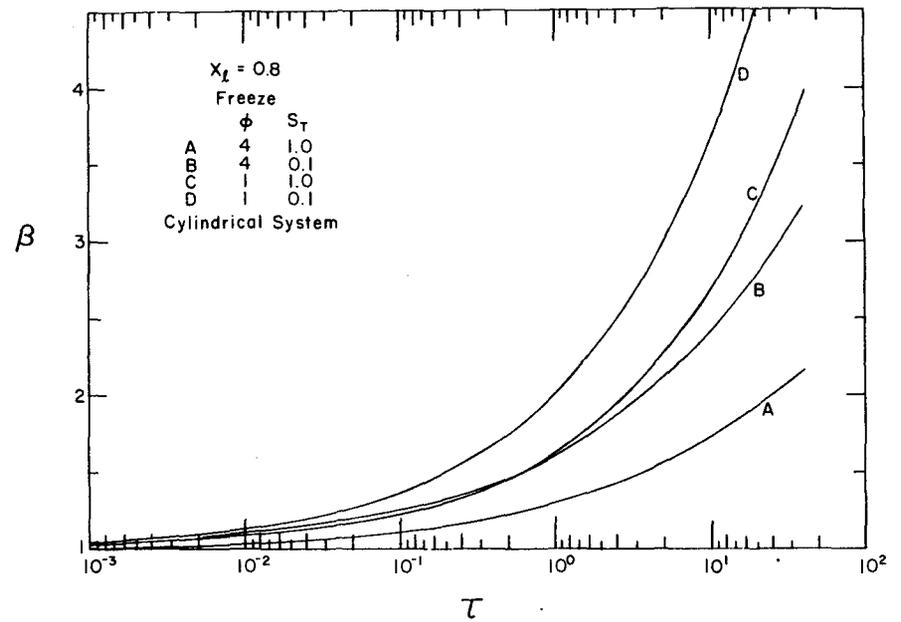


Figure 17. Phase change position vs time $x_0 = 0.8$, freeze.

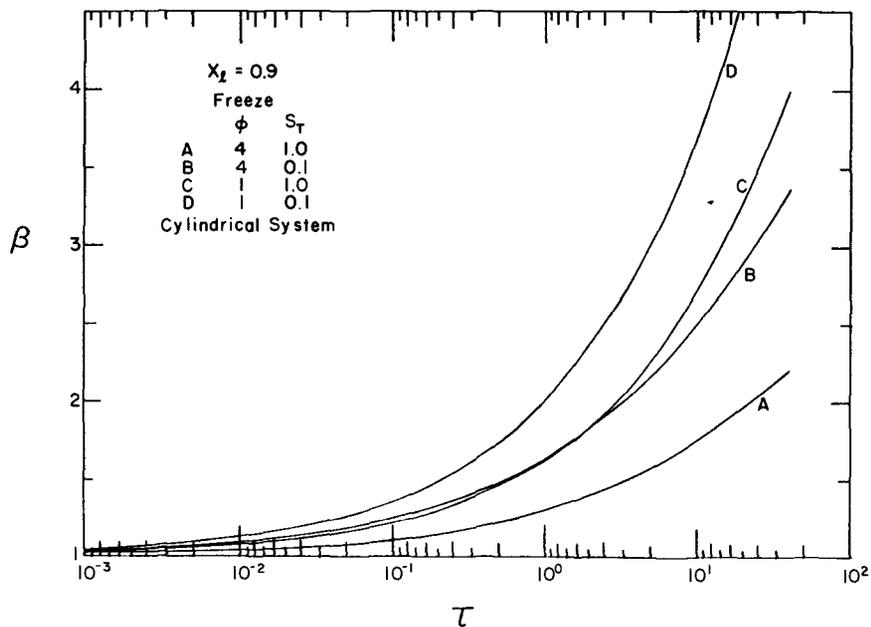


Figure 18. Phase change position vs time $x_0 = 0.9$, freeze.

