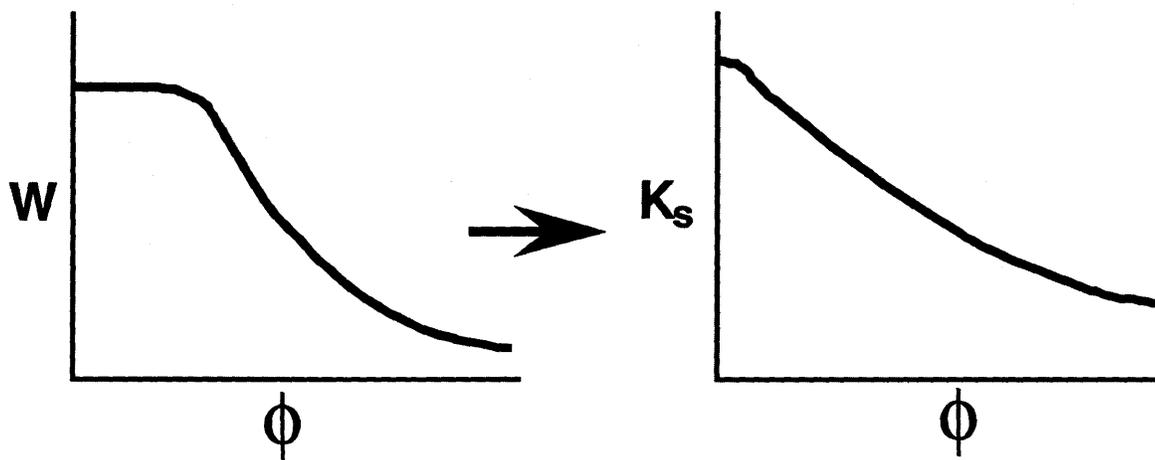




Three Functions That Model Empirically Measured Unfrozen Water Content Data and Predict Relative Hydraulic Conductivity

Patrick B. Black

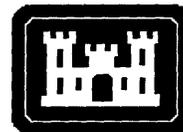
May 1990



For conversion of SI metric units to U.S./British customary units of measurement consult ASTM Standard E380, Metric Practice Guide, published by the American Society for Testing and Materials, 1916 Race St., Philadelphia, Pa. 19103.

Cover: The hydraulic conductivity (K_s) is a function of the unfrozen water content (W).

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**U.S. Army Corps
of Engineers**
Cold Regions Research &
Engineering Laboratory

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PREFACE

This report was prepared by Dr. Patrick B. Black, Research Physical Scientist, of the Applied Research Branch, Experimental Engineering Division, U.S. Army Cold Regions Research and Engineering Laboratory. Funding for this project was provided by DA Project 4A161102AT24, *Research in Snow, Ice and Frozen Ground; Task A, Properties of Cold Regions Materials; Work Unit 002, Properties of Frozen Soils.*

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PATRICK B. BLACK

INTRODUCTION

The most basic of all soil functions used to characterize the physical behavior of frozen soil is the soil freezing curve. It is a graphical representation of empirically measured changes in unfrozen water content occurring with changes of state of ice and water in soil (Fig. 1). The thermal (Farouki 1981), hydraulic (Black and Miller 1985, 1990) and stress (Black and Miller 1985) properties of frozen soil are all dependent upon the amount of unfrozen water. It is important, therefore, that the soil freezing curve be represented by a function that both contains few parameters and allows its use to determine the other physical properties of frozen soil. The thermal conductivity and stress partition factor are simple functions of unfrozen water content, whereas hydraulic conductivity is not.

While there are many possible classes of functions to model unfrozen water content, some functions clearly offer significant advantages over others. Currently, most soil freezing data are fitted to a power curve that appears to be statistically sufficient but lacks physical meaning or does not offer the capability to predict hydraulic conductivity. There are, though, three relationships commonly used for describing the soil-water retention and relative hydraulic conductivity behavior of ice-free soil (Gardner 1958, Brooks and Corey 1964, van Genuchten 1980) that do offer some physical interpretation and the capability to predict one type of data from the other.

