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Method of Determining Dynamic Properties of Visco-Elastic Solids Employing Forced Vibration

by Tung-Ming Lee

U.S. ARMY MATERIEL COMMAND
COLD REGIONS RESEARCH AND ENGINEERING LABORATORY

Hanover, New Hampshire

PREFACE

This work was performed as part of USA CRREL subtask 5028.21206 for the Applied Research Branch, Dr. A. Assur, Chief.

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This report has been reviewed and approved for publication by the Commander, U. S. Army Materiel Command.

W. L. NUNGESSER
Colonel, CE
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CONTENTS

ILLUSTRATIONS

SUMMARY

The dynamic properties of visco-elastic solids are evaluated by using the forced longitudinal and torsional vibration techniques. A method of eliminating experimental difficulties due mainly to the coupling of sample with supporting system is introduced in using R_{max} (the maximum amplitude ratio of the free end of a sample to the end attached to a driver) and f_{ro} (the corresponding vibration frequency) as a criterion. Experimental measurements of these values are sufficient to determine the dynamic properties of samples. The complex modulus is used to describe the stress-strain relationship for a visco-elastic solid. Simple expressions relating dynamic properties to R_{max} and f_{ro} are obtained. In the method presented, matching of natural frequencies of the sample and the driver is not necessary and the same driving unit may be used throughout the experiment. The expressions derived for longitudinal and torsional vibrations bear direct relationship between the measured items and the dynamic properties and are simple to use.

METHOD OF DETERMINING DYNAMIC PROPERTIES OF VISCO-ELASTIC SOLIDS EMPLOYING FORCED VIBRATION

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Tung-Ming Lee

INTRODUCTION

The forced-vibration technique of determining the dynamic properties of solids has been pursued diligently since Quimby (1925) first developed the idea. In general, the technique involves cementing a transducer to a specimen and subsequently exciting the composite rod to the resonant frequency. Using the resonance criterion, the dynamic moduli of the sample can be determined from the measured resonant frequency and the physical dimensions of the composite rod*. Experimental errors (Terry and Woods,1955) recognized in this method are introduced by cementing the transducer to the specimen, the difference in cross-sectional area of the driving oscillator and the specimen, and the presence of lateral motion during the test. The main difficulty, however, arises from coupling the specimen to the driver; consequently, the accuracy of the experiment is dependent upon that of the measurements pertaining to the physical properties of the vibrating unit, especially when a complex system of more than one simple transducer is used.

To avoid the difficulty due to coupling, a criterion using the maximum amplitude ratio and the corresponding vibration frequency is introduced. And, as is shown, these measurements are sufficient to determine the dynamic properties of a solid.

Both the longitudinal-vibration method and the torsional-vibration method are considered in the present work. Two basic models representing the constant driving-amplitude system and the constant driving-force system are used in the longitudinal-vibration study. A circular cylinder is assumed for the torsional-vibration analysis. It is demonstrated that the same criterion can be applied in all the cases.

LONGITUDINAL VIBRATIONS

General consideration

The equation of motion from a longitudinal disturbance along a thin filament is well known:

$$\frac{\partial \sigma}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2} \tag{1.1}$$

where σ and u are the stress and displacement, respectively, at distance z along the filament from the origin, and ρ is the density of the medium. While there is no simple relationship available to express stress in terms of strain for a real visco-elastic solid (Kolsky, 1960), the fact that the strain ε in a linear visco-elastic solid varies sinusoidally as the stress σ but lags behind it by a loss angle δ can be used to express the relationship conveniently by the complex modulus:

$$\sigma = (E' + iE'')\ \varepsilon = (E' + iE'')\ \frac{\partial u}{\partial z} \tag{1.2}$$

where E' and E'' are the real and imaginary parts of the complex modulus "E". The ratio E''/E' is a measure of the energy dissipation and is generally denoted by the loss factor $\tan\ \delta$.