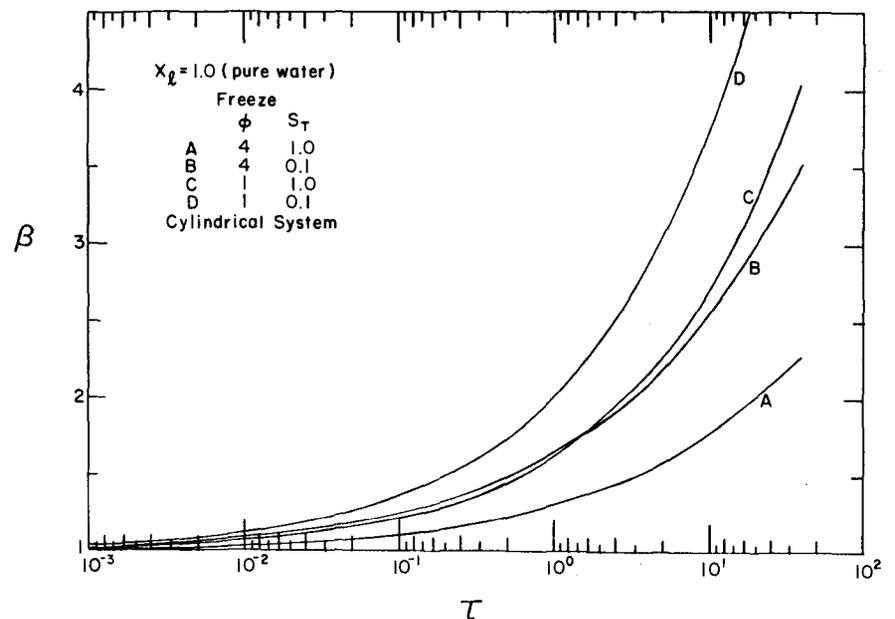


Figure 19. Phase change position vs time $x_0 = 1.0$ (pure water), thaw.

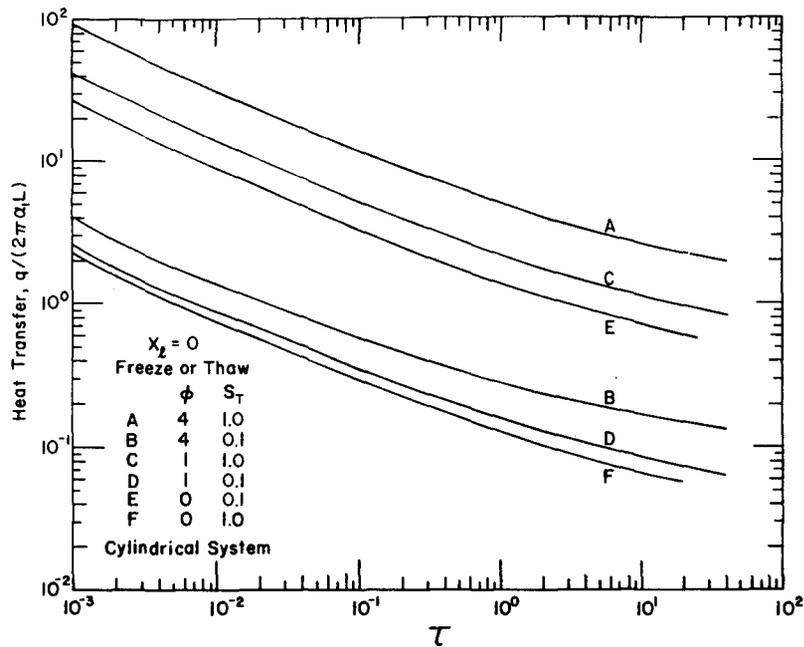


Figure 20. Phase change position vs time $x_q = 0.0$.

