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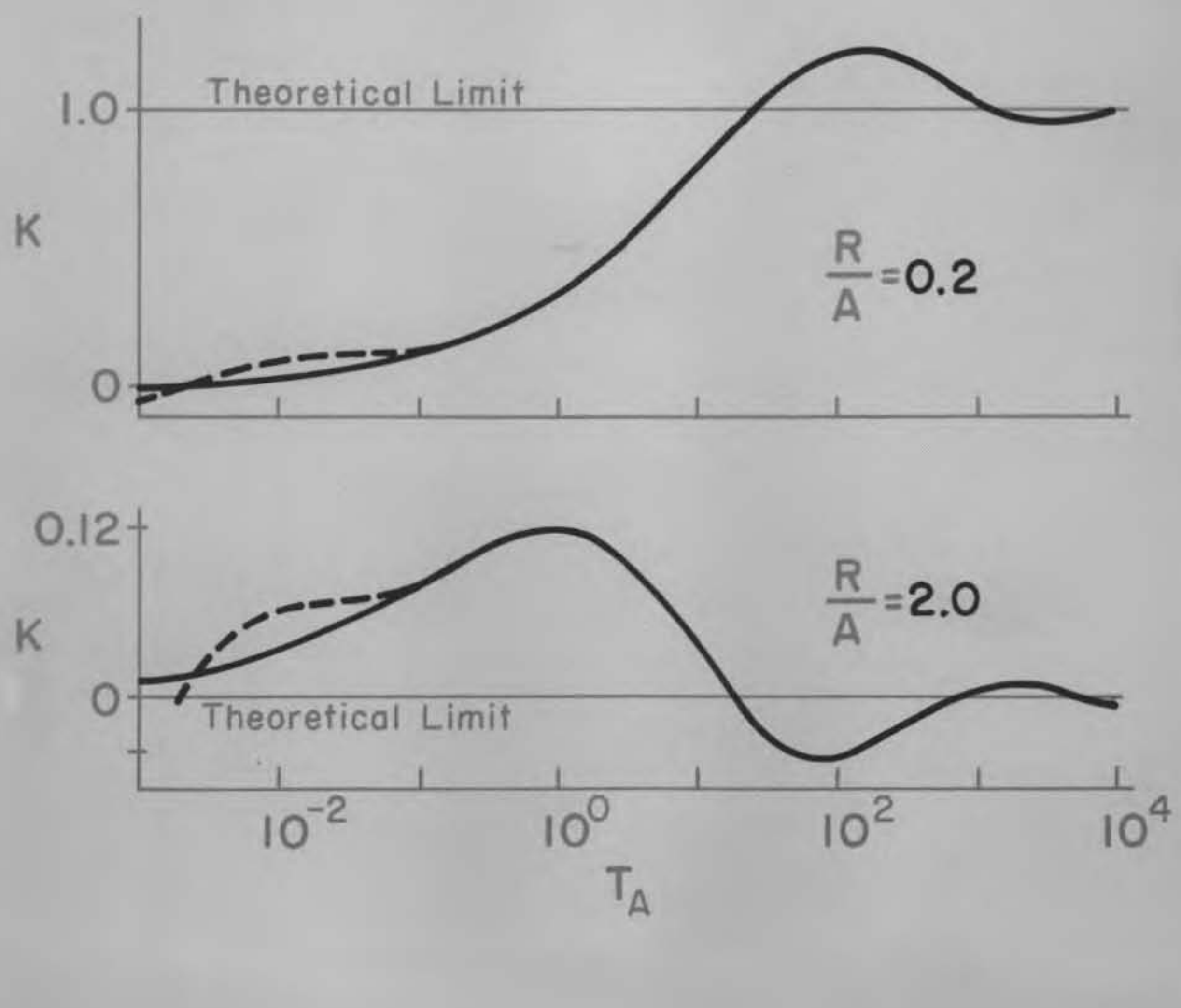
The viscoelastic deflection of an infinite floating ice plate subjected to a circular load



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We solved the viscoelastic deflection of an infinite floating ice plate subjected to a circular load, assuming the Maxwell-Voigt type four-element model. We developed an effective method of numerical integration of the solution integrals, of which each integrand contains a product of Bessel functions extending to infinity. We fitted the theoretical curve to the field data, but the material constants thus found varied with time and location.

PREFACE

This report was prepared by Dr. Shunsuke Takagi, Research Physical Scientist, Physical Sciences Branch, Research Division, U.S. Army Cold Regions Research and Engineering Laboratory.

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The report was technically reviewed by Dr. D.E. Nevel, G. Frankenstein, and Dr. A. Assur of CRREL.

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The viscoelastic deflection of an infinite floating ice plate subjected to a circular load

Shunsuke Takagi

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THE VISCOELASTIC DEFLECTION OF AN INFINITE FLOATING ICE PLATE SUBJECTED TO A CIRCULAR LOAD

Shunsuke Takagi

INTRODUCTION

Since ancient times floating ice plates have been used to cross rivers and lakes. During recent years traffic load on frozen rivers and lakes has greatly increased, and at the same time vehicles have become heavier. Aircraft landing and parking facilities also have added loads on these bodies of water. In addition, during the past several years, oil companies have started to use ice plates as drilling platforms. Thus, we now need to acquire a more detailed understanding of the creep of ice plates.

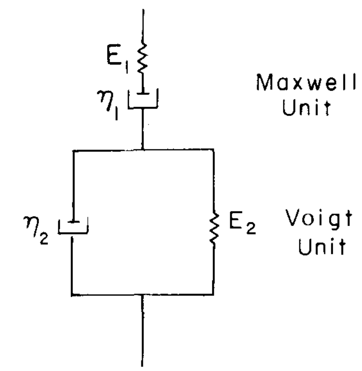

Figure 1. Maxwell-Voigt type four element model.

Formulation of the creep of a floating ice plate began after World War II with the intense development of the linear viscoelasticity theory. In 1947 Golushkevich (referred to by Kheysin[10]) presented an analysis assuming that ice behaves elastically for volumetric deformations and viscoelastically for deviatoric deformations. Kheysin[10] used a general viscoelastic thin-plate theory to analyze the infinite floating ice plate. He used the Maxwell unit (Fig. 1) only, and considered only a concentrated load. Nevel[11] also used the Maxwell unit only, but considered a distributed load. He limited his numerical computation only to the center of the load.

William L. Ko, as reported by Garbaccio,[4, 5] used the Maxwell-Voigt type four-element model (Fig. 1), which is known to represent the creep of ice satisfactorily (Jellinek and Brill[8]). In addition to thin-plate theory, Ko used Reissner's plate theory, which includes the deflection due to vertical shear forces. Garbaccio[5] numerically evaluated Ko's solution for specific values of material constants rather than for nondimensional parameters. Garbaccio's numerical answers show that the discontinuity of the load distribution yields a strong influence on the values of deflection. It is reasonable to suspect that his numerical evaluation may contain some errors.

IAkunin[6, 7] has solved the same problem as Ko, but he used only thin-plate theory. Unfortunately, only an abstract of IAkunin's work is available to western researchers.

Katona[9] and Vaudrey and Katona[17] solved the same problem with a finite-element viscoelastic computer program.

We solved this problem analytically by use of thin plate theory, and also developed an effective method of numerical integration of the solution integrals. However, the theoretical curves did not satisfactorily fit the field-test curves. It is now evident that a large scale laboratory test eliminating the variation due to natural conditions must be carried out and the theoretical assumptions must be tested.

1. THE PROBLEM

We shall consider the viscoelastic ice plate floating on water extending horizontally to infinity. We shall use the Maxwell-Voigt type four-element model (Fig. 1) to describe the viscoelastic deformation of ice.

Using the notation of Fig. 1, we can show that this model gives the stress-strain relationship which we show in an operator form,

$$\epsilon = \left[\frac{1}{E_1} + \frac{1}{\eta_1 \frac{\partial}{\partial t}} + \frac{1}{E_2 + \eta_2 \frac{\partial}{\partial t}} \right] \sigma \tag{1.1}$$

where t is time. To extend the one-dimensional relationship (1.1) to the three-dimensional relationship, we assume, as explained by Flügge,[2] that ϵ and σ are deviatoric and relate them by

$$\sigma = 2G\epsilon$$

where G is the rigidity modulus relative to the three-dimensional deformation. Using (1.1), $2G$ is given as an operator

$$\frac{1}{2G} = \frac{1}{E_1} + \frac{1}{\eta_1 \frac{\partial}{\partial t}} + \frac{1}{E_2 + \eta_2 \frac{\partial}{\partial t}} \ . \tag{1.2}$$

The differential equation describing the deflection w of an elastic plate on water is

$$D\nabla^4 w + \rho w = q \tag{1.3}$$

where ∇^4 is the biharmonic operator

$$\nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 \tag{1.4}$$

ρ the density of water, q the load per unit area, D the flexural rigidity defined by

$$D = 2Gh^3/[12(1-\nu)] \tag{1.5}$$

in which h is the thickness of the ice plate, and ν Poisson's ratio. Substituting $2G$ from (1.2) into (1.5), and D thus found into (1.3), we find the differential equation governing the viscoelastic deflection of a floating ice plate. We shall show this equation later in the nondimensional form.

We assume the load q to be a step loading applied at $t = 0$ and distributed uniformly over a circle of radius a with the center at origin. Then, letting r be the radial distance from origin

$$\begin{aligned} q &= q_0\, U(t) \qquad \text{for } 0 \leqq r < a \\ &= 0 \qquad\qquad\ \ \text{for } a < r \end{aligned} \tag{1.6}$$

where $U(t)$ is the step function, and t the time. Our problem is axisymmetric, and the biharmonic operator ∇^4 reduces to

$$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right)^2 .$$

We shall nondimensionalize our differential equation. We define the characteristic length ℓ by

$$\ell^4 = E_0 h^3/[12\rho(1-\nu)] \tag{1.7}$$

where

$$\frac{1}{E_0} = \frac{1}{E_1} + \frac{1}{E_2}\,. \tag{1.8}$$

We have chosen E_0, rather than E_1 or E_2, to define ℓ, because E_0 is related to the secondary creep (Nevel[12]), which is the main interest in our field observation.

Let D_1 be defined by

$$D_1 = D/(\rho\ell^4). \tag{1.9}$$

Use of (1.4) and (1.7) changes (1.9) to

$$D_1 = 2G/E_0. \tag{1.10}$$

Substituting G in (1.2), (1.10) becomes

$$D_1 = 1\Bigg/\left\{\frac{E_0}{E_1} + \frac{E_0}{\eta_1 \frac{\partial}{\partial t}} + \frac{E_0}{E_2 + \eta_2 \frac{\partial}{\partial t}}\right\}. \tag{1.11}$$

We choose nondimensional time T

$$T = E_0\, t/\eta_1 \tag{1.12}$$

and a parameter τ

$$\tau = \eta_1 E_2/(\eta_2 E_0). \tag{1.13}$$

Then (1.11) becomes

$$D_1 = 1\Bigg/\left\{E + \frac{1}{\frac{\partial}{\partial T}} + \frac{\eta_1/\eta_2}{\tau + \frac{\partial}{\partial T}}\right\} \tag{1.14}$$

where

$$E = E_0/E_1\,. \tag{1.15}$$

It is noted that

$$0 \leqq E \leqq 1. \tag{1.16}$$

Clearing the denominator, (1.14) becomes

$$D_1 = \frac{\partial}{\partial T}\left(\frac{\partial}{\partial T} + \tau\right)\Bigg/\left\{E\frac{\partial^2}{\partial T^2} + (1+\tau)\frac{\partial}{\partial T} + \tau\right\} \tag{1.17}$$

where use is made of the relation

$$E\tau + \eta_1/\eta_2 = \tau$$

which can be proved by use of (1.13), (1.15), and (1.8).

We define the nondimensional length R by

$$R = r/\ell. \tag{1.18}$$

We replace D in (1.3) with D in (1.9), and (1.3) becomes

$$D_1 \nabla_R^4 w + w = q/\rho \tag{1.19}$$

where

$$\nabla_R^4 = \left(\frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR}\right)^2 . \tag{1.20}$$

With D_1 given by (1.17), (1.19) is the differential equation to be solved.

2. THE SOLUTION

We denote the Hankel transform of $f(R)$ by $\tilde{f}(\beta)$

$$\tilde{f}(\beta) = \int_0^\infty f(R) J_0(\beta R) R\, dR \tag{2.1}$$

and the two-sided Laplace transform (Van der Pol and Bremmer[16]) of $g(T)$ by $\bar{g}(s)$

$$\bar{g}(s) = S \int_{-\infty}^{\infty} g(T)\, e^{-ST}\, dT . \tag{2.2}$$

We denote the inverse of (2.2) by

$$g(T) = L^{-1} [\bar{g}(S)] . \tag{2.3}$$

Applying these two transforms, (1.19) becomes

$$\bar{D}_1 \beta^4 \tilde{\bar{w}} + \tilde{\bar{w}} = \tilde{\bar{q}}/\rho$$

where

$$\bar{D}_1 = \frac{S(S+\tau)}{ES^2 + (1+\tau)S + \tau} . \tag{2.4}$$

Applying the two transforms to q defined by (1.6), we get

$$(1/\rho)\tilde{\bar{q}} = [P/(\pi A \rho \ell^2)]\,(1/\beta)\; J_1(\beta A) \tag{2.5}$$

where

$$P = \pi a^2 q \tag{2.6}$$

and

$$A = a/\ell . \tag{2.7}$$

Thus the transformed solution is given by

$$\tilde{\overline{w}} = \frac{P}{\pi A \rho \ell^2} \frac{1}{\beta(1 + \overline{D}_1 \beta^4)} J_1(\beta A).$$

Performing the Hankel inverse, we find

$$\overline{w} = \frac{P}{\pi A \rho \ell^2} \int_0^\infty \frac{1}{1 + \overline{D}_1 \beta^4} J_1(\beta A) \; J_0(\beta R) \; d\beta . \tag{2.8}$$

Performing the Laplace inverse, we find

$$w = \frac{P}{\pi A \rho \ell^2} \int_0^\infty L^{-1}\left(\frac{1}{1 + \overline{D}_1 \beta^4}\right) J_1(\beta A) \; J_0(\beta R) \; d\beta . \tag{2.9}$$