Hence, the longitudinal-wave-propagation equation for a linear visco-elastic solid subjected to a sinusoidal disturbance is obtained from substitution of eq 1.2 in eq 1.1

*See: Balamuth (1934), Rose (1936), Terry (1957), Marx and Sivertsen (1953).

$$(E' + iE'') \frac{\partial^2 u}{\partial z^2} = \rho \frac{\partial^2 u}{\partial t^2}. \tag{1.3}$$

In applying the forced-vibration technique for investigating the dynamic properties of a visco-elastic solid, eq 1.3 may be used to describe the wave propagation in the system if it undergoes harmonic motion. Consequently, the displacements in this type of problem may be expressed, from eq 1.3, in the following convenient form:

$$u = [A \cos \gamma z + B \sin \gamma z] \cos \omega t \tag{1.4}$$

where $\underline{A}$ and $\underline{B}$ are the constants to be determined from boundary conditions, and

$$\gamma = \left(1 - i \tan \frac{\delta}{2}\right) \frac{\omega}{c} \tag{1.5a}$$

c = the longitudinal phase velocity

$$= \left(\frac{E^*}{\rho}\right)^{\frac{1}{2}} \sec \frac{\delta}{2} \tag{1.5b}$$

$$E^* = (E'^2 + E''^2)^{\frac{1}{2}} \tag{1.5c}$$

and ω = the angular frequency of the oscillation.

Vibration systems

To facilitate the analysis, we group the various vibration systems for this testing purpose into the following two basic types:

1) The constant driving-amplitude system — the vibrating amplitude U_o of the driving unit can be set to the desired magnitude and kept at that value for various frequencies throughout the test.

2) The constant driving-force system — the amplitude of the driving stress σ_0 in the driving unit induced by the exciting field is maintained constant for all vibrating frequencies.

For the sake of simplicity, we represent the above-mentioned systems by the models given in Figures 1 and 2. In Figure 1, the driving unit is shown as a supporting floor which is oscillating at a fixed constant amplitude. Above the floor is the auxiliary unit "1" (or dummy) connecting the sample "2" to the driver. In Figure 2, the lower portion "1" represents the driving unit, with the exciting field not shown; the sample "2", in this case, is directly connected to the driver.

Constant driving amplitude

Let the motion of the floor, Figure 1, at time $\underline{t}$ be

$$U_o \cos \omega t \tag{2.1}$$

where U_o is a fixed constant and ω is the angular frequency of the oscillation. Using eq 1.4, we write the displacements in units "1" and "2":

$$u_1 = [A_1 \cos \gamma_1 z + B_1 \sin \gamma_1 z] \cos \omega t \tag{2.2}$$

$$u_2 = [A_2 \cos \gamma_2 z + B_2 \sin \gamma_2 z] \cos \omega t \tag{2.3}$$

where γ_1 and γ_2 take the same identity as the expressions in eq 1.5, with proper subscripts. If unit "1" is firmly attached to the floor and the top of unit "2" is free of stresses, then the boundary conditions and the continuity conditions at the interfaces are

$$(u_1)_{z = -\ell_1} = U_o \cos \omega t \tag{2.4a}$$

$$(u_1)_{z=0} = (u_2)_{z=0} \tag{2.4b}$$

$$\left[(E_1' + iE_1'') \frac{\partial u_1}{\partial z}\right]_{z=0} = \left[(E_2' + iE_2'') \frac{\partial u_2}{\partial z}\right]_{z=0} \qquad (2.4c),$$

$$\left(\frac{\partial u_2}{\partial z}\right)_{z=\ell_2} = 0. \qquad (2.4d)$$

Using these conditions, we obtain

$$u_1 = \frac{U_o}{D_0}\left[(E_1' + iE_1'')\gamma_1 \cos\gamma_1 z + (E_2' + iE_2'')\gamma_2 \tan\gamma_2\ell_2 \sin\gamma_1 z\right] \cos\omega t \qquad (2.5)$$

and

$$u_2 = \frac{U_o}{D_0}(E_1' + iE_1'')\gamma_1\left[\cos\gamma_2 z + \tan\gamma_2\ell_2 \sin\gamma_2 z\right] \cos\omega t \qquad (2.6)$$

where

$$D_0 = \left[(E_1' + iE_1'')\gamma_1 \cos\gamma_1\ell_1 - (E_2' + iE_2'')\gamma_2 \tan\gamma_2\ell_2 \sin\gamma_1\ell_1\right]. \qquad (2.7)$$