The purpose of this paper is to present a framework whereby the three commonly used deterministic relations for ice-free soil-water retention data can be used to describe unfrozen water changes in frozen soil. This will then allow the implementation of Mualem's (1976) model for unfrozen soil to predict the relative hydraulic conductivity for frozen soil.

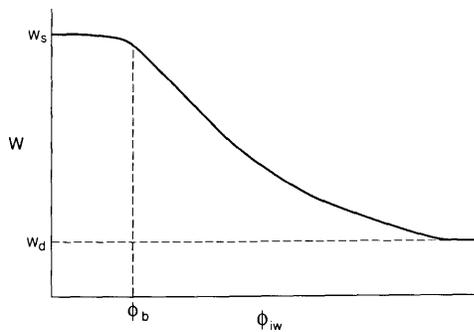


Figure 1. Representative unfrozen water content behavior.

AIR-WATER ICE-WATER SIMILARITY

When discussing the state of water in soil-water systems, it is convenient to introduce the variable ϕ_{bc}

$$\phi_{bc} = u_b - u_c \quad (1)$$

which is simply the pressure difference between water in the two phases *b* and *c* (Black and Miller 1985, 1990; Black and Tice 1989; Black 1989). Thus, the matric pressure for ice-free soil is expressed as

$$\phi_{wa} = u_w - u_a \quad (2)$$

where u_w and u_a refer to, respectively, soil-water and air pressures. Likewise, the state of water in an air-free frozen soil is expressed as

$$\phi_{iw} = u_i - u_w \quad (3)$$

where u_i is the ice pressure. Most often though, the state of water in air-free frozen soil is expressed in terms of temperature, θ . If the soil is devoid of solutes, the Clapeyron equation

$$u_w - \frac{u_i}{\gamma_i} = \frac{h}{273} \theta \quad (4)$$

can be used to relate ice and water pressures to temperature, the specific gravity γ_i of ice and volumetric latent heat of fusion h . Solving eq 3 and 4 for u_i gives the connection between water pressure u_w , ice temperature θ and ϕ_{iw}

$$\phi_{iw} = (\gamma_i - 1) u_w - \frac{\gamma_i h}{273} \theta. \quad (5)$$

The concepts of adsorption space and capillary space are other useful classifications to employ in discussing the physical behavior of soil. Adsorption space is that zone in which soil water is strongly affected by surface forces (real or virtual) emanating from the soil. Capillary space is the remaining region in which water is not affected by soil force fields but is governed by the laws of surface tension. Granular soil will tend to contain mostly capillary water, while highly colloidal soils will be dominated by adsorbed water. The nature and formulation of the adsorption space will not be addressed in this paper but will only be employed in a general classification scheme for soils.

Miller (1965) hypothesized that, under certain constraints, the behavior exhibited by soil upon freezing and thawing should be similar to its behavior upon drying and wetting. He proposed that if the same state of soil-water retention and distribution is achieved by a freezing and thawing process as by a drying and wetting process, then the two states should be similar and interchangeable if and only if the soil is either colloidal (adsorption space >> capillary space) or colloid-free (adsorption space << capillary space).

The laboratory experiments of Koopmans and Miller (1966) prove this hypothesis for colloidal and granular soils. They found that the ice-free soil-water retention data and air-free unfrozen water data for the same soil maintained to the same density and subjected to similar histories were directly related. They found a simple equivalence for the case of colloidal soils (i.e., $\phi_{wa} = \phi_{iw}$), while granular soils required an additional correction factor to account for differences in surface tensions (i.e., $\phi_{wa} = [\sigma_{wa} / \sigma_{iw}] \phi_{iw}$).

Recent work by Black and Tice (1989) showed that the strict similarity requirements imposed in the experiments of Koopmans and Miller can be relaxed and still result in agreement between ice-free soil-water retention and unfrozen water data. They successfully transposed their unfrozen water data onto ice-free soil-water retention data collected by other researchers a decade earlier for the same granular field soil.