To find $L^{-1} \, [1/(1 + \overline{D}_1 \beta^4)]$, we compute the partial fraction

$$\frac{1}{S} \frac{1}{1 + \overline{D}_1 \beta^4} = \frac{1}{S} + \frac{\beta^4 \, (\tau - \alpha_2)}{\sqrt{DESC}} \frac{1}{S + \alpha_2} - \frac{\beta^4 \, (\tau - \alpha_1)}{\sqrt{DESC}} \frac{1}{S + \alpha_1}$$

where $-\alpha_1$ and $-\alpha_2$ are the roots of the quadratic equation

$$(E + \beta^4) \, S^2 + (\tau \beta^4 + 1 + \tau) \, S + \tau = 0. \tag{2.10}$$

They are given by

$$\left.\begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix}\right\} = \frac{\tau \beta^4 + 1 + \tau \mp \sqrt{DESC}}{2(\beta^4 + E)} \tag{2.11}$$

where

$$DESC = (\tau \beta^4 + 1 + \tau)^2 - 4\tau \, (\beta^4 + E) \tag{2.12}$$

which transforms to

$$= [\tau (\beta^4 + 1) - 1]^2 + 4\tau (1 - E). \tag{2.13}$$

From (2.13), it is clear that

$$DESC > 0. \tag{2.14}$$

The roots α_1 and α_2 are therefore always real. Moreover, inspection of (2.11) and (2.12) shows that both α_1 and α_2 are always positive. Thus we find that

$$L^{-1}\left(\frac{1}{1+\bar{D}_1\beta^4}\right) = 1 + \frac{\beta^4(\tau-\alpha_2)}{\sqrt{DESC}} e^{-\alpha_2 T} - \frac{\beta^4(\tau-\alpha_1)}{\sqrt{DESC}} e^{-\alpha_1 T}. \tag{2.15}$$

Substituting (2.15) into (2.9), the solution for w is found:

$$w = \frac{P}{\pi A \rho \ell^2} \int_0^\infty \left\{ 1 + \frac{\beta^4(\tau-\alpha_2)}{\sqrt{DESC}} e^{-\alpha_2 T} - \frac{\beta^4(\tau-\alpha_1)}{\sqrt{DESC}} e^{-\alpha_1 T} \right\} J_1(\beta A)\, J_0(\beta R)\, d\beta. \tag{2.16}$$

The radial and hoop stresses are given by

$$\sigma_r = -\frac{6D}{h^2}\left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r}\right)$$

$$\sigma_\theta = -\frac{6D}{h^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \nu\frac{\partial^2 w}{\partial r^2}\right)$$

respectively. Changing D to D_1 by use of (1.9) and r to nondimensional R by use of (1.18), they become

$$\sigma_r = -\frac{6\rho\ell^2}{h^2} D_1\left(\frac{\partial^2 w}{\partial R^2} + \frac{\nu}{R}\frac{\partial w}{\partial R}\right)$$

$$\sigma_\theta = -\frac{6\rho\ell^2}{h^2} D_1\left(\frac{1}{R}\frac{\partial w}{\partial R} + \nu\frac{\partial^2 w}{\partial R^2}\right)$$

where D_1 is the operator on T given by (1.17). The two-sided Laplace transform yields

$$\bar{\sigma}_r = -\frac{6\rho\ell^2}{h^2}\left(\frac{\partial^2}{\partial R^2} + \frac{\nu}{R}\frac{\partial}{\partial R}\right)\overline{D_1 w} \tag{2.17}$$

$$\bar{\sigma}_\theta = -\frac{6\rho\ell^2}{h^2}\left(\frac{1}{R}\frac{\partial}{\partial R} + \nu\frac{\partial^2}{\partial R^2}\right)\overline{D_1 w} \tag{2.18}$$

where $\overline{D_1 w}$ is the Laplace transform of $D_1 w$.

Using (2.8) one gets

$$\overline{D_1 w} = \frac{P}{\pi A \rho \ell^2}\int_0^\infty \frac{\bar{D}_1}{1+\bar{D}_1\beta^4} J_1(\beta A)\, J_0(\beta R)\, d\beta.$$

The Laplace inverse of this is

$$D_1 w = \frac{P}{\pi A \rho \ell^2}\int_0^\infty L^{-1}\left(\frac{\bar{D}_1}{1+\bar{D}_1\beta^4}\right) J_1(\beta A)\, J_0(\beta R)\, d\beta.$$

To find $L^{-1}[\bar{D}_1/(1+\bar{D}_1\beta^4)]$, we compute the partial fraction,

$$\frac{1}{S}\,\frac{\bar{D}_1}{1+\bar{D}_1\beta^4} = \frac{\tau-\alpha_1}{\sqrt{DESC}}\,\frac{1}{s+\alpha_1} - \frac{\tau-\alpha_2}{\sqrt{DESC}}\,\frac{1}{s+\alpha_2}\,.$$

Thus we find

$$L^{-1}\left(\frac{\bar{D}_1}{1+\bar{D}_1\beta^4}\right) = \frac{\tau-\alpha_1}{\sqrt{DESC}}\,e^{-\alpha_1 T} - \frac{\tau-\alpha_2}{\sqrt{DESC}}\,e^{-\alpha_2 T}\,.$$

Thus the inverse of (2.17) is

$$\sigma_r = \frac{6P}{\pi A h^2}\int_0^\infty J_1(\beta A)\left\{J_0(\beta R) - \frac{1-\nu}{\beta R}\,J_1(\beta R)\right\}\frac{(\tau-\alpha_1)e^{-\alpha_1 T} - (\tau-\alpha_2)e^{-\alpha_2 T}}{\sqrt{DESC}}\,\beta^2\,d\beta. \qquad (2.19)$$

The inverse of (2.18) is

$$\sigma_\theta = \frac{6P}{\pi A h^2}\int_0^\infty J_1(\beta A)\left\{\nu J_0(\beta R) + \frac{1-\nu}{\beta R}\,J_1(\beta R)\right\}\frac{(\tau-\alpha_1)e^{-\alpha_1 T} - (\tau-\alpha_2)e^{-\alpha_2 T}}{\sqrt{DESC}}\,\beta^2\,d\beta. \qquad (2.20)$$

Tabulation of σ_r and σ_θ becomes easier if linear combinations of (2.19) and (2.20) that do not contain ν are computed.

3. METHOD OF NUMERICAL INTEGRATION

It is impossible to analytically integrate the solution integrals (2.16), (2.19) and (2.20). (See App. I.)

The direct numerical integration is inconvenient because of the slow convergence of the Bessel functions for large values of the independent variable β. We shall choose finite ranges of integration that give sufficiently close approximations. The essence of our method consists of the following integration procedure:

Consider the integral

$$I = \int_0^\infty \phi(\beta)\; J_1(\beta A)\; J_0(\beta R)\; d\beta \qquad (3.1)$$

where the non-Bessel factor $\phi(\beta)$ is finite in the range of integration, and asymptotically

$$\phi(\beta) \sim a\beta^{-n} \qquad (3.2)$$

in which a is constant. The value of n in our formulas in the previous section is $\geqq 4$. The general case is discussed in Appendix I.

We will replace the infinite integral (3.1) with a finite integral. Given a large value N, we can estimate an upper bound of the absolute integral,

$$\int_N^\infty \left|\phi(\beta)\; J_1(\beta A)\; J_0(\beta R)\right|\; d\beta \qquad (3.3)$$

called the absolute remainder, by substituting the asymptotic expansions of $\phi(\beta)$ and Bessel functions. We let the trigonometric functions in the latter equal one. Denoting the absolute remainder by $[Abs\ I]_N^\infty$, we find

$$[Abs\ I]_N^\infty < [2a/(\pi\sqrt{aA})]\ (nN^n)^{-1}\ . \tag{3.4}$$

Let ϵ be the error we can tolerate in our computation. In our actual computation, we chose

$$\epsilon = 10^{-5}.$$

The value of N is evaluated by equating the right hand side of (3.4) to ϵ:

$$[2a/(\pi\sqrt{aA})]\ (nN^n)^{-1} = \epsilon. \tag{3.5}$$

Then, integral I in (3.1) is approximated by

$$I \doteqdot \int_0^N \phi(\beta)\ J_1(\beta A)\ J_0(\beta R)\ d\beta. \tag{3.6}$$

The value of N was small in most of our computation [$N \leqq 10$ except in (6.4)], and our numerical scheme worked very effectively.

We list in the following the asymptotic expansions of the non-Bessel factors $\phi(\beta)$ contained in the integrands of our integral solutions (2.16), (2.19), and (2.20):

$$\begin{aligned} &\alpha_1 \sim \beta^{-4} \\ &\alpha_2 \sim \tau(1+\beta^{-4}) \\ &e^{-\alpha_1 T} \sim 1 - T\beta^{-4} \\ &e^{-\alpha_2 T} \sim e^{-\tau T} \\ &(\tau - \alpha_1)/\sqrt{DESC} \sim \beta^{-4} - \beta^{-8} \\ &(\tau - \alpha_2)/\sqrt{DESC} \sim -(1-E)\beta^{-8}\ . \end{aligned} \tag{3.7}$$

4. RAMP/STEADY LOADING

We used two load tests to fit our theoretical curves. One was the Sun Oil Corporation's (SUNOCO) data obtained during the winters of 1973-1974 and 1974-1975 at Resolute Bay, Northwest Territory (unpublished). The other was Frankenstein's data[3] obtained on Portage Lake, Michigan, and the Garrison Dam Reservoir, North Dakota, on 20 March 1956 and 18 January 1957. Among these tests, we chose the ideal ramp/steady loading for our numerical computation. In this loading, as illustrated in Figure 2, the load P is increased initially at a constant rate $\dot{P}$ and, after a certain time T_0, kept constant at $P = \dot{P}T_0$. However, since SUNOCO does not allow the publication of their data, we cannot include their data in this paper.

We will derive the ramp/steady formulas by use of the step-loading formulas given in the previous section. However, since both SUNOCO and Frankenstein measured only deflection, we derive only the deflection formulas.

Define the influence function $w_0(T)$ by letting $P = 1$ in (2.16):

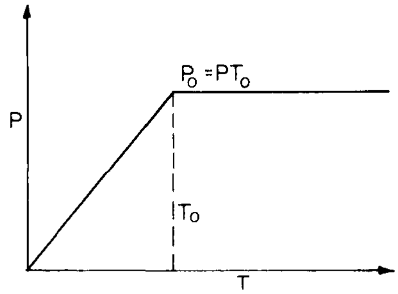

Figure 2. Definition of the ramp/steady loading.

$$w_0(T) = \frac{1}{\pi A \rho \ell^2} \int_0^\infty \left[1 + \frac{\beta^4 (\tau - \alpha_2)}{\sqrt{DESC}} e^{-\alpha_2 T} - \frac{\beta^4 (\tau - \alpha_1)}{\sqrt{DESC}} e^{-\alpha_1 T} \right] J_1(\beta A)\, J_0(\beta R)\, d\beta. \quad (4.1)$$

The deflection $w(T)$ for $0 \leqq T \leqq T_0$ is given by

$$w(T) = \int_0^T w_0(T - \lambda)\, \dot{P}\, d\lambda \quad (4.2)$$

and for $T_0 \leqq T$ by

$$= \int_0^{T_0} w_0(T - \lambda)\, \dot{P}\, d\lambda \quad (4.3)$$

where $\dot{P} = P/T_0$.

Substituting (4.1) into (4.2) and integrating with regard to λ, we get the deflection $w(T)$ for $0 \leqq T \leqq T_0$:

$$w(T) = [\dot{P}/(\pi A \rho \ell^2)]\, (U_1 - U_2 + U_3) \quad (4.4)$$

where

$$U_1 = \int_0^N \frac{1}{\alpha_1} (e^{-\alpha_1 T} - 1 + \alpha_1 T)\, J_1(\beta A)\, J_0(\beta R)\, d\beta \quad (4.5)$$

$$U_2 = \int_0^N \left[\frac{\beta^4 (\tau - \alpha_1)}{\sqrt{DESC}} - 1 \right] \frac{1}{\alpha_1} (1 - e^{-\alpha_1 T})\, J_1(\beta A)\, J_0(\beta R)\, d\beta \quad (4.6)$$

$$U_3 = \int_0^N \frac{\beta^4 (\tau - \alpha_2)}{\sqrt{DESC}} \frac{1}{\alpha_2} (1 - e^{-\alpha_2 T})\, J_1(\beta A)\, J_0(\beta R)\, d\beta. \quad (4.7)$$

The absolute remainders are as follows:

$$[Abs\, U_1]_N^\infty < [T^2/(4\pi\sqrt{AR})]\, N^{-4} \quad (4.8)$$

$$[Abs\, U_2]_N^\infty < [T/(2\pi\sqrt{AR})]\, N^{-4} \quad (4.9)$$

$$[Abs\, U_3]_N^\infty < [(1 - E)/(2\pi\sqrt{AR})]\, \frac{1}{\tau}\, (1 - e^{-\tau T})\, N^{-4}\,. \tag{4.10}$$

Substituting (4.1) into (4.3) and integrating with regard to λ, we get the deflection $w(T)$ for $T_0 \leqq T$:

$$w(T) = [P/(\pi A \rho \ell^2)]\, (I_1 + I_2 + I_3) \tag{4.11}$$

where

$$I_1 = \int_0^N \left\{ 1 - \frac{\beta^4 (\tau - \alpha_1)}{\sqrt{DESC}}\, \frac{1}{\alpha_1 T_0}\, (e^{\alpha_1 T_0} - 1) \right\} J_1(\beta A)\, J_0(\beta R)\, d\beta \tag{4.12}$$

$$I_2 = \int_0^N \frac{\beta^4 (\tau - \alpha_2)}{\sqrt{DESC}}\, (1 - e^{-\alpha_1 T})\, \frac{1}{\alpha_1 T_0}\, (e^{\alpha_1 T_0} - 1)\, J_1(\beta A)\, J_0(\beta R)\, d\beta \tag{4.13}$$

$$I_3 = \int_0^N \frac{\beta^4 (\tau - \alpha_2)}{\sqrt{DESC}}\, e^{-\alpha_2 T}\, \frac{1}{\alpha_2 T_0}\, (e^{\alpha_2 T_0} - 1)\, J_1(\beta A)\, J_0(\beta R)\, d\beta. \tag{4.14}$$

The absolute remainders are as follows:

$$[Abs\, I_1]_N^\infty < (2\pi\sqrt{AR})^{-1}\, N^{-4} \tag{4.15}$$

$$[Abs\, I_2]_N^\infty < [T/(2\pi\sqrt{AR})]\, N^{-4} \tag{4.16}$$

$$[Abs\, I_3]_N^\infty < [(1 - E)/(2\pi\sqrt{AR})]\, [(e^{\tau T_0} - 1)/(\tau T_0)]\, e^{-\tau T}\, N^{-4}\,. \tag{4.17}$$

Computer programs for these formulas are shown in Appendix II.