Equations 2.5 and 2.6 indicate that, for a given value of z in either u_1 or u_2, the magnitude of the displacement relies on both the values of "E_1" and "E_2" of the system. If, however, we consider the ratio of the absolute values of

$$u_2(\ell_2) = \frac{U_o}{D_0}(E_1' + iE_1'')\gamma_1 \sec\gamma_2\ell_2 \cos\omega t \qquad (2.8a)$$

and

$$u_2(o) = \frac{U_o}{D_0}(E_1' + iE_1'')\gamma_1 \cos\omega t \qquad (2.8b)$$

we find

$$R = \left|\frac{u_2(\ell_2)}{u_2(o)}\right| = \frac{1}{\left[\sin h^2\left(\frac{\omega\ell_2}{c_2}\tan\frac{\delta_2}{2}\right) + \cos^2\left(\frac{\omega\ell_2}{c_2}\right)\right]^{\frac{1}{2}}} \qquad (2.9)$$

containing only the properties in unit "2". Thus, it suggests that if we choose to use R_{max} instead of $[u_2(\ell_2)]_{max}$, the conventional way, as the criterion for determining c_2, then the difficulty due to coupling the sample with the supporting system will not enter the problem.

If $f_{ro} = \frac{\omega_{ro}}{2\pi}$ is the frequency of the fundamental mode when R reaches R_{max}, we find, from eq 2.9, that

$$c_2 = 4 f_{ro}\ell_2 \left(1 + \tan^2\frac{\delta_2}{2}\right) \qquad (2.10)$$

and

$$\tan\frac{\delta_2}{2} = \frac{2}{\pi R_{max}}. \qquad (2.11)$$

Using the relationship given in eq 1.5b for c and E^* and substituting it in eq 2.10 yields

$$E_2^* = 16 f_{ro}^2 \ell_2^2 \rho_2 (1 + \tan^2 \frac{\delta_2}{2}). \tag{2.12}$$

Hence, the dynamic properties of a visco-elastic material can be determined from R_{max} and f_{ro} by use of eqs 2.11 and 2.12.

It is obvious that, for elastic materials, eq 2.12 reduces to

$$E_2 = 16 f_{ro}^2 \ell_2^2 \rho_2. \tag{2.13}$$

It should be mentioned at this point that f_{ro} in most cases will not be the same as the resonance frequency, f_o, since the former is the indication for $R = R_{max}$ while the latter is obtained from $[u_2(\ell_2)]_{max}$ which, as mentioned before, depends on the coupling nature of the sample and the driver. When $\ell_1 = 0$ in Figure 1, we obtain the case of a specimen attached directly to the floor. Then, from eq 2.8a and eq 2.9, we have

$$\left| u_2(\ell_2) \right| = R U_0 \tag{2.14}$$

giving $f_{ro} = f_o$ in this special case.

Using eq 2.9, a chart showing R vs $\frac{\omega \ell_2}{c_2}$ for various values of tan $\frac{\delta_2}{2}$ is presented in Figure 3. It may be of interest to note that the presence of the factor tan $\frac{\delta_2}{2}$ damped the amplitude and also shifted the peak-point of R.