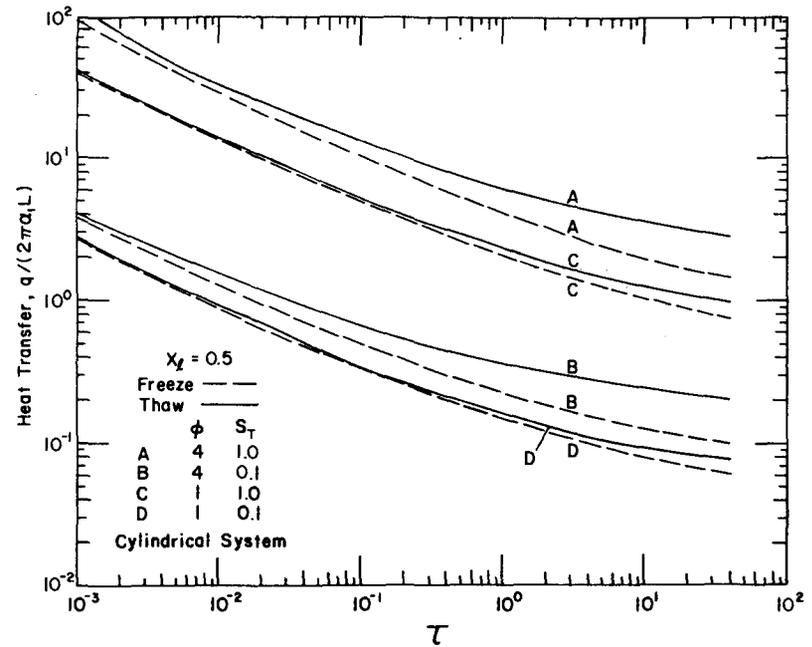


Figure 21. Heat flux at surface vs time $x_q = 0.5$.

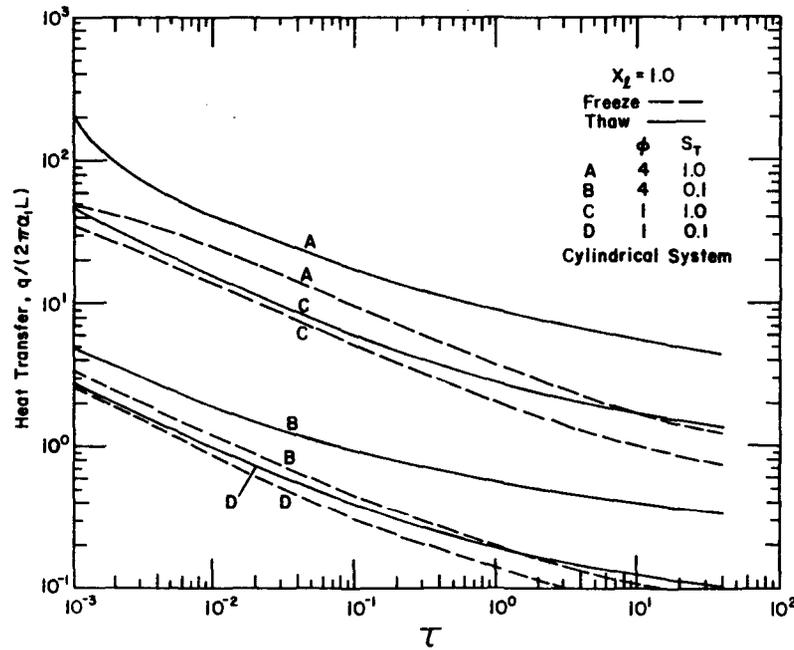


Figure 22. Heat flux at surface vs time $x_q = 1.0$.