5. CURVE FITTING TO TIME LAPSE DEFLECTIONS

Frankenstein[3] placed a 12-ft-diameter tank on the ice and pumped the adjoining water into the tank. (We call this the distributed load test.) However, the temperature of the water in the tank obviously disturbed the ice temperature. He then tried a variation by placing a 17.3-in.-diameter concrete block under the 12-ft-diameter tank. (We call this the concentrated load test.). The water in the tank was, in this case, isolated from the ice and did not disturb the ice temperature.

The load-vs-time curves of these tests and the measured deflections are shown in Figures 3 and 4. "TANK" designates the deflection of the edge of the tank. "RODS" are the sites where the measurements were taken. The distances of the measurement sites from the center of the load are listed in Tables I and II.

The material constants found by the curve fitting are shown in Tables I and II. They vary with the location of the measurement.

To show the significance of the material constant variation with the measurement sites, we chose the material constants determined at rod 1 of Frankenstein's concentrated-load time-lapse curve, and computed the deflections at the other measurement sites. Figure 5 shows the comparison of the computed curves and the measured data. The left and right columns show the ramp and steady portions of the deflection curves, respectively. They are designated by (r) and (s) respectively.

To express the degree of curve fitting we devised the trapezoidal error (TE). In Figure 6, a, A and b, B show two pairs of measured and computed deflections at two consecutive times t_1 and t_2,

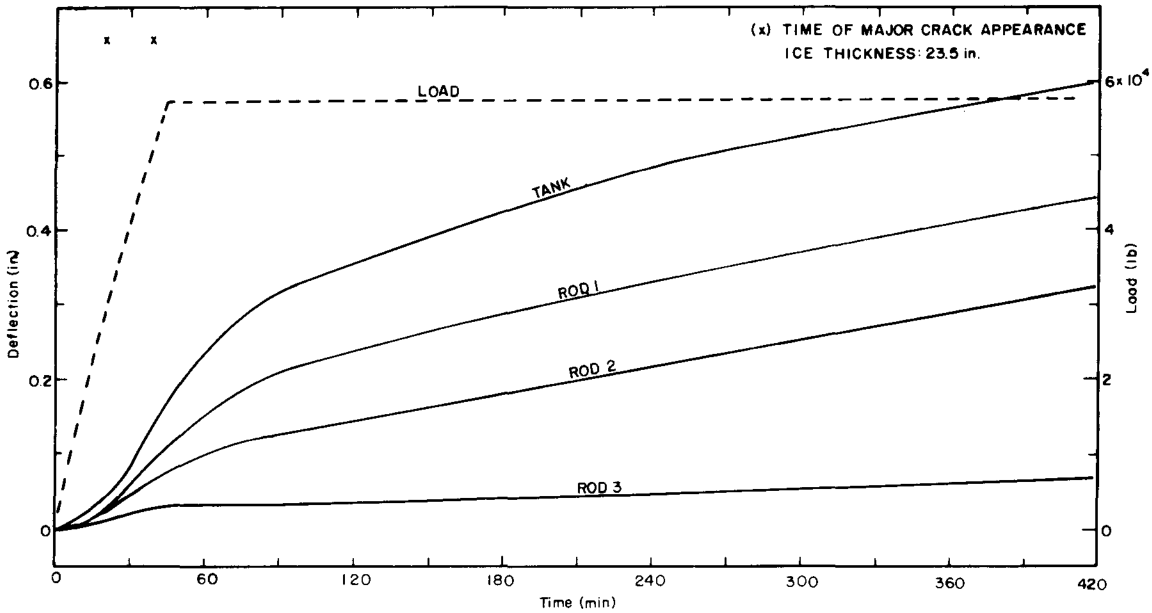

Figure 3. Distributed load test by Frankenstein (ref. 3, test 5).

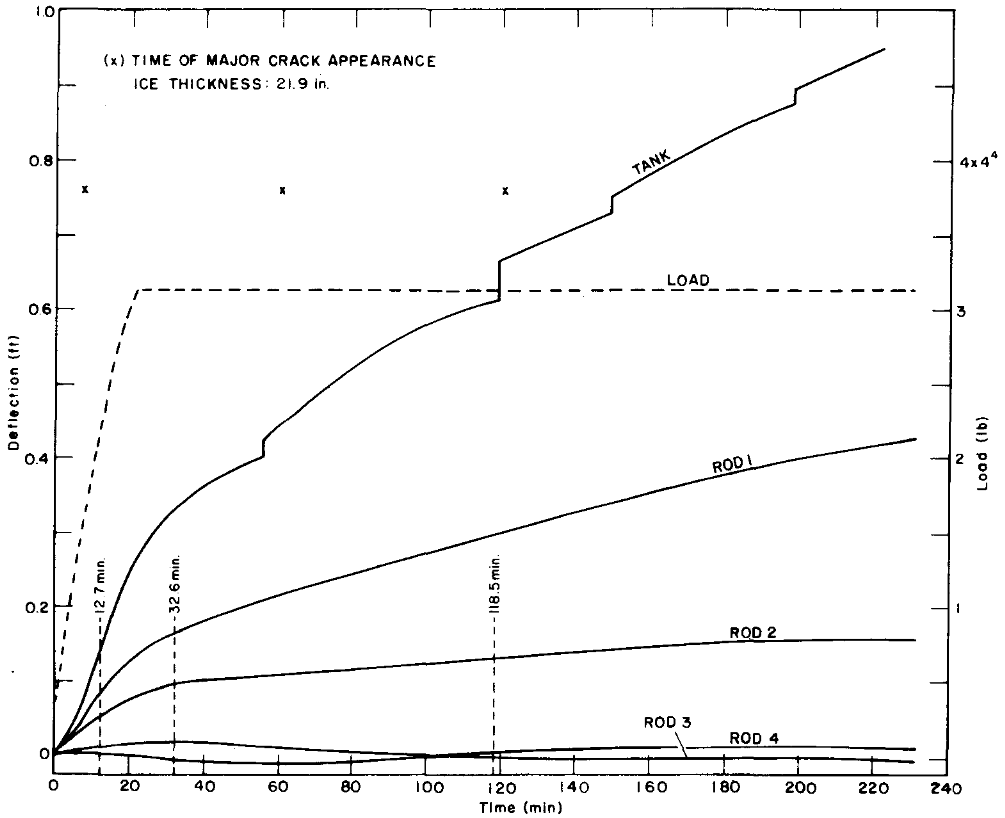

Figure 4. Concentrated load test by Frankenstein (ref. 3, test 8).

Table I. Material constants found by using the time-lapse curves of Frankenstein's distributed load test (ref. 3, test 5).

	TANK	*Rod 1*	*Rod 2*	*Rod 3*
Distance	*1.83 m*	*4.9 m*	*9.8 m*	*19.6 m*
τ	20	20	26	5
E	0.028	0.014	0.005	0.02
E_0 (kg/m^2)	2.159×10^8	3.729×10^8	9.812×10^8	3.925×10^9
η_1/E_0 (sec)	1.488×10^6	1.469×10^6	1.224×10^6	2.448×10^6
TE (ramp) (m)	4.928×10^{-3}	2.780×10^{-3}	3.117×10^{-3}	1.0541×10^{-3}
TE (flat) (m)	3.048×10^{-3}	2.195×10^{-3}	3.882×10^{-3}	1.393×10^{-3}

Table II. Material constants found by using the time-lapse curves of Frankenstein's concentrated load test (ref. 3, test 8).

	TANK	*Rod 1*	*Rod 2*	*Rod 3*
Distance	*0.22 m*	*4.9 m*	*9.8 m*	*19.6 m*
τ	2	6.5	20	5
E	0.0005	0.007	0.05	0.1
E_0 (kg/m^2)	1.766×10^6	9.813×10^7	6.869×10^8	9.813×10^{11}
η_1/E_0 (sec)	2.815×10^6	4.896×10^5	1.101×10^6	2.448×10^6
TE (ramp) (m)	4.718×10^{-3}	5.812×10^{-3}	2.063×10^{-3}	
TE (flat) (m)	4.727×10^{-3}	2.730×10^{-3}	2.884×10^{-4}	2.750×10^{-3}

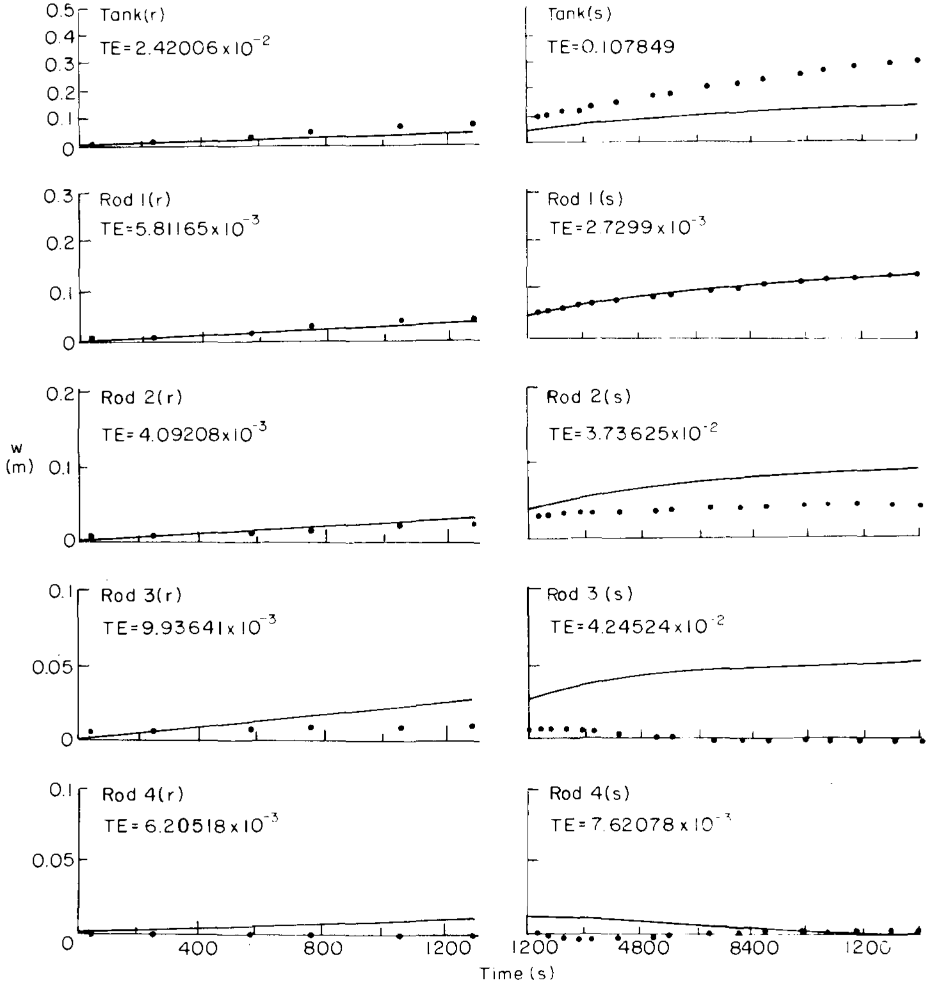

Figure 5. Comparison of the calculated curves and measured points of Frankenstein's[3] concentrated load test.

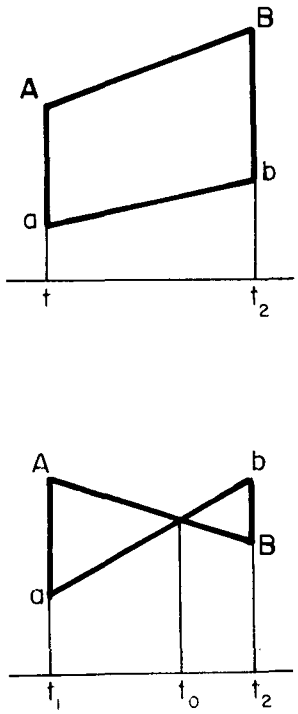

Figure 6. Elements of TE.

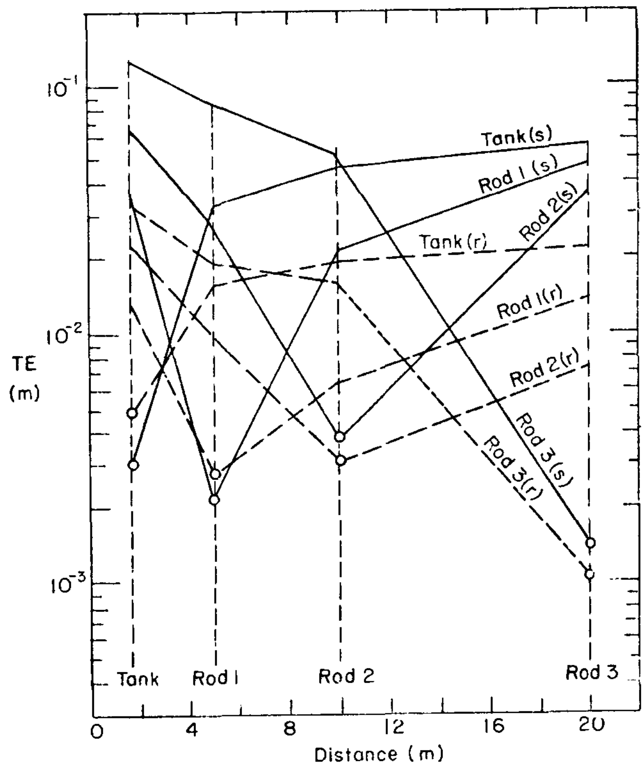

Figure 7. The TE of Frankenstein's distributed load test (ref. 3, test 5).

respectively. We squared A-a and B-b and, in case of the upper figure where the errors are of the same sign, computed the area of the trapezoid of which the bases are $(A\text{-}a)^2$ and $(B\text{-}b)^2$ and the height $t_2 - t_1$. In case of the lower figure where the errors change sign, we calculated the sum of the areas of the two triangles of which the bases are $(A\text{-}a)^2$ and $(b\text{-}B)^2$ and the heights $t_0 - t_1$ and $t_2 - t_0$ respectively, where t_0 is the abscissa of the intersection. Denoting by S the area of such a figure, we defined TE by

$$\mathrm{TE} = \sqrt{(\Sigma S)/T} \tag{5.1}$$

where the summation is over all the intervals and T the sum of the abscissa intervals.