With the theoretical and empirical basis for the similarity between ice-free soil-water retention and unfrozen water confirmed, relations that describe ice-free soil-water retention data should also describe unfrozen water data. This will allow the analytical methods for evaluating ice-free soil-water retention and unsaturated hydraulic conductivity characteristics to be used to describe the same properties in frozen soil.

HYDRAULIC MODELS

Moisture characteristics

For ease of use, mathematical models purporting to describe the behavior of soil-water retention data should contain few free parameters and still accurately predict data. A polynomial of large degree could be made to fit the data exactly, but would offer no advantage over using spline fits to the data

because of the large number of parameters required. There are two other functions with fewer parameters that have been observed to closely describe ice-free soil-water retention data (drying and wetting curves being treated as separate relations) as well as offering closed-form analytic expressions for predicting the unsaturated hydraulic conductivity. One function is that proposed by Brooks and Corey (1964). They found that, based upon a large number of observations for ice-free soils, the relative degree of saturation or dimensionless water content

$$S = \frac{W - W_d}{W_s - W_d} \quad (6)$$

could be reasonably described by the relationship

$$S = \begin{cases} \left[\frac{\phi_b}{\phi_{wa}} \right]^{-\alpha} & \phi_{wa} < \phi_b \\ 1 & \phi_{wa} \geq \phi_b \end{cases} \quad (7)$$

where W , W_d and W_s are, respectively, the water contents at a given ϕ_{wa} , at the lower limit of drying and at saturation. ϕ_b is the air entry value and α is a free parameter determined from a "curve fit" to the data. Equation 7 is valid for the range $\phi_{wa} < \phi_b$; otherwise, $S = 1$ (i.e., $W = W_s$) for $\phi_{wa} \geq \phi_b$.

To avoid the discontinuity at ϕ_b , van Genuchten (1980, 1978) proposed a different class of functions

$$S = \begin{cases} \left[\frac{1}{1 + (\lambda \phi_{wa})^n} \right]^m & \phi_{wa} < 0 \\ 1 & \phi_{wa} \geq 0 \end{cases} \quad (8)$$

in which λ , n and m are free parameters also determined from a curve fit to the data.

The free parameter α in eq 7 was found to be related to the pore-size distribution of the soil. Small values of α are found to correspond to soils with a wide span of pore sizes; large α values are appropriate when grain sizes are nearly uniform. The free parameters in eq 8 on the other hand have no physical significance except for large negative ϕ_{wa} values. In that case, eq 8 approaches

$$W = \left[\frac{1}{(\lambda \phi_{wa})^n} \right]^m \quad (9)$$

from which we find that $\alpha = mn$ and $1/\lambda = \phi_b$.

Hydraulic conductivity

Several models exist that purport to predict the relative hydraulic conductivity, K_r , from the behavior of soil-water retention data (Childs and Collis-George 1950, Burdine 1953, Millington and Quirk 1961, Jackson et al. 1965, Mualem 1976). The model proposed by Mualem is most commonly employed today in obtaining closed-form analytical expressions and is used here. In addition to soil-water retention data, this model requires the hydraulic conductivity at saturation K_s , from which the relative hydraulic conductivity is predicted to be

$$K_r(S) = S^B \left[\frac{\int_0^S dx/h(x)}{\int_0^1 dx/h(x)} \right]^2 \quad (10)$$

Table 1. Brooks and Corey, van Genuchten and Gardner equations for unfrozen water content and relative hydraulic conductivity.

BROOKS AND COREY

$$S = \begin{cases} \left[\frac{\phi_{iw}}{\phi_b} \right]^{-\alpha} & \phi_{iw} > \phi_b \\ 1 & \phi_{iw} \leq \phi_b \end{cases}$$

$$K_r(\phi_{iw}) = \begin{cases} K_s \left[\frac{\phi_b}{\phi_{iw}} \right]^{(2+B)\alpha+2} & \phi_{iw} > \phi_b \\ K_s & \phi_{iw} \leq \phi_b \end{cases}$$