Constant driving force

Let the stress introduced in unit "1" of Figure 2 by the exciting field be

$$\sigma_0 \cos \omega t \tag{3.1}$$

where σ_0 is a constant. The boundary conditions, in this case, become

$$\left[(E_1' + iE_1'')\frac{\partial u_1}{\partial z}\right]_{z = -\ell_1} + \sigma_0 \cos \omega t = 0 \tag{3.2a}$$

$$\left[(E_1' + iE_1'')\frac{\partial u_1}{\partial z}\right]_{z = 0} + \sigma_0 \cos \omega t = \left[(E_2' + iE_2'')\frac{\partial u_2}{\partial z}\right]_{z = 0} \tag{3.2b}$$

$$(u_1)_{z=0} = (u_2)_{z=0} \tag{3.2c}$$

$$\left(\frac{\partial u_2}{\partial z}\right)_{z = \ell_2} = 0. \tag{3.2d}$$

Using the same expressions given by equations 2.2 and 2.3 for u_1 and u_2 with the above boundary conditions, we find

$$u_1 = -\frac{\sigma_0}{D_0'}\left\{(1 - \cos \gamma_1 \ell_1) \cos \gamma_1 z + \left[\sin \gamma_1 \ell_1 + \frac{(E_2' + iE_2'')\gamma_2}{(E_1' + iE_1'')\gamma_1} \tan \gamma_2 \ell_2\right] \sin \gamma_1 z\right\} \cos \omega t \tag{3.3}$$

and

$$u_2 = -\frac{\sigma_0}{D_0'}(1 - \cos \gamma_1 \ell_1)[\cos \gamma_2 z + \tan \gamma_2 \ell_2 \sin \gamma_2 z] \cos \omega t \tag{3.4}$$

where

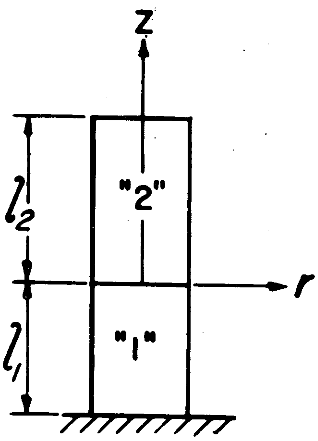

Figure 1. Constant driving-amplitude system.

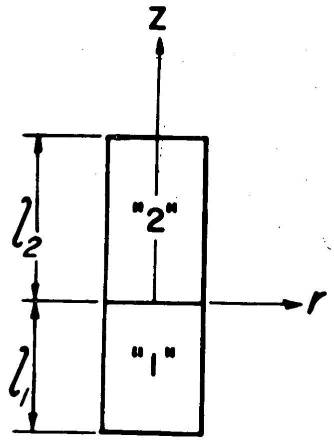

Figure 2. Constant driving-force system.

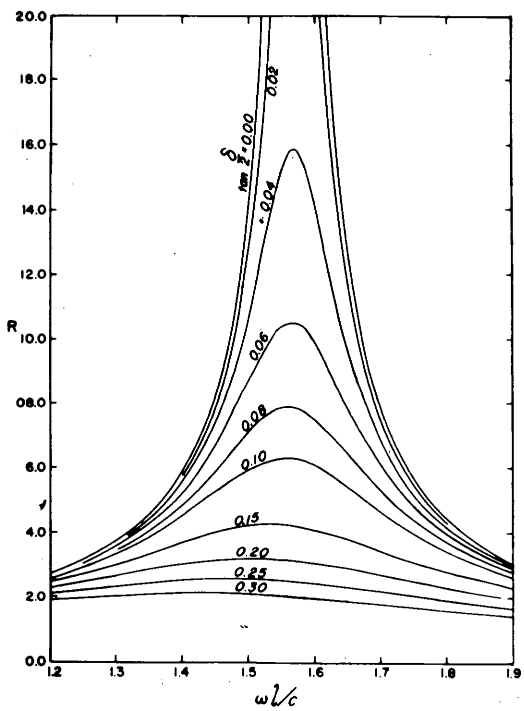

Figure 3. Damping effect of tan $\delta/2$ on amplitude ratio, R, and frequency ratio, $\omega\ell/c$.