The TE indicates a sort of absolute maximum error. Its unit is m. If the deflections are of ordinary magnitude, the TE of order 10^{-3} and 10^{-2} means a good and tolerable fit, respectively. If the deflections are very small, as in the case of rod 4, the smallness of the value of TE does not mean much. We did not list the computed values at rod 4 in Tables I and II.

We evaluated the TE for all the possible cases. They are shown in Figures 7 and 8. The abscissa is the distance from the center of the load. The measurement sites are noted on the abscissa axis. The circled points are those whose material constants are used to compute a set of TE. The sets of TE thus computed are connected with solid or broken lines and labeled with the appellations of the circled measurement sites.

Comparison of Figures 7 and 8 shows that the concentrated load test has smaller overall TE values than the distributed load test. However, we cannot recognize any significant effect of the temperature distribution due to the watertank temperature disturbance. Probably the cracks, whose appearances are noted in Figures 3 and 4 but are not considered in our formulation, were more detrimental

After the numerical computation was finished in 1976, Dr. Andrew Assur, an ice mechanics expert at CRREL, notified us that the variations of material constants in Tables I and II are in the

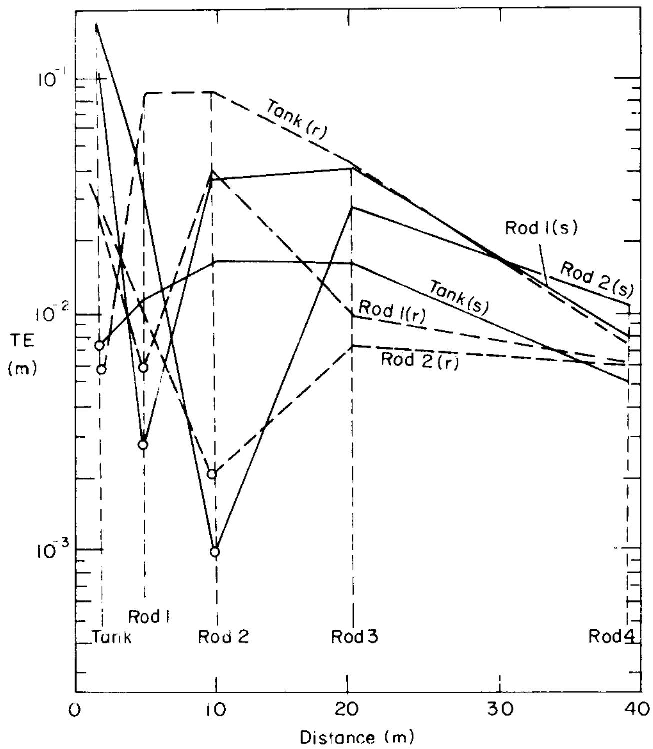

Figure 8. The TE of Frankenstein's concentrated load test (ref. 3, test 8).

range of reasonable values from the viewpoint of the nonlinear viscoelastic constants (Shumskij[14]). We tried in 1977 to reevaluate the material constants; we thought that, although the theoretical curve is formulated on the linear assumption, if we fit the theoretical curve in the narrow time interval and space span, we can find the material constants close to the incremental viscoelastic constants. However, this plan could not be executed because the distances between the measuring rods were too large.

6. ASYMPTOTIC DEFLECTION

We shall show in the following that only one material constant is contained in the asymptotic formulas. The curve fitting, therefore, must be carried out in the initial stage.

Referring to the asymptotic relationships in (3.7), we find that, when T is large, both the step-loading formulation (2.16) and the ramp/steady loading formulation (4.11) reduce to

$$w = [P/(\pi A \rho \ell^2)]\ (K/A) \tag{6.1}$$

where

$$\frac{1}{A} K = \int_0^\infty (1 - e^{-T\beta^{-4}}) J_1(\beta A) J_0(\beta R)\, d\beta. \tag{6.2}$$

It is assumed in this derivation that $\tau > 0$, and that only large values of β are effective in the integration. Letting

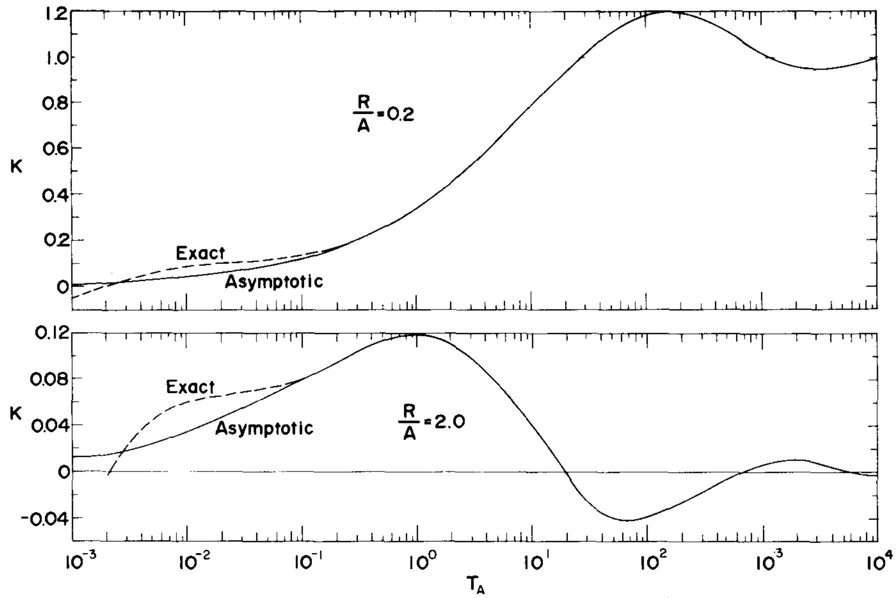

Figure 9. Graphs of asymptotic integral K in (6.4).

$$x = \beta A \tag{6.3}$$

(6.2) becomes

$$K = \int_0^\infty (1 - e^{-T_A x^{-4}}) J_1(x) J_0[(R/A)x]\, dx \tag{6.4}$$

where

$$T_A = TA^4 \tag{6.5}$$

$$= \frac{ta^4}{h^3} \frac{12\rho(1-\nu)}{\eta_1} . \tag{6.6}$$

Thus, all the material constants are lumped into the second factor of (6.6). The stress formulas, although not mentioned here, can be similarly transformed.

As shown in Appendix I, (6.4) cannot be analytically integrated; it must be numerically integrated. To effect the numerical integration, the non-Bessel factor in (6.4) is so chosen that it becomes zero at $x = \infty$. The absolute remainder is estimated:

$$[Abs\, K]_N^\infty < \left[T_A \frac{1}{2\pi} \sqrt{\frac{A}{R}} \right] N^{-4} . \tag{6.7}$$

Graphs of integral K for the values of $R/A = 0.2$ and 2.0 are shown in Figure 9. When $T_A = \infty$, the non-Bessel factor becomes equal to one. At this limit, therefore, $K = 1$ when $R < A$, and $K = 0$ when $R > A$. As shown in the graphs, this limit is almost reached when $T_A > 1000$.

Exact integral K was formulated for the ramp/steady loading, and evaluated by use of a set of constants: $T_0 = 6 \times 10^3$ sec, $\tau = 10$, $E = 1/6$, $\eta_1/E_0 = 6.12 \times 10^4$ sec $= 17$ hr, $E_0 = 7 \times 10^8$ kg/m^2, $\nu = 0.5$, and $A = 0.5$. These constants give $\ell = 29.31\ m$ and $T_A = t(2.48 \times 10^{-3}\ \text{day}^{-1})$. As shown in Figure 9, the asymptotic integral K is very close to the exact integral in the range $T_A > 0.1$. The above constants are the rough estimates used before starting the elaborate calculations.

They are not listed in the Tables. We did not use other sets of constants to evaluate the exact integral K. We expect that all the exact curves should show the similar coincidence with the asymptotic curve although with individual variations.

The values of T_A at the final time of the two tests are listed in Table III. These values are very small. However we experienced that the modification of some material constants was insensitive on the modification of the computed deflection values.

Table III. Final time of the three tests.

	in physical unit	*in T_A unit*
Frankenstein's distributed load test	420 min = 7 hr	2.67×10^{-4}
concentrated load test	240 min = 4 hr	1.2×10^{-8}

7. DEFLECTION PROFILES

We computed the deflection profiles of the concentrated load test (Frankenstein,[3] test 8) at 12.7, 32.6, and 118.5 min by use of the material constants, $\tau = 10, E = 0.02, \eta_1/E = 7 \times 10^5$ sec, and $E_0 = 2 \times 10^8$ kg/m^2, as shown in Figure 10. These material constants are round numbers intermediate between the material constants at rod 1 and rod 2 in Table II. The three chosen times mentioned above are marked in Figure 4. The computed profiles are quite different from the measured profiles. We varied the material constants but could not find values that make the theoretical curve assume a similarity to the measured curve. It is our impression that the measured profiles do not belong to the family of curves that our analytical formula can describe. The measured and computed curves intersect between rod 1 and rod 2, indicating the reliability of our computation, as may be expected from the choice of the material constants.

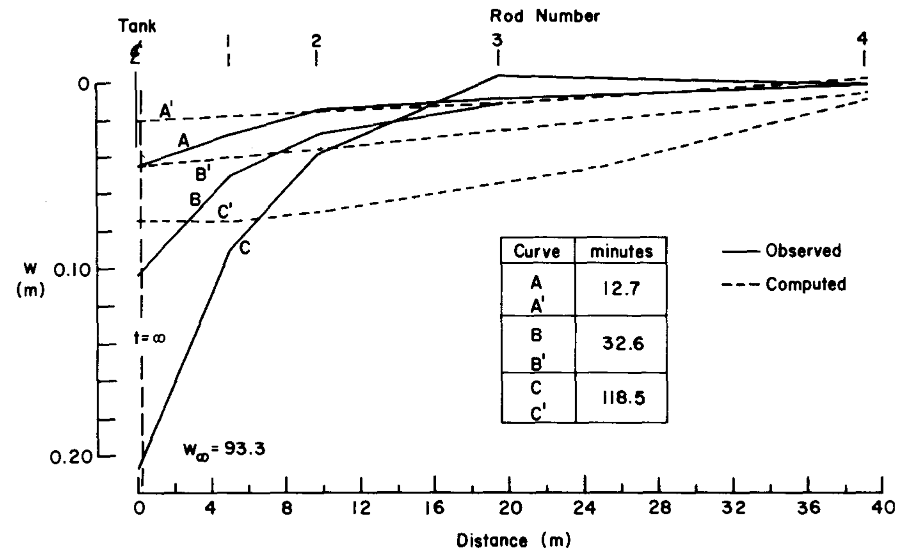

Figure 10. Deflection profile. (Frankenstein's concentrated load test, ref. 3, test 8.)

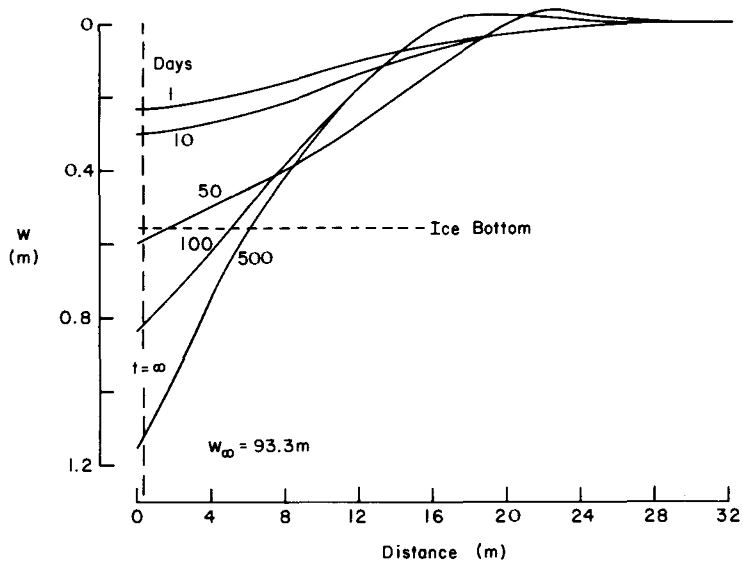

Figure 11. Asymptotic deflection profile. (Theoretical continuation of Frankenstein's concentrated load test, ref. 3, test 8.)

At $t = \infty$, the integral K in (6.4) becomes

$$K = 1 \quad \text{for } 0 < r < a$$
$$= 0 \quad \text{for } 0 < a < r. \tag{7.1}$$

The deflection w_∞ at $t = \infty$ is

$$w_\infty = q/\rho \quad \text{for } 0 < r < a$$
$$= 0 \quad \text{for } 0 < a < r. \tag{7.2}$$

Therefore, the water tank sinks theoretically to $w_\infty = 93.3$ m in the case of the concentrated load. However, the ice thickness h is 0.556 m. Our analytical formulas, therefore, become invalid beyond a certain elapsed time. (In the case of the distributed load test, $w_\infty = 1.350$ m and $h = 0.597$ m.)

Theoretical deflection profiles for large times are shown in Figure 11. At time infinity, our analytical deflection comes to the vertical line denoted by $t = \infty$. Because T_A $5.41 \times 10^{-8} \times t$ (days) in the case of the concentrated load test, the largest time, 500 days, chosen for this calculation is still too short. However, the mode of approach to the ultimate $t = \infty$ curve is observable with the curves in Figure 11. [In the case of the distributed load test, $T_A = 1.98 \times 10^{-4} \times t$ (days).]

ACKNOWLEDGEMENT

When I was struggling to understand the analytical background, Dr. D. Freitag, Technical Director, CRREL, requested Dr. Noble, Director, Mathematical Research Center, University of Wisconsin, to help me resolve some difficulties. Dr. Noble's encouragement enabled me to start the numerical integration in this paper. He later held a session for me at MRC with the participation of Drs. R.A. Askey and W. Gautschi, members of MRC, for the further clarification of the analytical background.

This research was carried out with the close cooperation of Dr. D.E. Nevel, CRREL. Especially, he contributed the nondimensional formulation of (1.19).

We appreciate SUNOCO's permission for the use of their data, although we cannot include the data in this paper.

Numerical computation was carried out in 1976 and 1977 with the help of J. Bagger, student at Dartmouth College.