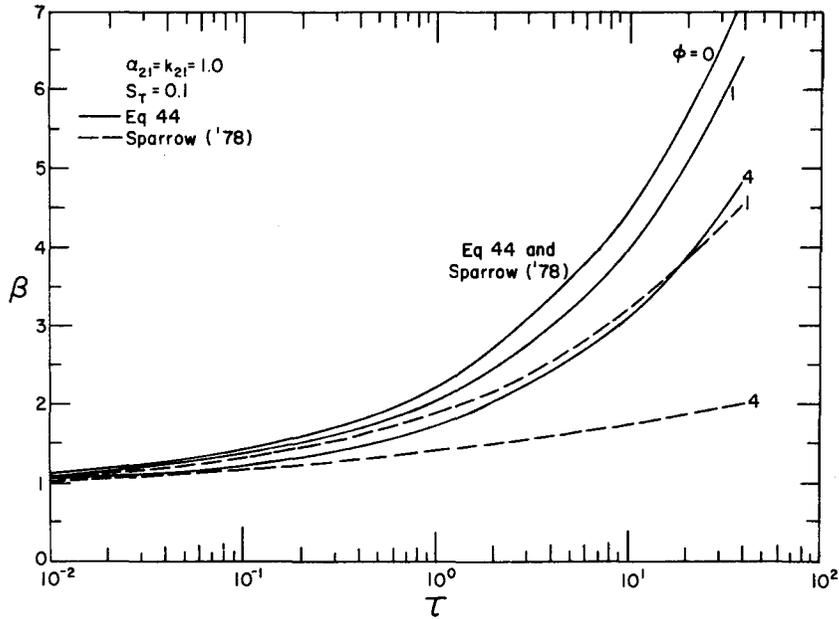


Figure 23. Phase change position vs time, $S_T = 0.10$; coordinate transformation method.

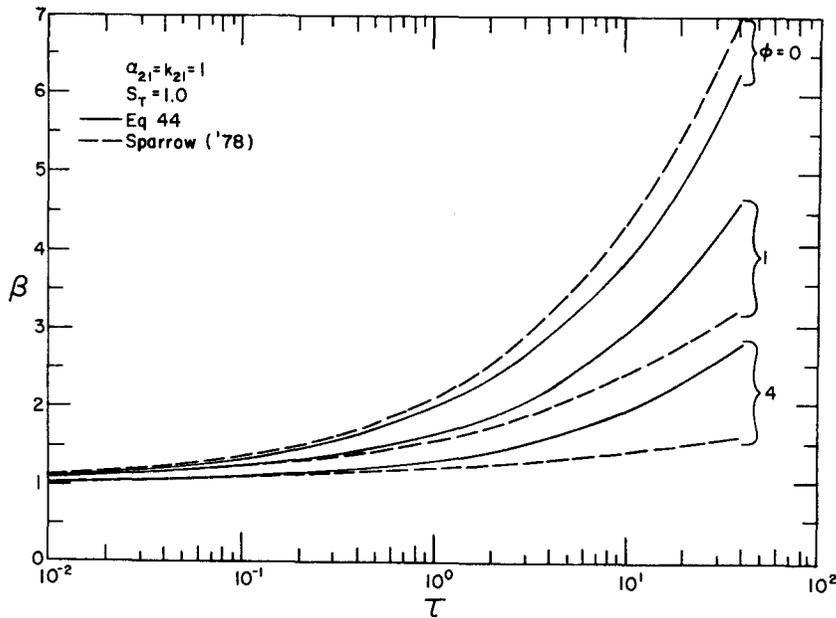


Figure 24. Phase change position vs time, $S_T = 1.0$; coordinate transformation method.

value of effective diffusivity is used, while if the surface heat flux density is desired a second thermal diffusivity is used.

The following relations can be used for the heat flux density and the phase change location, respectively.

$$\alpha_{eq} = \alpha_1 [(\phi + 1) \operatorname{erf} \gamma]^2 \quad (45)$$

$$\alpha_{ef} = \alpha_1 \left[\frac{\gamma}{\operatorname{erf}^{-1} \left(\frac{1}{\phi + 1} \right)} \right]^2 \quad (46)$$

where

$$\alpha_1 = \begin{cases} \alpha_{\text{thawed}} & \text{for a thawing problem} \\ \alpha_{\text{frozen}} & \text{for a freezing problem} \end{cases}$$

and erf is the error function.