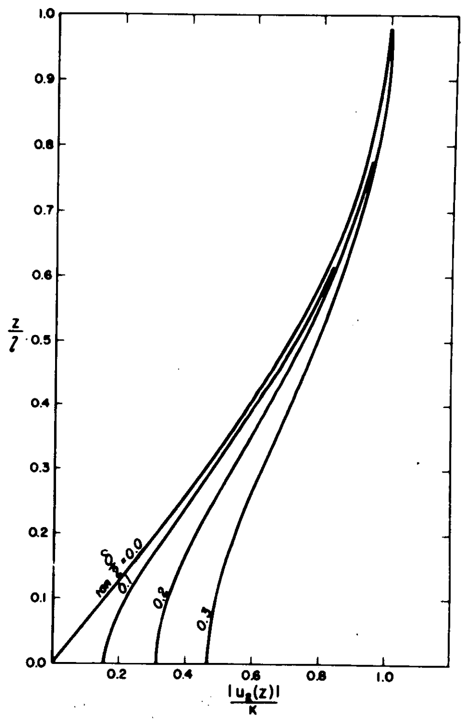

Figure 4. Standing wave in a sample when $f = f_{ro}$.

$$D_0' = (E_1' + iE_1'')\gamma_1 \sin\gamma_1 \ell_1 + (E_2' + iE_2'')\gamma_2 \tan\gamma_2 \ell_2 \cos\gamma_1 \ell_1 . \quad (3.5)$$

Note that these two expressions are very similar to those of eq 2.5 and 2.6. Therefore, it is expected that the same criterion can also be used here. In fact, the ratio of $|u_2(\ell_2)|$ over $|u_2(o)|$ from eq 3.4 is identical to that of eq 2.9. Thus, we may conclude that eq 2.11 and 2.12 are also applicable to this model in determining E_2^* and $\tan\frac{\delta_2}{2}$. And, as a matter of fact, these two equations can be applied to any other models if one end of the testing sample is free and the other end is firmly attached to a driving unit.

Rewriting the terms inside the square parenthesis of eq 3.4

$$[\cos\gamma_2 z + \tan\gamma_2\ell_2 \sin\gamma_2 z] = \frac{\cos\gamma_2(\ell_2 - z)}{\cos\gamma_2\ell_2} \quad (3.6)$$

and noting that $\cos\gamma_1\ell_1$, $\cos\gamma_2\ell_2$ and D_0' are all constants for a given system, we find

$$|u_2(z)| = K(\cos^2\alpha + \sinh^2\alpha')^{\frac{1}{2}} \quad (3.7)$$

where

$$\alpha = \frac{\omega}{c_2}(\ell_2 - z), \quad \alpha' = \alpha\tan\frac{\delta_2}{2} \quad (3.8a, b)$$

and K is a constant depending on σ_0 and the properties in the system. Equation 3.7 indicates that the configuration of the standing wave in unit "2" is the combination of a sinusoidal function and a hyperbolic function. To illustrate this phenomenon we plot (Figure 4) $|u_2(z)|/K$ for the cases of $\tan\frac{\delta_2}{2} = 0, 0.1, 0.2$ and 0.3 when $f = f_{ro}$. It is seen that for elastic materials the configuration is a quarter wave with the nodal point at $z = 0$, while those of the visco-elastic materials exhibit the minimum but non-zero values at this point.

TORSIONAL VIBRATIONS

When considering torsional vibrations, we shall only deal with samples of circular cylinders and use cylindrical coordinates for convenience. Let the coordinates be r, θ and z, with z as the axis of the cylinder, and the corresponding displacements be u_r, u_θ, and u_z. In the propagation of torsional waves, no longitudinal or lateral displacements are to be expected and the motion is symmetrical about the axis of the cylinder. Therefore, u_r and u_z must both vanish and we need to consider only the wave equation for u_θ which, in the elastic case, may be written as:

$$\rho\frac{\partial^2 u_\theta}{\partial t^2} = \mu\left(\frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial r^2}\right) \quad (4.1)$$

where μ is Lame's constant

= G the elastic shear modulus.