VAN GENUCHTEN

$$S = \begin{cases} \left[\frac{1}{1 + (\lambda \phi_{iw})^n} \right]^m & \phi_{iw} > 0 \\ 1 & \phi_{iw} \leq 0 \end{cases}$$

$$K_r(\phi_{iw}) = \begin{cases} K_s \frac{[1 - (\lambda \phi_{iw})^{n-1} [1 + (\lambda \phi_{iw})^n]^{-(1-1/n)}]^2}{[1 + (\lambda \phi_{iw})^n]^{B(1-1/n)}} & \phi_{iw} > 0 \\ K_s & \phi_{iw} \leq 0 \end{cases}$$

GARDNER

$$S = \begin{cases} \left(e^{-0.5\beta\phi_{iw}} [1 + 0.5\beta \phi_{iw}] \right)^{2/(B+2)} & \phi_{iw} > 0 \\ 1 & \phi_{iw} \leq 0 \end{cases}$$

$$K_r(\phi_{iw}) = \begin{cases} K_s e^{-\beta\phi_{iw}} & \phi_{iw} > 0 \\ K_s & \phi_{iw} \leq 0 \end{cases}$$

where B is a new soil parameter that accounts for the effects of tortuosity and multiple interconnections at the pore level on water content. It can be either positive or negative.

Mualem applied eq 6 and 7 to eq 9 and obtained the relative hydraulic conductivity for the Brooks and Corey relationship

$$K_r(\phi_{wa}) = \begin{cases} K_s \left[\frac{\phi_b}{\phi_{wa}} \right]^{(2+B)\alpha + 2} & \phi_{wa} < \phi_b \\ K_s & \phi_{wa} \geq \phi_b \end{cases} \quad (11)$$

In order to apply eq 9 to eq 10, certain restrictions on n and m must be imposed. van Genuchten (1978) found that for the case of $m = 1 - 1/n$, eq 10 obtained the relative hydraulic conductivity

$$K_r(\phi_{wa}) = \begin{cases} K_s \frac{[1 - (\lambda |\phi_{wa}|)^{n-1} [1 + (\lambda |\phi_{wa}|)^n]^{-(1-1/n)}]^2}{[1 + (\lambda |\phi_{wa}|)^n]^{B(1-1/n)}} & \phi_{wa} < 0 \\ K_s & \phi_{wa} \geq 0 \end{cases} \quad (12)$$

The third, and last, function often used to describe relative hydraulic conductivity as a function of ϕ_{wa} is the exponential model proposed by Gardner (1958). He suggested that relative hydraulic conductivity be expressed in terms of a single soil parameter

$$K_r(\phi_{wa}) = K_s e^{-\beta |\phi_{wa}|} \quad (13)$$

In this case, the single parameter β assumes the importance that both α and ϕ_b have in eq 7 and the three soil parameters have in eq 8. Unfortunately, Gardner did not develop an equivalent expression for soil-water retention data.

Russo (1988) found a class of functions that, when applied to eq 10, resulted in eq 13. The expression for soil-water retention data that he suggests, which can be thought of as the Gardner equation for water retention, is

$$S = \begin{cases} \left(e^{-0.5\beta |\phi_{wa}|} [1 + 0.5\beta |\phi_{wa}|] \right)^{2/(B+2)} & \phi_{wa} < 0 \\ 1 & \phi_{wa} \geq 0 \end{cases} \quad (14)$$

All three type of functions are presented in Table 1 in terms of ϕ_{iw} . Since unfrozen water content data are collected in terms of temperature, a value of ϕ_{iw} is obtained by applying eq 5. Most often the soil specimens are exposed to the atmosphere so u_w can be assumed to be zero and ϕ_{iw} is simply 11.1 bars/°C (Black and Tice 1989). The influence of air in frozen soil has been ignored so far. There will definitely be an effect in ice redistribution as the soil undergoes freezing (Miller 1973). However, it appears that, in the case of unfrozen water content measurements, the presence of air merely offsets