The method consists of first solving the linear conduction problem and then finding the relation between the space coordinate and the freezing (or simply the

32°F) isotherm. In this relation one replaces the thermal diffusivity by eq 46 to obtain the phase change interface location for the phase change problem. It is likely that numerical evaluation will be necessary because of the complex inverse relation between the space coordinate and the freezing isotherm in the linear problem. The method is powerful enough to handle complicated problems but the range of validity is not known. The method cannot take into account the difference in thermal properties of the regions for which the temperature is above or below 32°F when finding the no-phase change solution.

Churchill and Gupta (1977) applied the method to cylinders and to freezing in a corner with good results. They used the exact solution for the cylindrical, no-phase change problem given by Carslaw and Jaeger (1959). This required the use of tabulated, numerical values. The following section will derive a convenient approximate solution to the problem which yields good results.

The first step in the solution of the phase change problem is to solve for the temperature of the pure conduction problem with zero latent heat. For the case of a cylinder surrounded by an infinite medium, an exact solution for the surface step change situation is given by Carslaw and Jaeger (1959). However, the solution involves an infinite series of Bessel functions which were approximated by error functions for small values of time. The final results are in graphical form. An approximate solution to this problem can be found by using the heat balance integral methods and the equations are given below, referring again to Figure 1. The melting case will be examined but the results apply to freezing also.

$$\frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = \frac{r}{\alpha} \frac{\partial T}{\partial t} \quad (47)$$

$$T(\delta, t) = T_0 \quad (47a)$$

$$T(r_0, t) = T_s \quad (47b)$$

$$\frac{\partial T}{\partial r}(\delta, t) = 0. \quad (47c)$$

The heat balance integral of eq 47 is a single integration over space and reduces to

$$-\alpha r_0 \frac{\partial v(r_0, t)}{\partial r} = \frac{d\theta}{dt} \quad (48)$$

$$\theta = \int_{r_0}^{\delta} r v dr \quad (49)$$

where $v = (T - T_0)/(T_s - T_0)$. For simplicity, a polynomial in r is assumed for the temperature

$$v = \left(\frac{\delta - r}{\delta - r_0} \right)^n; \quad r_0 < r < \delta \quad (50)$$

which satisfies eq 47a-47c and also the smoothness relations

$$\frac{\partial^{n-1} v(\delta, t)}{\partial r^{n-1}} = 0. \quad (51)$$

Boundary layer theory has shown that the additional smoothness relations of eq 51 may improve the accuracy of integral methods to a certain extent. The initial condition, $v(r, 0) = 0$, cannot be met but this will not seriously affect the final results.

Equations 40 through 50 lead to a differential equation for δ as follows

$$\left(\frac{2\Delta^2}{n+2} + \Delta \right) \frac{d\Delta}{d\tau} = \frac{n(n+1)}{S_T} \quad (52)$$

where $\Delta = (\delta - r_0)/r_0$. Equation 52 is easily integrated to give

$$\frac{4}{3(n+2)} \Delta^3 + \Delta^2 = 2n(n+1) \frac{\tau}{S_T}. \quad (53)$$

Volkov and Li-Orlov (1970) noted that the accuracy of the integral method could be improved by integrating eq 42 twice over the space coordinate. El-Genk and Cronenberg (1979) applied this idea to the Neumann problem with apparent success. However, for the cylindrical system this resulted in considerably poorer results than the integral heat balance for any given value of $n > 2$.

Equations 50 and 53 complete the solution of the no-phase change problem. These equations are acceptable when compared to the exact results of Carslaw and Jaeger (1959). The location of the phase change interface is found by using the movement of the isotherm with the fusion value. Thus eq 50 gives

$$v_f = \frac{T_f - T_0}{T_s - T_0} = \left(\frac{\delta - r_f}{\delta - r_0} \right)^n. \quad (54)$$

The location of the fusion value isotherm is then

$$\Delta_1 = \Delta \left[1 - \left(\frac{\phi}{\phi+1} \right)^{1/n} \right] \quad (55)$$

where $\Delta_1 = (r_f - r_0)/r_0$. Finally, the effective thermal diffusivity (eq 46) will replace the thermal diffusivity