LITERATURE CITED

1. Dougall, J. (1903-04) An analytical theory of the equilibrium of an isotropic elastic plate. *Transaction of the Royal Society of Edinburgh*, vol. 41, Part I, no. 8, p. 129-228.
2. Flügge, W. (1967) *Viscoelasticity.* Blaisdell Pub. Co.
3. Frankenstein, G.E. (1963) Load test data for lake ice sheets. CRREL Technical Report 89. (AD 414955).
4. Garbaccio, D.H. (1967) Creep of floating ice sheet. Naval Civil Engineering Laboratory, Port Hueneme, Calif., Report No. CR-67.025.
5. Garbaccio, D.H. (1968) Creep of floating ice sheets computer calculations. Naval Civil Engineering Laboratory, Port Hueneme, Calif., Report No. CR-69014.
6. IAkunin, A. Ye. (1968) K voprosu ob izgibe ledianogo prokrova s uchetom viazkokh svolstv l'da (Calculation of ice-cover bending allowing for viscous properties of ice). *Transactions, Novosibirskiy Institut Inzhenerov Zheleznodoroznogo Transporta*, vol. 79, p. 79-82. (Available as CRREL Draft Translation 425).
7. IAkunin, A. Ye (1970) Issledovaniye vliyaniya vremeni deystviya nagruzki na nesushchuyu sposobnost' ledyanogo pokrova (The investigation of the effect of the loading time on the bearing capacity of an ice cover). Novosibirskiy Institut Inzhenerov Zheleznodoroznogo Transporta, Speciality 01.022, Resistance of Materials and Structural Mechanics, p. 1-22.
8. Jellinek, H.H.G. and R. Brill (1956) Viscoelastic properties of ice. *Journal of Applied Physics*, vol. 27, p. 1198.
9. Katona, M.G. (1974) Ice engineering: Viscoelastic finite element formulation. Naval Civil Engineering Laboratory, Port Hueneme, Calif., Technical Report R803.
10. Kheysin, D. Ye. (1964) K zadache uprugo-plasticheskogo izgiba ledyanogo pokrova (On the problem of the elastic-plastic bending of an ice cover). *Trudy Arkticheskogo i Antarkticheskogo Nauchno-Issledovatel' skgo Instituta*, Tom. 267, p. 143.
11. Nevel, D.E. (1966) Time-dependent deflection of a floating ice sheet. CRREL Research Report 196. (AD 638717).
12. Nevel, D.E. (1976) Creep theory for a floating ice sheet. CRREL Special Report 76-4. (AD A026122).
13. Nevel, D.E., Personal communication.
14. Shumskij, P.A. (1974) *Über das fliessgesetz in polykristallinem Eis.* Polarforschung, vol. 44, no. 2, p. 105-116.
15. Titchmarsh, E.C. (1939) *The theory of functions.* Oxford University Press.
16. Van der Pol, B. and H. Bremmer (1959) *Operational calculus based on the two-sided Laplace integral.* Cambridge at the University Press.
17. Vaudry, K.D. and M.G. Katona (1975) Viscoelastic finite element analysis of sea ice sheets. International Association of Hydraulic Research, Third International Symposium on Ice Problems, CRREL.
18. Watson, G.N. (1962) *A treatise on the theory of Bessel functions.* Cambridge at the University Press.
19. Wyman, M. (1950) Deflections of an infinite water. *Canadian Journal of Research*, vol. A28, p. 293.

APPENDIX I. ANALYTICAL BACKGROUND

A. The following theorem shows the condition under which the integral (3.1) becomes either discontinuous or continuous at $R = A$.

Theorem 1. The integral (3.1) is discontinuous or continuous at $R = A$ when n in (3.2) is equal to or larger than zero, respectively.

Proof. We can rewrite (3.1) to a one-parameter integral

$$I(\alpha) = \int_0^\infty f(x, \alpha)\, dx \tag{A.1}$$

by letting $x = \beta A$, i.e. $\alpha = R/A$, where $f(x, \alpha)$ is continuous with regard to x and α. The condition that $I(\alpha)$ is a continuous function of α is that the integral (A.1) converges uniformly with respect to α (c.f. Titchmarch,[15] p. 25). The integral (A.1) uniformly converges when $n > 0$, but does not when $n = 0$.

B. We shall consider in the following the integral (3.1) whose non-Bessel factor $\phi(\beta)$ is finite in the range of integration but asymptotically becomes zero in a more general form than in the specific form (3.2).

Let an asymptotic expansion of $\phi(\beta)$ be

$$\phi(\beta) \sim \sum_{n=0}^{m} \phi_n(\beta)\,. \tag{B.1}$$

Rewrite (3.1) as

$$I = I_0 + \sum_{n=0}^{m} K_n \tag{B.2}$$

where

$$I_0 = \int_0^\infty \left\{ \phi(\beta) - \sum_{n=0}^{m} \phi_n(\beta) \right\} J_1(\beta A)\; J_0(\beta R)\; d\beta \tag{B.3}$$

and

$$K_n = \int_0^\infty \phi_n(\beta)\; J_1(\beta A)\; J_0(\beta R)\; d\beta. \tag{B.4}$$

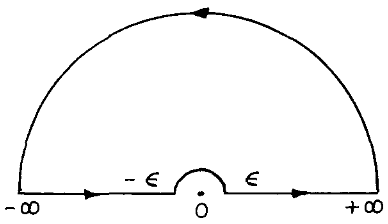

Figure 12. Contour of integrations (B.5) and (B.6).

We choose such an integer m that makes I_0 rapidly convergent. We choose such a function $\phi_n(\beta)$ that makes (B.4) analytically integrable. The following theorem is useful for the choice of $\phi_n(\beta)$.

Theorem 2. Let $F(z)$ be an even function of the complex variable $z = x + iy$ that becomes zero at $z = \infty$ and possesses only algebraic singularities (pole or branch points) on the upper half plane but no poles on the real axis. Then

$$\int_0^\infty F(x)\, J_1(ax)\, J_0(bx)\, dx$$

$$= \frac{1}{a} F(0) + \frac{1}{2} \oint_{-\infty}^{+\infty} F(z)\, H_1^{(1)}(az)\, J_0(bz)\, dz \qquad \text{when } a > b > 0 \tag{B.5}$$

$$= \frac{1}{2} \oint_{-\infty}^{+\infty} F(z)\, J_1(az)\, H_0^{(1)}(bz)\, dz \qquad \text{when } 0 < a < b \tag{B.6}$$

where $\oint_{-\infty}^{+\infty} dz$ means the integral along the contour in Figure 12, where radius ϵ is infinitesimal, and the z-plane is cut along the negative real axis.

Proof. Consider the contour integrals

$$I(a > b) = \frac{1}{2} \oint_{-\infty}^{+\infty} F(z)\, H_1^{(1)}(az)\, J_0(bz)\, dz \tag{i}$$

where $a > b > 0$, and

$$I(a < b) = \frac{1}{2} \oint_{-\infty}^{+\infty} F(z)\, J_1(az)\, H_0^{(1)}(bz)\, dz \tag{ii}$$

when $0 < a < b$. Use of the asymptotic formulas show that $H_1^{(1)}(az)\, J_0(bz)$ and $J_1(az)\, H_0^{(1)}(bz)$ are zero on the infinitely large circle when $a > b > 0$ and $0 < a < b$, respectively. Therefore we may consider only the contour along the real axis

$$I(a > b) = \frac{1}{2} \int_{-\infty}^{+\infty} F(z)\, H_1^{(1)}(az)\, J_0(bz)\, dz \tag{iii}$$

$$I(a < b) = \frac{1}{2} \int_{-\infty}^{+\infty} F(z)\, J_1(az)\, H_0^{(1)}(bz)\, dz. \tag{iv}$$

We divide the real axis in three parts: $-\infty \sim -\epsilon$, $-\epsilon \sim \epsilon$, and $\epsilon \sim \infty$. We let $z = -x$ in the region $-\infty \sim -\epsilon$, and $z = x$ in the region $\epsilon \sim \infty$, neglect the infinitesimal terms, and let

$$F(z) = F(0)$$

$$H_1^{(1)}(az) = -2i/(\pi a z)$$

$$H_0^{(1)}(bz) = [2i/(bz)] \log(bz/2).$$

Then (iii) and (iv) become

$$I(a > b) = -\frac{1}{a} F(0) + \int_0^\infty F(x)\, J_1(ax)\, J_0(bx)\, dx \tag{v}$$

$$I(a < b) = \int_0^\infty F(x)\, J_1(ax)\, J_0(bx)\, dx. \tag{vi}$$

Equations (v) and (vi) prove (B.5) and (B.6), respectively.

C. The need of Theorem 2 appears frequently in the mathematical study of the problems of a floating ice plate and the problems of an elastic plate on an elastic foundation. A similar integral including only one Bessel function was proved by Dougal (ref. 1, p. 138 and 147) as early as in 1903.

When $t = 0$, our solution of the viscoelastic plate reduces to the solution of the elastic plate. The elastic solution thus found is composed of the following integrals:

$$M_0 = \int_0^\infty \frac{1}{1+x^4} J_1(ax)\, J_0(bx)\, dx$$

$$M_1 = \int_0^\infty \frac{x}{1+x^4} J_1(ax)\, J_1(bx)\, dx$$

$$M_2 = \int_0^\infty \frac{x^2}{1+x^4} J_1(ax)\, J_0(bx)\, dx$$

where $a = AE^{1/4}$ and $b = RE^{1/4}$. We can carry out these integrals by direct or indirect application of Theorem 2:

$$\begin{aligned}
M_0 &= \operatorname{ber} b \operatorname{ker}' a - \operatorname{bei} b \operatorname{kei}' a + a^{-1} && \text{when } b \leqq a \\
&= \operatorname{ber}' a \operatorname{ker} b - \operatorname{bei}' a \operatorname{kei} b && \text{when } a \leqq b \\
M_1 &= -\operatorname{ber}' b \operatorname{ker}' a + \operatorname{bei}' b \operatorname{kei}' a && \text{when } b \leqq a \\
&= -\operatorname{ber}' a \operatorname{ker}' b + \operatorname{bei}' a \operatorname{kei}' b && \text{when } a \leqq b \\
M_2 &= \operatorname{bei} b \operatorname{ker}' a + \operatorname{ber} b \operatorname{kei}' a && \text{when } b \leqq a \\
&= \operatorname{bei}' a \operatorname{ker} b + \operatorname{ber}' a \operatorname{kei} b && \text{when } a \leqq b.
\end{aligned}$$

M_0 and M_2 are found by directly applying the theorem. M_1 is found by differentiating M_0 with regard to B. Wyman[19] derived M_0 by integrating a concentrated-load elastic-plate solution over the loading circle.

The continuity of M_0 and M_2 at $a = b$ is obvious on the strength of Theorem 1. We shall show, however, a direct proof in the following. We shall prove that

$$\mathrm{ber}x \ \mathrm{ker}'x - \mathrm{bei}x \ \mathrm{kei}'x + x^{-1} = \mathrm{ber}'x \ \mathrm{ker}x - \mathrm{bei}'x \ \mathrm{kei}x \tag{C.1}$$

and

$$\mathrm{bei}x \ \mathrm{ker}'x + \mathrm{ber}x \ \mathrm{kei}'x = \mathrm{bei}'x \ \mathrm{ker}x + \mathrm{ber}'x \ \mathrm{kei}x. \tag{C.2}$$

To prove this, note that

$$w_1(x) = \mathrm{ber}x + i \ \mathrm{bei}x \tag{C.3}$$

and

$$w_2(x) = \mathrm{ker}x + i \ \mathrm{kei}x \tag{C.4}$$

are the solutions of the differential equation

$$\frac{d^2w}{dx^2} + \frac{1}{x}\frac{dw}{dx} - iw = 0.$$

This can be proved by decomposing the equation

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}\right)^2 w + w = 0$$

of which (C.3) and (C.4) are the solutions.

We can find that the Wronskian

$$\begin{vmatrix} w_1(x) & w_2(x) \\ w_1'(x) & w_2'(x) \end{vmatrix}$$

is equal to

$$= -x^{-1}.$$

Thus we have the identity

$$\begin{vmatrix} \mathrm{ber}x + i \ \mathrm{bei}x & \mathrm{ker}x + i \ \mathrm{kei}x \\ \mathrm{ber}'x + i \ \mathrm{bei}'x & \mathrm{ker}'x + i \ \mathrm{kei}'x \end{vmatrix} = -\frac{1}{x}$$

of which the real part gives (C.1) and the imaginary part gives (C.2).

Theorem 2 can be extended in many ways. Nevel[18] found that

$$\int_0^\infty F(x)\, dx = \frac{i}{\pi} \oint_{-\infty}^{+\infty} F(z) \log z \, dz \tag{C.3}$$

for an odd function $F(z)$ that does not have any pole on the real axis and vanishes at $z = \infty$.

D. It is impossible to apply Theorem 2 to the integrals of w in (2.16), σ_r in (2.19), and σ_θ in (2.20) for the following reason.

The function $\exp(-\alpha_2 T)$ has essential singularities at the roots of

$$\beta^4 + E = 0,$$

because

$$\lim_{\beta^4 \to -E} \alpha_2 = \infty$$

The function $\exp(-\alpha_1 T)$ does not possess any essential singularities because the limit of

$$\alpha_1 = 2\tau / [\tau\beta^4 + 1 + \tau + \sqrt{(\tau\beta^4 + 1 + \tau)^2 - 4\tau(\beta^4 + E)}\ ]$$

is finite. However, the real part of α_1 becomes negative, and $\exp(-\alpha_1 T)$ diverges, as $|\beta| \to \infty$ in a certain range of direction.

Theorem 2 does not apply to integral K in (6.4) because the point $x = 0$ is an essential singularity.

The only alternative we can find for the integration of (3.1) is the use of Barnes' integral method. It consists in substituting the integrals

$$J_v(x) = \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{\Gamma(-S)(\frac{1}{2}x)^{v+2S}}{\Gamma(\nu + S + 1)}\, dS \tag{D.1}$$

$$\pi e^{\frac{1}{2}(v+1)\pi i} H_v^{(1)}(z) = \frac{1}{2\pi i} \int_{-c-\infty i}^{-c+\infty i} \Gamma(-\nu - S)\, \Gamma(-S) \left(-\frac{i}{2} z\right)^{v+2S} ds \tag{D.2}$$

for $J_v(x)$ and $H_v^{(1)}(z)$, respectively, where c is a real number satisfying $c > R(\nu)$, z is complex, and x is real. We can usually exchange the order of integration to carry out the integration with regard to x or z. Then, we can carry out the rest of the integration in most cases by use of the theorem of residue. Only the forms (D.1) and (D.2) serve this purpose. The other Barnes' representations of $J_v(x)$ and $H_v^{(1)}(z)$ do not enable us to carry out the above two procedures.

