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Laminar Heat Transfer Over a Melting Plate - the Modified Leveque Problem

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PREFACE

This report was prepared by Dr. Yen, Research Engineer of Materials Research Branch, and Dr. Tien, Expert of Research Division.

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CONTENTS

	Page
Preface -----	ii
Summary -----	iv
Introduction -----	1
Formulation and solution of the problem -----	1
Discussion -----	6
References -----	10
Appendix A: Convergence of the sequence (a_n) -----	A1
Appendix B: Numerical evaluation of the integral $\int_0^{\infty} \exp(-y^3+wy)dy$ -----	B1
Appendix C: Evaluation of the limiting value a_N of the sequence (a_n) -----	C1
Appendix D: Typical calculation on the melting rate -----	D1

ILLUSTRATIONS

Figure	
1. Schematic representation of the melting problem -----	2
2. Relationship between a_n and w -----	5
3. Relationship between dimensionless temperature distribution and X as a function of β -----	7
4. Relationship between ψ , θ , and β -----	8
5. Relationship between melting-velocity ratio and ratio of temperature difference -----	10

TABLES

Table	
I. Numerical values of a_N as a function of β -----	6
II. Calculation of melting-velocity ratio as function of temperature-difference ratio -----	9
BI. Numerical approximation of the integral $\int_0^{\infty} \exp(-y^3+wy) dy$ -----	
as a function of w -----	B4
CI. Evaluation of the limiting value a_N as function of β -----	C2

SUMMARY

The temperature distribution in a fluid flowing over a melting plate and having a linear tangential velocity profile is determined theoretically. Melting at the interface is shown to result in a decrease in the Nusselt number. Consideration is given to the practical application of the results. A simple numerical calculation is given in Appendix D.

LAMINAR HEAT TRANSFER OVER A MELTING PLATE — THE MODIFIED LEVEQUE PROBLEM

by

Yin-Chao Yen and Chi Tien

INTRODUCTION

The study of heat transfer where melting and freezing are involved is of considerable interest from a practical as well as a theoretical viewpoint. However, the majority of the literature on this subject has been restricted to the conduction aspect of the problem. Also, the boundary conditions investigated are usually those of a prescribed temperature or heat flux at the interface, and little indication is given of the conditions required to sustain such a temperature or heat flux.

To study the aforementioned problem, the classical treatment of convective heat transfer between a solid boundary and a fluid in relative motion can only be used as an approximation. This is due to the fact that the melting (or freezing) which takes place at the boundary creates a finite interfacial velocity, and it is well known from the study of boundary layer control (Schlichting, 1955) and mass transfer (Bird, et al., 1960) that the presence of such a velocity can greatly affect the transfer rate.

This paper investigates the heat transfer in a system consisting of a fluid flowing over a melting plate composed of the same material as the fluid. This model approximates the heat transfer phenomena occurring when an iceberg drifts in warm sea water. The results of this study, applicable to a two-dimensional, two-phase system with an interfacial velocity, show that the melting rate can be expressed in terms of a product consisting of the usual flow parameters and an additional factor that is dependent upon thermal conditions.

Formulation and solution of the problem. The following model was used to describe the physical problem. Flow of a liquid, initially at a uniform temperature T_∞ ($T_\infty > T_m$ the melting point of the liquid), over a flat plate composed of the solid phase of the flowing liquid at constant temperature T_m was assumed. For simplification of subsequent analysis the flow was stipulated as the Leveque type (Leveque, 1928), $v_x = cy$; consequently, the problem was considered a modified Leveque problem. Once the above assumptions were made, the problem was to obtain an expression for the rate of melting resulting from the transfer of heat from the liquid to the surface of the plate. With the x -axis taken along the surface of the plate and the y -axis perpendicular to it as shown in Figure 1, the problem was formulated as:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (1)$$

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (2)$$

$$v_x = cy \quad (3)$$

$$v_y = \frac{k}{L_p} \cdot \frac{\partial T}{\partial y} \Big|_0 \quad y = 0 \quad (4)$$

$$x = 0 \quad y > 0 \quad T = T_\infty \quad (5)$$

$$x > 0 \quad y = 0 \quad T = T_m \quad (6)$$

Figure 1. Schematic representation of the melting problem.

where

- x = coordinate along plate
- y = coordinate normal to plate
- v_x = component of velocity in x -direction
- v_y = component of velocity in y -direction
- T = temperature of fluid
- α = thermal diffusivity
- c = constant in velocity distribution
- k = thermal conductivity
- L = latent heat of fusion
- ρ = density of fluid

In addition to the usual assumptions such as two-dimensional, steady-flow, constant-fluid properties, insignificant heat loss due to friction, etc., an added assumption was introduced; i. e., the energy equation for a fixed-coordinate system is still valid although there is some movement of the interface due to melting.