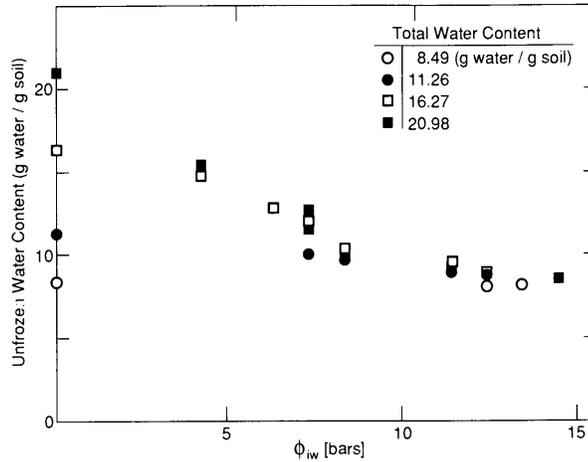


Figure 2. Changes in unfrozen water content in Fox Tunnel Silt for four different total gravimetric water contents attributable to changes in ϕ_{iw} .

the onset of freezing. That is, nonsaturated soils start freezing at a lower temperature than saturated soils, but then follow the unfrozen water content curve of the saturated soil (Fig. 2). This process is being studied in more detail in our laboratory.

DISCUSSION

Russo (1988) presented a statistical study of the relative predictive properties of these equations for ice-free soil. His results, along with those of others (van Genuchten 1980, Milly 1987, Lenhard et al. 1989), suggest that the van Genuchten equations offer superior performance. In the case of ice-free soil, especially field soils, the lack of an apparent pronounced ϕ_b value reduces the predictive ability of the Brooks and Corey equation (eq 7) and favors the van Genuchten equation (eq 8). Since the appropriate form of the Gardner equation for water retention data is relatively new, little can be said for eq 14 except that it has the fewest free-parameters and would therefore be the simplest to use.

In order to apply eq 6, which is used in eq 7, 8 and 14, a value for the lower limit of drying, W_d , is required. Unfortunately, data at very low ϕ_{wa} are seldom collected because of experimental complications. As a result, its value usually must be inferred by extrapolating outside the range of data through numerical or graphical procedures. Black and Tice (1989) have found that unfrozen water content data, transformed by eq 5, offer a source of low ϕ_{wa} data. Using this approach, they found compelling experimental evidence to suspect that $W_d \rightarrow 0$ for $\phi_{wa} \rightarrow -\infty$. This observation has the potential for minimizing the number of parameters in eq 6 by one.

Finally, and most important for the predictive property of these equations, is the uncertainty in the B parameter. While the other soil parameters are obtained from the easily acquired soil-water retention curve, B must initially be found by examining relative hydraulic conductivity data. This presents a problem owing to the experimental complexities of obtaining such data. Mualem determined that a value for B of $1/2$ minimized the error between predicted and measured data for his data set. From the work of Bresler et al. (1978), on correctly scaling the data of Reichardt et al. (1972), an optimal value for B of $-2/15$ is obtained. Russo found that each of the soils he investigated might display a unique value. It is hoped that this parameter will turn out to be more general than the others; otherwise, there is no benefit to using eq 10.

CONCLUSIONS

While this paper is intended to discuss the proper way to express unfrozen water content data, most of the references have been to ice-free soil. This circumstance is caused by the meager effort that has been made to utilize unfrozen water content data to the level of sophistication that is exercised with soil-water retention data. Perhaps this can be singly attributed to the extreme experimental complications that are encountered in measuring the relative hydraulic conductivity of frozen soil and the resulting lack of data.

Black and Miller (1990) were able to make a limited number of simultaneous measurements of unfrozen water content and relative hydraulic conductivity. They found that the presence of a pronounced ice entry pressure allowed the Brooks and Corey type of equation to give results superior to the van Genuchten equation. The Gardner equation was not tested. They also found that the $-2/15$ value of B provided by Bresler et al. (1978) gave satisfactory prediction of the measured relative hydraulic conductivity when only the unfrozen water content data were used to predict.

It is clear that much more experimental work is needed to determine the strong connection (B parameter) between unfrozen water content data and relative hydraulic conductivity. Until this task is accomplished, these two types of data will continue to be treated separately.

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