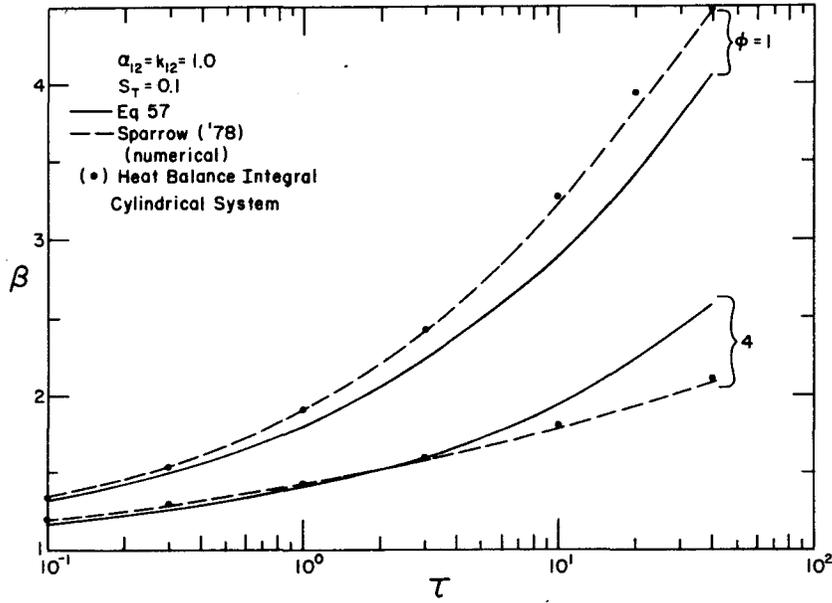


Figure 25. Phase change vs time, accuracy $S_T = 0.10$, $\alpha_{12} = k_{12} = 1.0$ of effective diffusivity method.

in eq 53. The solution for the actual phase change interface location is then

$$\frac{4}{3(n+2)}\beta^{*3} = \beta^{*2} - 2n(n+1)\tau^* = 0 \quad (56)$$

where

$$\tau^* = \frac{\alpha_{ef} t}{r_o^2} = \frac{\alpha_{ef}}{\alpha} \frac{\tau}{S_T}$$

$$\beta^* = \frac{\beta - 1}{1 - \frac{\phi}{\phi + 1} \frac{1}{n}}$$

The accuracy of the solution increases as n increases. Above $n = 20$ the improvement is slight and thus $n = 20$ was used for the numerical evaluations. With $n = 20$ the solution of eq 56, explicitly for β as a function of time, is

$$\beta = \left[1 - \left(\frac{\phi}{\phi + 1} \right)^{0.05} \right] \left[(a+d)^{1/3} + (a-d)^{1/3} - 5.5 \right] + 1 \quad (57)$$

$$a = 6930 \frac{\alpha_{ef}}{\alpha} \frac{\tau}{S_T} - 166.375$$

$$d = (a^2 - 27680.6406)^{1/2}$$

Equation 57, combined with eq 46 for the effective thermal diffusivity and eq 2 for γ , is a simple relation for freezing or thawing about a cylinder which is accurate enough for most engineering purposes. A comparison of eq 57 with previous numerical results is shown in Figures 25 and 26. These figures are limited to values of $k_{12} = \alpha_{12} = 1.0$ but eq 57 can be used for any ratios of the frozen and thawed properties. It is thus considerably more flexible than the numerical results. For most cases it will be accurate to within 10% of the numerical results. This is felt to be satisfactory for engineering calculations.

Heat transfer

The instantaneous heat flow from the cylinder and the total heat loss or gain during a given time are also quantities of interest. The surface heat transfer rate is given by

$$q = -k_1 A \frac{\partial T(r_o, t)}{\partial r} \quad (58)$$

Use of eq 50 and 57 leads to the following nondimensional heat flow

$$q^* = \frac{qc_1}{2\pi k_1 l} = \frac{20 S_T (1 + \phi)}{e} \quad (59)$$

where $e = (a+d)^{1/3} + (a-d)^{1/3} - 5.5$ as before, and eq 45 ($\alpha_{eq}/\alpha_1 = [(\phi+1) \operatorname{erf} \gamma]^2$) is used with a and d .