However, as mentioned by Watson (ref. 18, p. 192), (D.1) does not hold true for $\nu = 0$, and (D.2) does not hold true when $\nu = 0$ and z is real. In these two cases, the integrands of (D.1) and (D.2) become proportional to s^{-1} as s approaches $i\infty$ as the limit on the imaginary axis. Therefore we cannot use Barnes' integral method to carry out our integrals.

APPENDIX II. COMPUTER PROGRAMS

Ramp Time Profiles

```
100 ' NUMERICAL STUDY OF THE VISCOELASTIC DEFORMATION
110 ' OF THE ICE PLATE UNDER A CIRCULAR LOAD
120 ' S. TAKAGI, 1976 (J. BAGGER)
130 ' [RAMP FORMULATION]
140
150
160 ' ************************************
170 ' ***PLOTTING OF EXPERIMENTAL DATA***
180 ' ************************************
190
200 ' THIS SECTION OF THE PROGRAM PLOTS THE FIELD DATA FROM
210 ' GUENTHER FRANKENSTEIN'S TESTS ON LAKE ICE SHEETS.  THE
220 ' DEFORMATION OF THE ICE FROM A CIRCULAR LOAD WAS MEASURED
230 ' UNDER VARIOUS CONDITIONS.  THIS DATA IS COMPARED BELOW
240 ' WITH THE RAMP FORMULATION OF THE VISCOELASTIC THEORY
250 ' FOR DEFORMATION.
260
270
280 ' **********SYMBOL TABLE**********
290 '          A              DEFINED BELOW IN MAIN PROGRAM SYMBOL TABLE
300 '          A9             PHYSICAL LOAD RADIUS
310 '          A$             DATA FILE NAME
320 '          E              DEFINED BELOW IN MAIN PROGRAM SYMBOL TABLE
330 '          E8             E0 -- EQUATION 1.8
340 '          H9             PHYSICAL ICE THICKNESS
350 '          I              COUNTER
360 '          L9             CHARACTERISTIC LENGTH
370 '          N7             N1/E0
380 '          N8             NU -- EQUATION 1.5
390 '          P              NON-DIMENSIONALIZED P-DOT -- EQUATION 4.3
400 '          P9             PHYSICAL P-DOT
410 '          R              DEFINED BELOW IN MAIN PROGRAM SYMBOL TABLE
420 '          R8             RHO -- EQUATION 1.3
430 '          R9             PHYSICAL RADIUS OF OBSERVATION
440 '          T              DEFINED BELOW IN MAIN PROGRAM SYMBOL TABLE
450 '          T1             DEFINED BELOW IN MAIN PROGRAM SYMBOL TABLE
460 '          T5             MAXIMUM NON-DIMENSIONALIZED T
470 '          W              DEFINED BELOW IN MAIN PROGRAM SYMBOL TABLE
480 '          X5             MAXIMUM X-PLOT
490 '          Y5             MAXIMUM Y-PLOT
500 ' ********************************
510
520
530 ' SET UP PLOTTER DETAILS
540
550 LIBRARY "PHYSLIB***:FLABEL"
560 LIBRARY "PLOTLIB***:TEK10"
570 DIM C(600)
580 DIM C$(600)
590 PRINT "INPUT FILE";
600 INPUT A$
610 FILE#1:A$
620 PRINT "XMAX";
630 INPUT X5
640 PRINT "YMAX";
650 INPUT Y5
660 CALL "PLABEL":C(),0,X5,-Y5,Y5,"TIME (SEC)"," W (METERS)",-1
670 CALL "CONNECT":C(),0,0,X5,0
680
690 ' INPUT TEST SITE DATA
700
710 INPUT#1: P9,H9,R9
720 ' P9 = P-DOT
730 ' H9 = ICE THICKNESS
740 ' R9 = RADIUS
750
760 INPUT #1: A9
770 ' A9 = PHYSICAL LOAD RADIUS
780
790 ' PARAMETERS (VARY TO FIT)
800
810 LET T1 = 10
820 LET E = .02
830 LET N7 = 7.E+5
840 LET E8 = 2.E+8
850 ' T1 = TAU
860 ' E = E
870 ' N7 = N1/E0
880 ' E8 = E0
890
900 ' CONSTANTS (STANDARD)
910
920 LET R8 = 1000                                    'RHO
930 LET N8 = .5                                      'NU
940
950 ' CONVERT TO NONDIMENSIONALIZED FORM
960
970 LET L9 = (E8*H9↑3)/(12*R8*(1-N8))    'CHAR. LENGTH
980 LET L9 = L9↑(1/4)
990 LET R = R9/L9
1000 LET A = A9/L9
1010 LET P = P9*N7
1020
1030 ' PLOT DIMENSIONALIZED EXPERIMENTAL DATA
1040
1050 INPUT#1:N1
1060 FOR I = 1 TO N1
1070    INPUT#1: T,W
1080    ' CONVERT TO MKS UNITS
1090    LET T = T*60
1100    LET W = W*.3048
1110    CALL "MOVE":C(),T,W
1120    CALL "LABEL":C(),C$(),"+",.1,0,0
1130 NEXT I
1140 CALL "LIFT":C()
1150 LET T5 = T                                'SAVE MAXIMUM TIME
1160
1170 ' ************************************
1180 ' ***PLOTTING OF THEORETICAL CURVE***
1190 ' ************************************
1200
1210 ' THIS SECTION OF THE PROGRAM PLOTS THE THEORETICAL
1220 ' CURVE FOR COMPARISON WITH THE EXPERIMENTAL DATA.
1230
1240
1250 ' **********SYMBOL TABLE**********
```

```
1260 '          A             EQUATION 2.7
1270 '          A1            ALPHA1 -- EQUATION 2.11
1280 '          A2            ALPHA2 -- EQUATION 2.11
1290 '          E             EQUATION 1.15
1300 '          E6            NORMALIZED TRAPEZOIDAL ERROR
1310 '          FNA           ALPHA1(BETA)
1320 '          FNB           ALPHA2(BETA)
1330 '          FNI           J0(X)
1340 '          FNJ           J1(X)
1350 '          FNS           SQR(DESC) -- EQUATION 2.12
1360 '          FNU           INTEGRAND OF U1
1370 '          FNV           INTEGRAND OF U2
1380 '          FNW           INTEGRAND OF U3
1390 '          I             TIME COUNTER
1400 '          N             UPPER BOUND OF INTEGRATION -- EQUATION 4.8
1410 '          N1            UPPER BOUND OF INTEGRATION -- EQUATION 4.9
1420 '          N2            UPPER BOUND OF INTEGRATION -- EQUATION 4.10
1430 '          R             EQUATION 1.18
1440 '          T             EQUATION 1.12
1450 '          T1            TAU -- EQUATION 1.13
1460 '          U1            EQUATION 4.5
1470 '          U2            EQUATION 4.6
1480 '          U3            EQUATION 4.7
1490 '          W             DEFORMATION
1500 ' ********************************
1510
1520
1530 LET P1 = 3.141593
1540
1550 ' COMPUTE UPPER BOUNDS OF INTEGRATION
1560
1570 LET T5=T5/N7                         'NONDIMENSIONALIZE MAXIMUM TIME
1580 LET N = ((T5↑2*1E5)/(4*P1*SQR(A*R)))↑(1/4)
1590 LET N1 = ((T5*1E5)/(2*P1*SQR(A*R)))↑(1/4)
1600 LET N2 = (((1 - E)*1E5)/(2*P1*SQR(A*R)*T1)*(1 - EXP(-T5*T1)))↑(1/4)
1610
1620 FOR I=0 TO T5 STEP T5/20             'NONDIMENSIONAL TIME LOOP
1630
1640    LET T=I
1650    CALL "SIMP": 0,N,FNU,U1,A,E,R,T,T1
1660    CALL "SIMP": 0,N1,FNV,U2,A,E,R,T,T1
1670    CALL "SIMP": 0,N2,FNW,U3,A,E,R,T,T1
1680
1690    LET W = U1 - U2 + U3
1700    LET T=T*N7                        'CONVERT TIME TO SECONDS
1710    LET W=W/(3.14159*A*R8*L9↑2/P)     'CONVERT DEFLECTION TO METERS
1720    CALL "LINE":C(),T,W               'DRAW THEORETICAL CURVE
1730
1740 NEXT I
1750
1760 ' PRINT TITLE BLOCK
1770
1780 CALL "FINISH":C()
1790 PRINT "","   SITE ";SEG$(A$,1,1);"   TEST ";SEG$(A$,2,2);"   POSITION ";SEG$(A$,3,LEN(A$)-1)
1800 PRINT
1810 PRINT "", "TAU: ";T1,"N1/E0: ";N7
1820 PRINT "", "E: ";E,"E0: ";E8
1830 CALL "ERROR": N,N1,N2,FNU,FNV,FNW,A,E,R,T1,N7,R8,L9,P,E6,#1
1840 PRINT "","NORMALIZED TRAPEZOIDAL ERROR ";E6
1850 PRINT "              "
1860
1870 ' FUNCTION DEFINITIONS AND SUBPROGRAMS
1880
1890 DEF FNA(A,E,R,T,T1,X)
1900    ' ALPHA1
1910    LET FNA = (T1*X↑4) + 1 + T1 - FNS(A,E,R,T,T1,X)
1920    LET FNA = FNA/(2*(X↑4 + E))
1930 FNEND
1940
1950 DEF FNB(A,E,R,T,T1,X)
1960    ' ALPHA2
1970    LET FNB = (T1*X↑4) + 1 + T1 + FNS(A,E,R,T,T1,X)
1980    LET FNB = FNB/(2*(X↑4 + E))
1990 FNEND
2000
2010 DEF FNS(A,E,R,T,T1,X)
2020    ' SQR(DESC)
2030    LET FNS = ((T1*X↑4) + 1 + T1)↑2 - 4*T1*(X↑4 + E)
2040    LET FNS = SQR(FNS)
2050 FNEND
2060
2070 DEF FNU(A,E,R,T,T1,X)
2080    LET A1 = FNA(A,E,R,T,T1,X)
2090    LET FNU = T - (1/A1)*(1 - EXP(-T*A1))
2100    LET FNU = FNU*FNJ(X*A)*FNI(X*R)
2110 FNEND
2120
2130 DEF FNV(A,E,R,T,T1,X)
2140    LET A1 = FNA(A,E,R,T,T1,X)
2150    LET FNV = (X↑4)*(T1 - A1)/FNS(A,E,R,T,T1,X) - 1
2160    LET FNV = FNV*(1 - EXP(-A1*T))/A1
2170    LET FNV = FNV*FNJ(X*A)*FNI(X*R)
2180 FNEND
2190
2200 DEF FNW(A,E,R,T,T1,X)
2210    LET A2 = FNB(A,E,R,T,T1,X)
2220    LET FNW = (X↑4)*(T1 - A2)/FNS(A,E,R,T,T1,X)
2230    LET FNW = FNW*(1 - EXP(-A2*T))/A2
2240    LET FNW = FNW*FNJ(A*X)*FNI(X*R)
2250 FNEND
2260
2270 DEF FNI(X)
2280
2290    '    ***FUNCTION SUBPROGRAM:  J0(X)***
2300
2310    '    POLYNOMIAL APPROXIMATIONS FROM:
2320    '    HANDBOOK OF MATHEMATICAL FUNCTIONS,
2330    '    US DEPARTMENT OF COMMERCE, 1964
2340    '    (9.4.1, 9.4.3)
2350
2360
2370    '    *********** SYMBOL TABLE ***********
2380    '          J0            J0(X) ZEROTH-ORDER BESSEL FUNCTION
2390    '          F             F(X) IN NBS BOOK
2400    '          T3            THETA(X) IN NBS BOOK
2410    '          X             ARGUMENT
```

```
2420 '        Y                    X/3. OR 3./X
2430 '     ***********************************
2440
2450
2460 IF X >= 0 THEN 2510
2470 PRINT "FUNCTION J0(X):  ARGUMENT MUST BE >= 0"
2480 PRINT "X=";X
2490 STOP
2500
2510 IF X>3 THEN 2590
2520
2530 ' POLYNOMIAL APPROX, 0<=X<=3.
2540 LET Y=X/3.
2550 LET J0 = 1 - 2.24999 97*Y↑2 + 1.26562 08*Y↑4
2560 LET J0 = J0 - .31638 66*Y↑6 + .04444 79*Y↑8
2570 GOTO 2730
2580
2590 ' POLYNOMIAL APPROX, X>3.
2600 LET Y=3./X
2610 '  F = F IN NBS BOOK
2620 LET F = .79788 456 - .00000 077*Y
2630 LET F = F - .00552 740*Y↑2 - .00009 512*Y↑3
2640 LET F = F + .00137 237*Y↑4 - .00072 805*Y↑5
2650 LET F = F + .00014 476*Y↑6
2660 ' T3 = THETA IN NBS BOOK
2670 LET T3 = X - .78539 816 - .04166 397*Y
2680 LET T3 = T3 - .00003 954*Y↑2 + .00262 573*Y↑3
2690 LET T3 = T3 - .00054 125*Y↑4 - .00029 333*Y↑5
2700 LET T3 = T3 + .00013 558*Y↑6
2710 LET J0 = (1./SQR(X))*F*COS(T3)
2720
2730 LET FNI = J0
2740 FNEND
2750
2760 DEF FNJ(X)
2770
2780 '    ***FUNCTION SUBPROGRAM:  J1(X)***
2790
2800 '    POLYNOMIAL APPROXIMATIONS FROM:
2810 '    HANDBOOK OF MATHEMATICAL FUNCTIONS,
2820 '    US DEPARTMENT OF COMMERCE, 1964
2830 '    (9.4.4, 9.4.6)
2840
2850
2860 '    ********** SYMBOL TABLE **********
2870 '    J1        J1(X) FIRST-ORDER BESSEL FUNCTION
2880 '    F1        F1(X) IN NBS BOOK
2890 '    T2        THETA1(X) IN NBS BOOK
2900 '    X         ARGUMENT
2910 '    Y         X/3. OR 3./X
2920 '    ***********************************
2930
2940
2950 IF X >= 0 THEN 3000
2960 PRINT "FUNCTION J1(X):  ARGUMENT MUST BE >= 0"