If the torsional stress applied to the specimen varies sinusoidally with time and the strain thus induced also varies sinusoidally but with a phase difference, then we may rewrite eq 4.1 for visco-elastic materials as:

$$\rho\frac{\partial^2 u_\theta}{\partial t^2} = (G' + iG'')\left(\frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial r^2}\right) \quad (4.2)$$

where G' and G'' are the real and imaginary parts of the complex shear modulus "G".

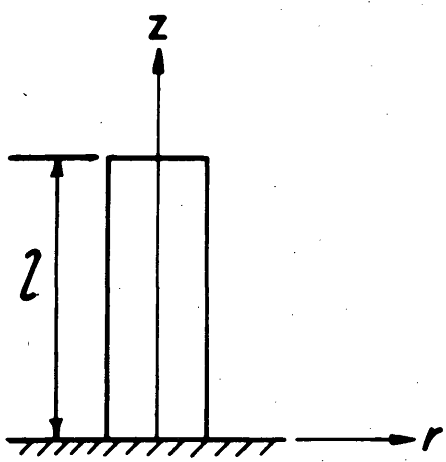

Figure 5. Circular cylinder for torsional vibration.

Note that u_θ is a function of r, z and t. If we assume

$$u_\theta = r\ F\ (z, t), \tag{4.3}$$

then it leads to

$\tau_{r\theta}$, the shearing stress in θ-direction on r·plane, =

$$= (G' + iG'')\ r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) = 0,$$

which together with $\sigma_r = 0$ and $\tau_{rz} = 0$, from $u_r = 0$ and $u_z = 0$, will satisfy the boundary conditions of free stresses along the surface of the cylinder. Substituting eq 4.3 in eq 4.2, we obtain

$$\rho \frac{\partial^2 F}{\partial t^2} = (G' + iG'' \frac{\partial^2 F}{\partial z^2} \tag{4.4}$$

with F being a function of z and t. Since eq 4.4 is identical with the longitudinal-wave equation previously obtained, it suggests that a similar procedure can be followed.

Let us now use eq 4.4 to consider the torsional-vibration problem (Figure 5) in which a cylindrical sample is attached to a floor that is oscillating according to

$$\theta_0 \cos \omega t \tag{4.5}$$

where θ_0 is the amplitude of the angular rotation and is constant for various vibrating frequencies.

The function F satisfying eq 4.4, in view of its similarity to eq 1.3, may be assumed of the form

$$F(z, t) = [A \cos \beta z + B \sin \beta z] \cos \omega t \tag{4.6}$$

where

$$\beta = \left(1 - i \tan \frac{\delta'}{2}\right) \frac{\omega}{c_s} \tag{4.7a}$$

c_s = shear wave velocity

$$= \left(\frac{G^*}{\rho}\right)^{\frac{1}{2}} \sec \frac{\delta'}{2} \tag{4.7b}$$

$$G^* = \left(G'^2 + G''^2\right)^{\frac{1}{2}}, \quad \tan \delta' = \frac{G''}{G'}, \tag{4.7c, d}$$

and A and B are constants to be determined from boundary conditions.