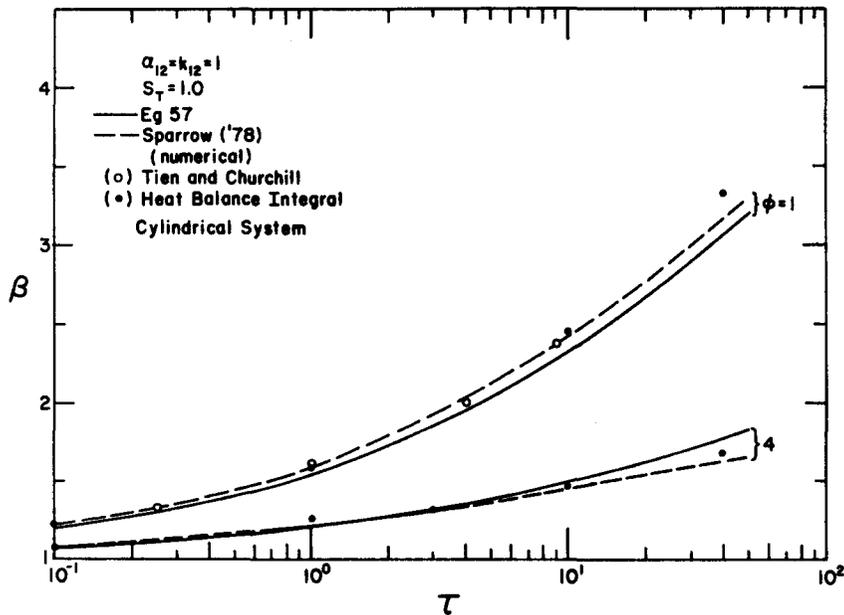


Figure 26. Phase change vs time, accuracy $S_T = 1.0$, $\alpha_{12} = k_{12} = 1.0$ of effective diffusivity method.

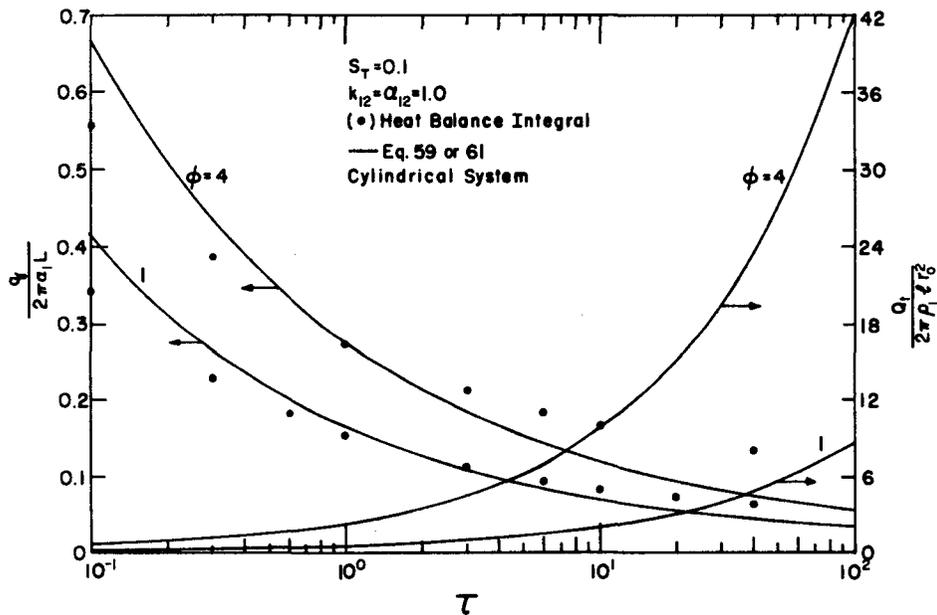


Figure 27. Heat flux vs time, accuracy of $\alpha_{12} = k_{12} = 1.0$, $S_T = 0.1$, effective diffusivity method.