From eq 1, it follows

$$v_y = v_y|_0 - \int_0^y \frac{\partial v_x}{\partial x} dy = v_y|_0. \quad (7)$$

Eq 7 is valid because v_x is only a function of y . Combining eq 2, 3, and 7, the following holds:

$$cy \frac{\partial T}{\partial x} + v_y|_0 \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}. \quad (8)$$

The solution of eq 8, subject to the conditions given by eq 4, 5, and 6, was achieved by an iterative process. The 0-th approximation was obtained by setting $v_y|_0 = 0$. This approximation is the classical Leveque problem, and its solution is given as

$$\frac{T - T_m}{T_\infty - T_m} = \frac{\int_0^X \exp(-y^3) dy}{\int_0^\infty \exp(-y^3) dy} = \frac{\int_0^X \exp(-y^3) dy}{a_0} \quad (9)$$

where

$$a_0 = \int_0^{\infty} \exp(-y^3) dy.$$

The temperature gradient at the wall is given as

$$\left. \frac{\partial T^{(0)}}{\partial y} \right|_{y=0} = \frac{T_{\infty} - T_m}{a_0} \left(\frac{c}{9a x} \right)^{\frac{1}{3}} \quad (10)$$

With this information a first order approximation of $v_y|_0$ can be written as

$$v_y|_0 = \frac{k(T_{\infty} - T_m)}{\rho L a_0} \left(\frac{c}{9a x} \right)^{\frac{1}{3}} = \frac{\phi}{a_0} \left(\frac{c}{9a x} \right)^{\frac{1}{3}} \quad (11)$$

where

$$\phi = k \Delta T / \rho L \quad (12)$$

and

$$\Delta T = T_{\infty} - T_m.$$

Substituting eq 1 into eq 8, the energy equation becomes

$$c y \frac{\partial T}{\partial x} + \frac{\phi}{a_0} \left(\frac{c}{9a x} \right)^{\frac{1}{3}} \frac{\partial T}{\partial y} = a \frac{\partial^2 T}{\partial y^2} \quad (13)$$

Let

$$X = \left(\frac{c}{9a x} \right)^{\frac{1}{3}} y. \quad (14)$$

Eq 13 is then transformed into

$$\frac{d^2 T^{(1)}}{dX^2} + \left(3X^2 - \frac{\phi}{a_0 a} \right) \frac{dT^{(1)}}{dX} = 0 \quad (15)$$

with boundary conditions

$$X = 0 \quad T = T_m \quad (16)$$

$$X = \infty \quad T = T_{\infty}. \quad (17)$$

The solution to eq 15 is expressed as

$$\frac{T - T_m}{T_{\infty} - T_m} = \frac{\int_0^X \exp(-y^3 + \frac{\beta}{a_0} y) dy}{a_1} \quad (18)$$

where

$$\beta = \frac{\phi}{a} = \frac{c_p \Delta T}{L \rho} \quad (18^1)$$

c_p = specific heat at constant pressure, and

$$a_1 = \int_0^{\infty} \exp(-y^3 + \frac{\beta}{a_0} y) dy. \quad (19)$$

By the same process the solution is found to be

$$\frac{T - T_m}{T_\infty - T_m} = \frac{\int_0^X \exp(-y^3 + \frac{\beta}{a_N} y) dy}{a_N} \quad (20)$$

where a_N is the limiting value of the sequence (a_n) as $n \rightarrow \infty$. This sequence is given as

$$a_0 = \int_0^\infty \exp(-y^3) dy \quad (21)$$

$$a_1 = \int_0^\infty \exp(-y^3 + \frac{\beta}{a_0} y) dy \quad (22)$$

$$a_2 = \int_0^\infty \exp(-y^3 + \frac{\beta}{a_1} y) dy \quad (23)$$

$$\vdots$$

$$a_n = \int_0^X \exp(-y^3 + \frac{\beta}{a_{n-1}} y) dy.$$

This sequence converges since one can show very easily that the magnitude of a_n diminishes in an oscillatory manner as n increases. Appendix A indicates the proof of the convergence of the sequence. It is clear that a_N is a unique value and dependent only on β . To evaluate the limiting value of a_N , numerical evaluation of the integral