2970 PRINT "X=",X
2980 STOP
2990
```



```
3000 IF X>3 THEN 3090
3010
3020 ' POLYNOMIAL APPROX, 0<=X<=3.
3030 LET Y=X/3.
3040 LET J1 = .5 - .56249 985*Y↑2 + .21093 573*Y↑4
3050 LET J1 = J1 - .03954 289*Y↑6 + .00443 319*Y↑8
3060 LET J1 = X*J1
3070 GOTO 3230
3080
3090 ' POLYNOMIAL APPROX, X>3.
3100 LET Y=3./X
3110 ' F1 = F1 IN NBS BOOK
3120 LET F1 = .79788 456 + .00000 156*Y
3130 LET F1 = F1 + .01659 667*Y↑2 + .00017 105*Y↑3
3140 LET F1 = F1 - .00249 511*Y↑4 + .00113 653*Y↑5
3150 LET F1 = F1 -  .00020 033*Y↑6
3160 ' T2 = THETA1 IN NBS BOOK
3170 LET T2 = X - 2.35619 449 + .12499 612*Y
3180 LET T2 = T2 + .00005 650*Y↑2 - .00637 879*Y↑3
3190 LET T2 = T2 + .00074 348*Y↑4 + .00079 824*Y↑5
3200 LET T2 = T2 - .00029 166*Y↑6
3210 LET J1 = (1./SQR(X))*F1*COS(T2)
3220
3230 LET FNJ = J1
3240 FNEND
3250 END
3260
3270 SUB "SIMP":X1,X2,FNF,A4,A,E,R,T,T1
3280 ' *** SUBPROGRAM:  SIMPSON'S RULE INTEGRATION ***
3290
3300 ' SIMPSON'S RULE FORMULA FROM:
3310 ' NUMERICAL CALCULUS
3320 ' WILLIAM MILNE, 1949
3330
3340 ' ********** SYMBOL TABLE **********
3350 '              A3            SIMPSON APPROXIMATION FOR PREVIOUS TRIAL
3360 '              A4            SIMPSON APPROXIMATION FOR CURRENT TRIAL
3370 '              FNF           FUNCTION SUBPROGRAM FOR INTEGRAND
3380 '              H             INTERVAL WIDTH
3390 '              I             COUNTER
3400 '              N             2*N = NUMBER OF INTERVALS
3410 '              S1            PARTIAL SUM OF THE ODD TERMS
3420 '              S2            PARTIAL SUM OF THE EVEN TERMS
3430 '              X             VARIABLE OF INTEGRATION
3440 '              X2            UPPER BOUND OF INTEGRATION
3450 '              X1            LOWER BOUND OF INTEGRATION
3460 ' ***********************************
3470
3480 IF X2>X1 THEN 3530
3490 PRINT "SUBPROGRAM SIMP:  XMAXIMUM MUST BE > XMIN"
3500 PRINT "XMIN=";X1,"XMAX=";X2
3510 STOP
3520
3530 ' INITIALIZE OLD APPROXIMATION
3540 LET A3=0
3550 ' INTIALIZE TO 100 INTERVALS
3560 LET N = 50
3570 ' CALCULATE INTERVAL WIDTH
```

```
3580    LET H=(X2-X1)/(2*N)
3590    ' INITIALIZE PARTIAL SUMS
3600    LET S1=S2=0
3610    ' CALCULATE PARTIAL SUMS
3620    LET X=X1
3630    FOR I=0 TO N
3640       LET S2=S2+FNF(A,E,R,T,T1,X)
3650       LET X=X+2*H
3660    NEXT I
3670    LET X=X1+H
3680    FOR I=1 TO N
3690       LET S1=S1+FNF(A,E,R,T,T1,X)
3700       LET X=X+2*H
3710    NEXT I
3720    ' CALCULATE NEW APPROXIMATION
3730    LET A4=H*(4*S1+2*S2-FNF(A,E,R,T,T1,X1)-FNF(A,E,R,T,T1,X2))/3
3740
3750    IF ABS(A3-A4)<=1.E-6 THEN 3850
3760
3770    ' PREPARE TO TRY AGAIN
3780
3790    LET A3=A4
3800    LET S2=S1+S2
3810    LET S1=0
3820    LET N=2*N
3830    LET H=(X2-X1)/(2*N)
3840    GOTO 3670
3850 SUBEND
3860
3870 SUB "ERROR":N,N1,N2,FNU,FNV,FNW,A,E,R,T1,N7,R8,L9,P,E6,#1
3880    RESET#1
3890    INPUT#1: N3                      'DISCARD FIRST LINE
3900    INPUT#1: N3                      'DISCARD SECOND LINE
3910    INPUT#1: N3                      'NUMBER OF DATA POINTS
3920    INPUT#1: T2,W2                   'TIME, DEFORMATION
3930    LET T2 = T2*60                   'TIME IN SECONDS
3940    LET W2 = W2*.3048                'DEFORMATION IN METERS
3950    LET T3 = T2/N7                   'NONDIMENSIONALIZE TIME
3960    CALL "SIMP": 0,N,FNU,U1,A,E,R,T3,T1
3970    CALL "SIMP": 0,N1,FNV,U2,A,E,R,T3,T1
3980    CALL "SIMP": 0,N2,FNW,U3,A,E,R,T3,T1
3990    LET W3 = U1 - U2 + U3            'COMPUTE NONDIMENSIONAL DEFORMATION
4000    LET W3 = W3/(3.14159*A*R8*L9↑2/P)  'DIMENSIONALIZE IT
4010    LET E2 = (W3 - W2)               'COMPUTE ERROR
4020    LET T6 = 0                       'INITIALIZE TIME SUM FOR NORMALIZATION
4030    LET S2 = 0                       'INITIALIZE TRAPEZOIDAL AREA SUM
4040    FOR I = 1 TO N3 - 1
4050       INPUT#1 :T4,W4                'NEXT TIME, DEFORMATION PAIR
4060       LET T4 = T4*60
4070       LET W4 = W4*.3048
4080       LET T5 = T4/N7
4090       CALL "SIMP": 0,N,FNU,U1,A,E,R,T5,T1
4100       CALL "SIMP": 0,N1,FNV,U2,A,E,R,T5,T1
4110       CALL "SIMP": 0,N2,FNW,U3,A,E,R,T5,T1
4120       LET W5 = U1 - U2 + U3
4130       LET W5 = W5/(3.14159*A*R8*L9↑2/P)
4140       LET E4 = (W5-W4)              'NEXT ERROR
4150       LET T6 = T4 - T2 + T6         'INCREMENT TIME SUM
4160          IF E2*E4 > 0 THEN 4190            'CURVES DON'T CROSS => TRAP
4170          LET S2 = S2 + .5*(T4 - T2)*(E2↑4 + E4↑4)/(E2↑2 + E4↑2)
4180          GOTO 4200
4190          LET S2 = S2 + .5*(T4 - T2)*(E4↑2 + E2↑2)  'INCREMENT TRAP. AREA SUM
4200          LET E2 = E4                       'NEW BECOMES OLD
4210          LET T2 = T4                       'NEW BECOMES OLD
4220       NEXT I
4230       LET E6 = S2/T6                       'NORMALIZE SUM
4240       LET E6 = SQR(E6)
4250 SUBEND
```

```
100 ' NUMERICAL STUDY OF THE VISCOELASTIC DEFORMATION
110 ' OF THE ICE PLATE UNDER A CIRCULAR LOAD
120 ' S. TAKAGI, 1976    (J. BAGGER)
130 ' [RAMP FORMULATION -- STEADY STATE SOLUTION]
140
150
160 ' ************************************
170 ' ***PLOTTING OF EXPERIMENTAL DATA***
180 ' ************************************
190
200 ' THIS SECTION OF THE PROGRAM PLOTS THE FIELD DATA
210 ' GUENTHER FRANKENSTEIN'S TESTS ON LAKE ICE SHEETS. THE
220 ' DEFORMATION OF THE ICE FROM A CIRCULAR LOAD WAS MEASURED
230 ' UNDER VARIOUS CONDITIONS.  THIS DATA IS COMPARED BELOW
240 ' WITH THE STEADY-STATE FORMULATION OF THE VISCOELASTIC
250 ' THEORY FOR DEFORMATION.
260
270
280 ' **********SYMBOL TABLE**********
290 '         A             DEFINED BELOW IN MAIN PROGRAM SYMBOL TABLE
300 '         A9            PHYSICAL LOAD RADIUS
310 '         A$            DATA FILE NAME
320 '         E             DEFINED BELOW IN MAIN PROGRAM SYMBOL TABLE
330 '         E0            E0 -- EQUATION 1.8
340 '         H9            PHYSICAL ICE THICKNESS
350 '         I             COUNTER
360 '         L9            CHARACTERISTIC LENGTH
370 '         N7            N1/E0
380 '         N8            NU -- EQUATION 1.5
390 '         P             NON-DIMENSIONALIZED P-DOT -- EQUATION 4.3
400 '         P9            PHYSICAL P-DOT
410 '         R             DEFINED BELOW IN MAIN PROGRAM SYMBOL TABLE
420 '         R8            RHO -- EQUATION 1.3
430 '         R9            PHYSICAL RADIUS OF OBSERVATION
440 '         T             DEFINED BELOW IN MAIN PROGRAM SYMBOL TABLE
450 '         T0            TIME OF CONSTANT LOAD
460 '         T1            DEFINED BELOW IN MAIN PROGRAM SYMBOL TABLE
470 '         T5            MAXIMUM NON-DIMENSIONALIZED T
480 '         W             DEFINED BELOW IN MAIN PROGRAM SYMBOL TABLE
490 '         X5            MAXIMUM X-PLOT
500 '         Y5            MAXIMUM Y-PLOT
510 ' ********************************
520
530
540 ' SET UP PLOTTER DETAILS
550
560 LIBRARY "PHYSLIB***:FLABEL"
570 LIBRARY "PLOTLIB***:TEK10"
580 DIM C(600)
590 DIM C$(600)
600 PRINT "INPUT FILE";
610 INPUT A$
620 FILE#1:A$
630 PRINT "XMAX";
640 INPUT X5
650 PRINT "YMAX";
660 INPUT Y5
670
680 ' INPUT TEST SITE DATA
690
700 INPUT #1: P9,H9,R9,T0
710 ' P9 = P-DOT
720 ' H9 = ICE THICKNESS
730 ' R9 = RADIUS
740 ' T0 = TIME WHEN LOAD BECOMES CONSTANT
750
760 INPUT#1: A9
770 ' A9 = PHYSICAL LOAD RADIUS
780
790 ' PARAMETERS (VARY TO FIT)
800
810 LET T1 = 10
820 LET E = .02
830 LET N7 = 7.E+5
840 LET E8 = 2.E+8
850 ' T1 = TAU
860 ' E = E
870 ' N7 = N1/E0
880 ' E8 = E0
890
900 ' CONSTANTS (STANDARD)
910
920 LET R8 = 1000                             'RHO
930 LET N8 = .5                               'NU
940
950 ' CONVERT TO NONDIMENSIONALIZED FORM
960
970 LET L9 = (E8*H9↑3)/(12*R8*(1-N8))    'CHAR. LENGTH
980 LET L9 = L9↑(1/4)
990 LET R = R9/L9
1000 LET A = A9/L9
1010 LET P = P9*N7
1020
1030 ' PLOT DIMENSIONALIZED EXPERIMENTAL DATA
1040
1050 CALL "PLABEL":C(),T0,X5,-Y5,Y5,"TIME (SEC)"," W (METERS)",1
1060 CALL "CONNECT":C(),T0,0,X5,0
1070
1080 INPUT#1:N1
1090 FOR I = 1 TO N1
1100    INPUT#1: T,W
1110    LET T = T*60
1120    LET W = W*.3048
1130    CALL "MOVE":C(),T,W
1140    CALL "LABEL":C(),C$(),"+",.1,0,0
1150 NEXT I
1160 CALL "LIFT":C()
1170 LET T5 = T                               'SAVE MAXIMUM TIME
1180
1190 ' ************************************
1200 ' ***PLOTTING OF THEORETICAL CURVE***
1210 ' ************************************
1220
1230 ' THIS SECTION OF THE PROGRAM PLOTS THE THEORETICAL
1240 ' CURVE FOR COMPARISON WITH THE EXPERIMENTAL DATA.
1250
```

```
1260
1270 '**********SYMBOL TABLE**********
1280 '          A          EQUATION 2.7
1290 '          A1         ALPHA1 -- EQUATION 2.11
1300 '          A2         ALPHA2 -- EQUATION 2.11
1310 '          E          EQUATION 1.15
1320 '          FNA        ALPHA1(BETA)
1330 '          FNB        ALPHA2(BETA)
1340 '          FNI        J0(X)
1350 '          FNJ        J1(X)
1360 '          FNS        SQR(DESC) -- EQUATION 2.12
1370 '          FNU        INTEGRAND OF I1
1380 '          FNV        INTEGRAND OF I2
1390 '          FNW        INTEGRAND OF I3
1400 '          I          TIME COUNTER
1410 '          N          UPPER BOUND OF INTEGRATION -- EQUATION 4.15
1420 '          N1         UPPER BOUND OF INTEGRATION -- EQUATION 4.16
1430 '          N2         UPPER BOUND OF INTEGRATION -- EQUATION 4.17
1440 '          R          EQUATION 1.18
1450 '          T          EQUATION 1.12
1460 '          T1         TAU -- EQUATION 1.13
1470 '          I1         EQUATION 4.12
1480 '          I2         EQUATION 4.13
1490 '          I3         EQUATION 4.14
1500 '          W          DEFORMATION
1510 '*********************************
1520
1530
1540 LET P1 = 3.141593
1550
1560 ' COMPUTE UPPER BOUNDS OF INTEGRATION
1570
1580 LET T0=T0/N7                         'NONDIMENSIONALIZE MIN TIME