The integrated heat transfer at the cylinder surface is

$$Q_t = \int_0^t q dt. \quad (60)$$

This can be written in nondimensional form as

$$Q_t^* = \frac{Q_t}{2\pi r_0^2 \rho_1 l} = (1+\phi) \frac{S_T}{21} (e^2 + e). \quad (61)$$

In eq 59 and 61, the value $n = 20$ has again been used. Figures 27 and 28 show q^* and Q_t^* plotted for $S_T = 0.1$ and 1.0 with $\alpha_{12} = k_{12} = 1.0$. The results compare fairly well with published numerical values. The effective diffusivity for heat transfer does not seem as accurate as that for phase change location but it still gives

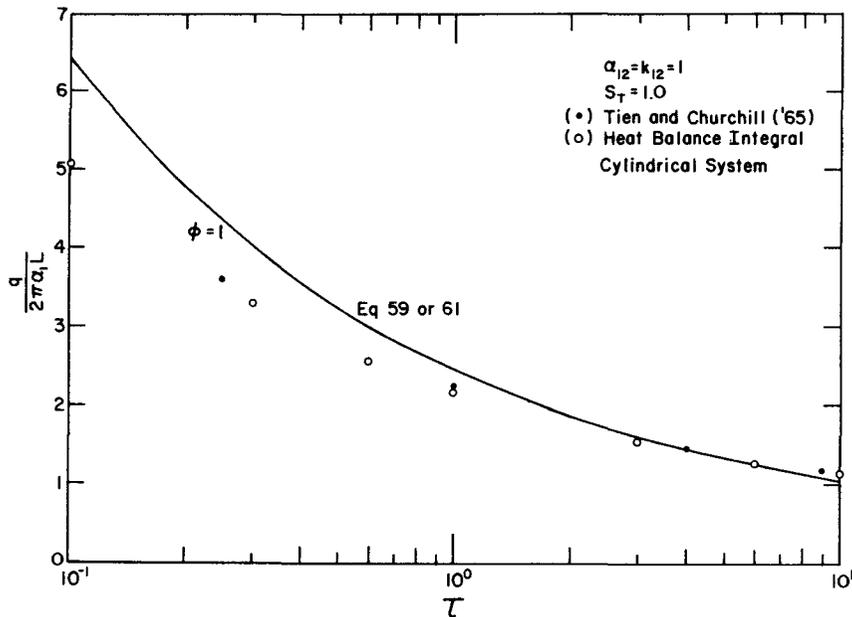


Figure 28. Heat flux vs time, accuracy of $\alpha_{12} = k_{12} = 1.0$, $S_T = 1.0$; effective diffusivity method.

reasonable results. Equations 59 and 61 can be applied to any values of α_{12} and k_{12} . The method of this section cannot be applied when the superheat parameter ϕ is zero but for this case there are sufficiently accurate solutions available, as given earlier.

CONCLUSIONS

A number of procedures have been examined for the problem of freezing or thawing about a cylinder. The heat balance integral method was applied to the general problem and yielded excellent results with a simple numerical scheme. The phase change interface and the cylinder heat flows can be evaluated from Figures 3-22 for certain values of the Stefan number, the superheat parameter, and the property value ratios of the surrounding medium, as functions of the volumetric water content of a soil. These are essentially exact results using numerical evaluation but are limited to specific values of S_T and ϕ .

Simple, reasonably accurate, closed form solutions have also been derived, given by eq 57 and 59. These can be used for any combination of S_T , ϕ , α_{12} , k_{12} and will be accurate enough for most permafrost calculations. These solutions are limited to cases for which $\phi > 0$. If the superheat is zero, then eq 14 or eq 44 can be used with good results. The effective thermal diffusivity method used here should be adaptable to many problems of freezing and thawing.

The problem of evaluating the effects of freeze/thaw about a cylinder surrounded by a homogeneous, infinite medium has been completed. The solutions, while not exact (except for the straight numerical results), are sufficiently accurate for engineering design.

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