The boundary conditions at both ends of the cylinder are

$$u_\theta (r, o, t) = r\,\theta_0 \cos \omega t \tag{4.8a}$$

and

$$\left(\frac{\partial u_\theta}{\partial z}\right)_{z = \ell} = 0. \tag{4.8b}$$

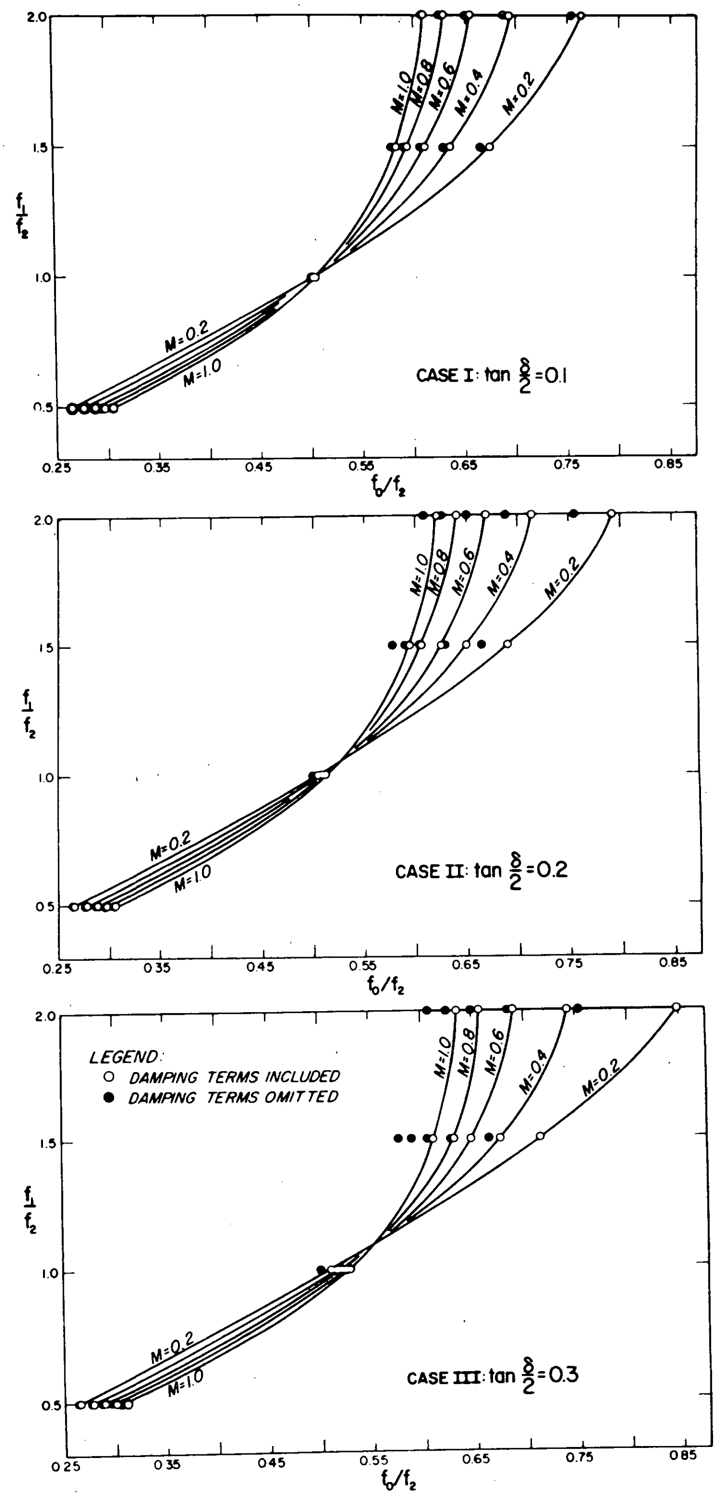

Figure 6. f_1/f_2 vs f_0/f_2 for a given system.

Using eq 4.3, we transform eq 4.8a and 4.8b to

$$F(o, t) = \theta_0 \cos \omega t \tag{4.9a}$$

and

$$\left(\frac{\partial F}{\partial z}\right)_{z = \ell} = 0, \tag{4.9b}$$

which are exactly the same conditions as those in the first longitudinal-vibration problem when $\ell_1 = 0$. Thus, from analogy, if

$$R' = \left|\frac{u_\theta(r, \ell, t)}{u_\theta(r, o, t)}\right| \tag{4.10}$$

and f_{so} is the frequency when R' reaches R'_{max} for the fundamental mode, then

$$G^* = 16 f_{so}^2 \ \ell^2 \rho\left(1 + \tan^2 \frac{\delta'}{2}\right) \tag{4.11}$$

and

$$\tan\frac{\delta'}{2} = \frac{2}{\pi R'_{max}} \tag{4.12}$$

giving the components of a complex shear modulus "G".