1590 LET T5=T5/N7                         'NONDIMENSIONALIZE MAXIMUM TIME
1600
1610 LET N=(1E5/(2*P1*SQR(A*R)))↑(1/4)
1620 IF FNA(A,E,R,T,T0,T1,N)<.005/T0 THEN 1650
1630 LET N=N+1
1640 GOTO 1620
1650 LET N1=(1E5*T5/(2*P1*SQR(A*R)))↑(1/4)
1660 IF FNA(A,E,R,T,T0,T1,N1)<.005/T0 THEN 1690
1670 LET N1=N1+1
1680 GOTO 1660
1690 LET N2=(1E5*(EXP(T1*T0)-1)*(1-E)/(T1*T0*2*P1*SQR(A*R)))↑(1/4)
1700
1710 CALL "SIMP": 0,N,FNU,I1,A,E,R,T,T0,T1
1720
1730 FOR I = T0 TO T5 STEP (T5-T0)/20     'NONDIMENSIONAL TIME LOOP
1740
1750    LET T=I
1760    CALL "SIMP": 0,N1,FNV,I2,A,E,R,T,T0,T1
1770    CALL "SIMP": 0,N2,FNW,I3,A,E,R,T,T0,T1
1780
1790    LET W = I1 + I2 + I3
1800    LET T=T*N7                         'CONVERT TIME TO SECONDS
1810    LET W=W/(3.14159*A*R8*L9↑2/P)      'CONVERT DEFLECTION TO METERS
1820    CALL "LINE":C(),T,W                'DRAW THEORETICAL CURVE
1830
1840 NEXT I
1850 '
1860 ' PRINT TITLE BLOCK
1870
1880 CALL "FINISH":C()
1890 PRINT "","   SITE ";SEG$(A$,1,1);"   TEST ";SEG$(A$,2,2);"   POSITION ";SEG$(A$,3,LEN(A$)-1)
1900 PRINT
1910 PRINT "", "TAU: ";T1,"N1/E0: ";N7
1920 PRINT "", "E: ";E,"E0: ";E8
1930 CALL "ERROR": N,N1,N2,I1,FNV,FNW,A,E,R,T0,T1,N7,R8,L9,P,E6,#1
1940 PRINT "","NORMALIZED TRAPEZOIDAL ERROR ";E6
1950 PRINT "          "
1960
1970 ' FUNCTION DEFINITIONS AND SUBPROGRAMS
1980
1990 DEF FNA(A,E,R,T,T0,T1,X)
2000    ' ALPHA1
2010    LET FNA = (T1*X↑4) + 1 + T1 - FNS(A,E,R,T,T0,T1,X)
2020    LET FNA = FNA/(2*(X↑4 + E))
2030 FNEND
2040
2050 DEF FNB(A,E,R,T,T0,T1,X)
2060    ' ALPHA2
2070    LET FNB = (T1*X↑4) + 1 + T1 + FNS(A,E,R,T,T0,T1,X)
2080    LET FNB = FNB/(2*(X↑4 + E))
2090 FNEND
2100
2110 DEF FNS(A,E,R,T,T0,T1,X)
2120    ' SQR(DESC)
2130    LET FNS = ((T1*X↑4) + 1 + T1)↑2 - 4*T1*(X↑4 + E)
2140    LET FNS = SQR(FNS)
2150 FNEND
2160
2170 DEF FNU(A,E,R,T,T0,T1,X)
2180    LET A1 = FNA(A,E,R,T,T0,T1,X)
2190    LET FNU = X↑4*(T1 - A1)/FNS(A,E,R,T,T0,T1,X)
2200    IF A1*T0 > .005 THEN 2230
2210    LET FNU = FNU*(1 + A1*T0/2)
2220    GOTO 2240
2230    LET FNU = FNU*(EXP(A1*T0)-1)/(A1*T0)
2240    LET FNU = T0*(1 - FNU)
2250    LET FNU = FNU*FNJ(X*A)*FNI(X*R)
2260 FNEND
2270
2280 DEF FNV(A,E,R,T,T0,T1,X)
2290    LET A1 = FNA(A,E,R,T,T0,T1,X)
2300    LET FNV = X↑4*(T1 - A1)/FNS(A,E,R,T,T0,T1,X)
2310    LET FNV = FNV*(1 - EXP(-A1*T))/A1
2320    LET FNV = FNV*(EXP(A1*T0) - 1)
2330    LET FNV = FNV*FNJ(X*A)*FNI(X*R)
2340 FNEND
2350
2360 DEF FNW(A,E,R,T,T0,T1,X)
2370    LET A2 = FNB(A,E,R,T,T0,T1,X)
2380    LET FNW = X↑4*(T1 - A2)/FNS(A,E,R,T,T0,T1,X)
2390    LET FNW = FNW*EXP(-A2*T)/A2
2400    LET FNW = FNW*(EXP(A2*T0) - 1)
2410    LET FNW = FNW*FNJ(A*X)*FNI(X*R)
```

```
2420 FNEND
2430
2440 DEF FNI(X)
2450
2460    '    ***FUNCTION SUBPROGRAM:  JO(X)***
2470
2480    '    POLYNOMIAL APPROXIMATIONS FROM:
2490    '    HANDBOOK OF MATHEMATICAL FUNCTIONS,
2500    '    US DEPARTMENT OF COMMERCE, 1964
2510    '    (9.4.1, 9.4.3)
2520
2530
2540    '    ********** SYMBOL TABLE **********
2550    '          J0              J0(X) ZEROTH-ORDER BESSEL FUNCTION
2560    '          F               F(X) IN NBS BOOK
2570    '          T3              THETA(X) IN NBS BOOK
2580    '          X               ARGUMENT
2590    '          Y               X/3. OR 3./X
2600    '    ***********************************
2610
2620
2630    IF X >= 0 THEN 2680
2640    PRINT "FUNCTION JO(X):  ARGUMENT MUST BE >= 0"
2650    PRINT "X=";X
2660    STOP
2670
2680    IF X>3 THEN 2770
2690
2700    ' POLYNOMIAL APPROX, 0<=X<=3.
2710    LET Y=X/3.
2720    LET J0 = 1 - 2.24999 97*Y↑2 + 1.26562 08*Y↑4
2730    LET J0 = J0 - .31638 66*Y↑6 + .04444 79*Y↑8
2740    LET J0 = J0 - .00394 44*Y↑10 + .00021 00*0
2750    GOTO 2910
2760
2770    ' POLYNOMIAL APPROX, X>3.
2780    LET Y=3./X
2790    '  F = F IN NBS BOOK
2800    LET F = .79788 456 - .00000 077*Y
2810    LET F = F - .00552 740*Y↑2 - .00009 512*Y↑3
2820    LET F = F + .00137 237*Y↑4 - .00072 805*Y↑5
2830    LET F = F + .00014 476*Y↑6
2840    ' T3 = THETA IN NBS BOOK
2850    LET T3 = X - .78539 816 - .04166 397*Y
2860    LET T3 = T3 - .00003 954*Y↑2 + .00262 573*Y↑3
2870    LET T3 = T3 - .00054 125*Y↑4 - .00029 333*Y↑5
2880    LET T3 = T3 + .00013 558*Y↑6
2890    LET J0 = (1./SQR(X))*F*COS(T3)
2900
2910    LET FNI = J0
2920 FNEND
2930
2940 DEF FNJ(X)
2950
2960    '    ***FUNCTION SUBPROGRAM:  J1(X)***
2970
2980    '    POLYNOMIAL APPROXIMATIONS FROM:
2990    '    HANDBOOK OF MATHEMATICAL FUNCTIONS,
3000    '    US DEPARTMENT OF COMMERCE, 1964
3010    '    (9.4.4, 9.4.6)
3020
3030
3040    '    ********** SYMBOL TABLE **********
3050    '    J1          J1(X) FIRST-ORDER BESSEL FUNCTION
3060    '    F1          F1(X) IN NBS BOOK
3070    '    T2          THETA1(X) IN NBS BOOK
3080    '    X           ARGUMENT
3090    '    Y           X/3. OR 3./X
3100    '    ***********************************
3110
3120
3130    IF X >= 0 THEN 3180
3140    PRINT "FUNCTION J1(X):  ARGUMENT MUST BE >= 0"
3150    PRINT "X=",X
3160    STOP
3170
3180    IF X>3 THEN 3280
3190
3200    ' POLYNOMIAL APPROX, 0<=X<=3.
3210    LET Y=X/3.
3220    LET J1 = .5 - .56249 985*Y↑2 + .21093 573*Y↑4
3230    LET J1 = J1 - .03954 289*Y↑6 + .00443 319*Y↑8
3240    LET J1 = J1 - .00031 761*Y↑10 + .00001 109*0
3250    LET J1 = X*J1
3260    GOTO 3420
3270
3280    ' POLYNOMIAL APPROX, X>3.
3290    LET Y=3./X
3300    ' F1 = F1 IN NBS BOOK
3310    LET F1 = .79788 456 + .00000 156*Y
3320    LET F1 = F1 + .01659 667*Y↑2 + .00017 105*Y↑3
3330    LET F1 = F1 - .00249 511*Y↑4 + .00113 653*Y↑5
3340    LET F1 = F1 -  .00020 033*Y↑6
3350    ' T2 = THETA1 IN NBS BOOK
3360    LET T2 = X - 2.35619 449 + .12499 612*Y
3370    LET T2 = T2 + .00005 650*Y↑2 - .00637 879*Y↑3
3380    LET T2 = T2 + .00074 348*Y↑4 + .00079 824*Y↑5
3390    LET T2 = T2 - .00029 166*Y↑6
3400    LET J1 = (1./SQR(X))*F1*COS(T2)
3410
3420    LET FNJ = J1
3430 FNEND
3440 END
3450
3460 SUB "SIMP":X1,X2,FNF,A4,A,E,R,T,T0,T1
3470    ' *** SUBPROGRAM:  SIMPSON'S RULE INTEGRATION ***
3480
3490    ' SIMPSON'S RULE FORMULA FROM:
3500    ' NUMERICAL CALCULUS
3510    ' WILLIAM MILNE, 1949
3520
3530    ' ********** SYMBOL TABLE **********
3540    '            A3             SIMPSON APPROXIMATION FOR PREVIOUS TRIAL
3550    '            A4             SIMPSON APPROXIMATION FOR CURRENT TRIAL
3560    '            FNF            FUNCTION SUBPROGRAM FOR INTEGRAND
3570    '            H              INTERVAL WIDTH
```

```
3580 '             I                 COUNTER
3590 '             N                 2*N = NUMBER OF INTERVALS
3600 '             S1                PARTIAL SUM OF THE ODD TERMS
3610 '             S2                PARTIAL SUM OF THE EVEN TERMS
3620 '             X                 VARIABLE OF INTEGRATION
3630 '             X2                UPPER BOUND OF INTEGRATION
3640 '             X1                LOWER BOUND OF INTEGRATION
3650 ' ************************************
3660
3670 IF X2>X1 THEN 3720
3680 PRINT "SUBPROGRAM SIMP:  XMAXIMUM MUST BE > XMIN"
3690 PRINT "XMIN=";X1,"XMAX=";X2
3700 STOP
3710
3720 ' INITIALIZE OLD APPROXIMATION
3730 LET A3=0
3740 ' INITIALIZE TO 100 INTERVALS
3750 LET N = 50
3760 ' CALCULATE INTERVAL WIDTH
3770 LET H=(X2-X1)/(2*N)
3780 ' INITIALIZE PARTIAL SUMS
3790 LET S1=S2=0
3800 ' CALCULATE PARTIAL SUMS
3810 LET X=X1
3820 FOR I=0 TO N
3830    LET S2=S2+FNF(A,E,R,T,T0,T1,X)
3840    LET X=X+2*H
3850 NEXT I
3860 LET X=X1+H
3870 FOR I=1 TO N
3880    LET S1=S1+FNF(A,E,R,T,T0,T1,X)
3890    LET X=X+2*H
3900 NEXT I
3910 ' CALCULATE NEW APPROXIMATION
3920 LET A4=H*(4*S1+2*S2-FNF(A,E,R,T,T0,T1,X1)-FNF(A,E,R,T,T0,T1,X2))/3
3930
3940 IF ABS(A3-A4)<=1.E-6 THEN 4050
3950
3960 ' PREPARE TO TRY AGAIN
3970
3980 LET A3=A4
3990 LET S2=S1+S2
4000 LET S1=0
4010 LET N=2*N
4020 LET H=(X2-X1)/(2*N)
4030 GOTO 3860
4040
4050 SUBEND
4060
4070 SUB "ERROR":N,N1,N2,I1,FNV,FNW,A,E,R,T0,T1,N7,R8,L9,P,E6,#1
4080 RESET#1
4090 INPUT#1: N3                          'DISCARD FIRST LINE
4100 INPUT#1: N3                          'DISCARD SECOND LINE
4110 INPUT#1: N3                          'NUMBER OF DATA POINTS
4120 INPUT#1: T2,W2                       'TIME, DEFORMATION
4130 LET T2 = T2*60                       'TIME IN SECONDS
4140 LET W2 = W2*.3048                    'DEFORMATION IN METERS
4150 LET T3 = T2/N7                       'NONDIMENSIONALIZE TIME
4160 CALL "SIMP": 0,N1,FNV,I2,A,E,R,T3,T0,T1
4170 CALL "SIMP": 0,N2,FNW,I3,A,E,R,T3,T0,T1
4180 LET W3 = I1 + I2 + I3                'COMPUTE NONDIMENSIONAL DEFORMATION
4190 LET W3 = W3/(3.14159*A*R8*L9↑2/P)    'DIMENSIONALIZE IT
4200 LET E2 = (W3 - W2)                   'COMPUTE ERROR
4210 LET T6 = 0                           'INITIALIZE TIME SUM FOR NORMALIZATION
4220 LET S2 = 0                           'INITIALIZE TRAPEZOIDAL AREA SUM
4230 FOR I = 1 TO N3 - 1
4240    INPUT#1 :T4,W4                    'NEXT TIME, DEFORMATION PAIR
4250    LET T4 = T4*60
4260    LET W4 = W4*.3048
4270    LET T5 = T4/N7
4280    CALL "SIMP": 0,N1,FNV,I2,A,E,R,T5,T0,T1
4290    CALL "SIMP": 0,N2,FNW,I3,A,E,R,T5,T0,T1
4300    LET W5 = I1 + I2 + I3
4310    LET W5 = W5/(3.14159*A*R8*L9↑2/P)
4320    LET E4 = (W5-W4)                  'NEXT ERROR
4330    LET T6 = T4 - T2 + T6             'INCREMENT TIME SUM
4340    IF E2*E4 > 0 THEN 4370            'CURVES DON'T CROSS => TRAP
4350    LET S2 = S2 + .5*(T4 - T2)*(E2↑4 + E4↑4)/(E2↑2 + E4↑2)
4360    GOTO 4380
4370    LET S2 = S2 + .5*(T4 - T2)*(E4↑2 + E2↑2)   'INCREMENT TRAP. AREA SUM
4380    LET E2 = E4                       'NEW BECOMES OLD
4390    LET T2 = T4                       'NEW BECOMES OLD
4400 NEXT I
4410 LET E6 = S2/T6                       'NORMALIZE SUM
4420 LET E6 = SQR(E6)
4430 SUBEND
```