It should be mentioned, from the results obtained in the longitudinal-vibration problems, that equations 4.11 and 4.12 can also be applied to more complicated cases, where the amplitude of rotation at the base of the sample varies with the vibrating frequency.

DISCUSSION

Expressing $C_1 = 2f_1 \ell_1^*$, $C_2 = 2 f_2 \ell_2 \left(1 + \tan^2 \frac{\delta_2}{2}\right)$ and $x = \pi \frac{f}{f_2}$, we write the vibration amplitude at the top of the sample from eq 3.4 in a suitable form for computation

$$\frac{|u_2(\ell_2)|}{\sigma_0 \ell_1 / \rho_1 C_1^2} = \frac{(1 - \cos Nx)}{Nx(S^2 + W^2)^{\frac{1}{2}}} \tag{5.1}$$

where

$$S = \sin Nx \sin Lx \sinh L'x - \frac{N \cos Nx}{M}\left(\cos Lx \sinh L'x + \tan\frac{\delta_2}{2} \sin Lx \cosh L'x\right), \tag{5.2a}$$

$$W = \sin Nx \cos Lx \cosh L'x + \frac{N \cos Nx}{M}\left(\sin Lx \cosh L'x - \tan\frac{\delta_2}{2} \cos Lx \sinh L'x\right), \tag{5.2b}$$

$$L = \frac{1}{1 + \tan^2 \frac{\delta_2}{2}}, \qquad L' = L \tan\frac{\delta_2}{2} \tag{5.2c, d}$$

$$N = f_2 / f_1, \qquad M = \rho_1 \ell_1 / \rho_2 \ell_2. \tag{5.2e, f}$$

f_i is the natural frequency of a cylinder vibrating alone in its fundamental longitudinal mode with i = 1 and 2 for the driver and the specimen, respectively. For a given system, the value of $|u_2(\ell_2)|$ in eq 5.1 varies only with $\underline{x}$. Thus, when $|u_2(\ell_2)|$ reaches maximum, we obtain the resonance-frequency ratio $x_o = \pi \frac{f_o}{f_2}$. Arbitrarily assigning the values of $\underline{N}$ and $\underline{M}$, we then obtain the plot of $\frac{f_1}{f_2}$ against $\frac{f_o}{f_2}$ (Figure 6) which may be used to compare

* The driving unit is assumed as elastic material to simplify the calculation.

with the results from the conventional method based upon the equation

$$M \tan N x_o + N \tan x_o = 0 \qquad (5.3)$$

given by Quimby (1925) and used by many others*.

It is seen from Figure 6 that results from the conventional method, eq 5.3, shown in solid dots, are not accurate in calculating f_2 from f_0 for materials of large internal friction. The accuracy is also affected by the natural frequency ratio of the driving unit to the sample. Therefore, if one should choose to use the conventional method, the testing system should be designed with $\frac{f_1}{f_2}$ close to unity† to limit the effect from the variations of M and $\tan\frac{\delta}{2}$. However, if the method presented here is used, the matching of the natural frequencies will not be necessary, allowing us to use the same driving unit throughout the entire work. Furthermore, eq 2.11 and 2.12 for longitudinal vibrations and eq 4.11 and 4.12 for torsional vibrations bear direct relationship between the measured items and the dynamic properties and are simple to use.

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* Balamuth (1934), Rose (1936), Terry (1957).

† It agrees with the finding from Terry (1957) that for accurate measurements the transducer has to have a natural frequency approximately equal to or higher than that of the specimen.

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Unclassified Report

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