$$a_n = \int_0^\infty \exp(-y^3 + wy) dy \quad (w = \beta/a_{n-1})$$

becomes a necessity. Detailed determination of this integral as a function of w is shown in Appendix B. Figure 2 shows the value of a_n as function of w . Table CI indicates the iterative process to find the limiting value of a_N . Numerical values of a_N to four-digit accuracy are given in Table I. Dimensionless temperature based on the a_N values are calculated from eq 20 for β ranging from 0 to 1. The effect of melting on the temperature distribution is shown in Figure 3.

The melting rate is given as

$$v_y|_0 = \frac{\phi}{a_N} \left(\frac{c}{9\alpha x} \right)^{\frac{1}{3}} \quad (24)$$

and the Nusselt number is

$$Nu = \frac{h\ell}{k} = \frac{k}{\Delta T} \left(\frac{\partial T}{\partial y} \right)_{y=0} \frac{\ell}{k} = \frac{\ell}{a_N} \left(\frac{c}{9\alpha x} \right)^{\frac{1}{3}} \quad (25)$$

where

h = heat transfer coefficient

ℓ = characteristic length of the system.

The ratio of the Nusselt number, given by eq 25, to the number where there is no interfacial velocity is found to be

$$\psi(\beta) = \frac{(Nu)_{\text{with surface flux}}}{(Nu)_{\text{with no surface flux}}} = \frac{a_0}{a_N} \quad (26)$$

Figure 2. Relationship between a_n and w .

neglecting the effect of the term containing $\left(\frac{c}{9ax}\right)^{\frac{1}{3}}$. Values of ψ as a function of β are shown in Figure 4.

Table I. Numerical values of a_N as a function of β .

β	a_N	β	a_N
0.0	0.893	0.6	1.176
0.1	0.941	0.7	1.219
0.2	0.990	0.8	1.262
0.3	1.041	0.9	1.302
0.4	1.085	1.0	1.364
0.5	1.131		

DISCUSSION

From consideration of the numerical values of a_N , it becomes obvious that the surface flux reduces the steepness of the temperature gradient and, consequently, decreases the transfer coefficient.

For application of the present results the value of c in the velocity-distribution equation must be known. A reasonable estimate of c can be obtained by equating c to

$\frac{\tau_w}{\mu}$ where τ_w is the wall stress and μ is the coefficient of viscosity of the fluid.

Eq 24 and 25 thus become

$$v_y|_0 = \frac{\phi}{a_N} \left(\frac{\tau_w}{9ax\mu} \right)^{\frac{1}{3}} \quad (27)$$

$$Nu = \frac{\ell}{a_N} \left(\frac{\tau_w}{9ax\mu} \right)^{\frac{1}{3}}. \quad (28)$$

Thus the ratio of the Nusselt number with surface flux to that when there is no surface flux is given by

$$\theta(\beta) = \frac{(Nu)_{\text{with surface flux}}}{(Nu)_{\text{with no surface flux}}} = \left(\frac{a_0}{a_N} \right) \left(\frac{\tau_w, \text{ with surface flux}}{\tau_w, \text{ with no surface flux}} \right)^{\frac{1}{3}}. \quad (29)$$

In view of the fact that τ_w is proportional to the velocity gradient and the assumption that the distribution of the velocity is comparable to that of the temperature profile, it follows

$$\theta(\beta) = \frac{(Nu)_{\text{with surface flux}}}{(Nu)_{\text{with no surface flux}}} = \left(\frac{a_0}{a_N} \right) \left(\frac{1/a_N}{1/a_0} \right)^{\frac{1}{3}} = \left(\frac{a_0}{a_N} \right)^{\frac{4}{3}}. \quad (30)$$

Values of θ are also included in Figure 4. The melting rate can be rewritten as

$$v_y = \frac{k\Delta T}{\rho L a_N} \left(\frac{c}{9ax} \right)^{\frac{1}{3}} \propto \frac{k\Delta T}{\rho L a_N} \left(\frac{1}{a_N} \right)^{\frac{1}{3}}. \quad (31)$$

Similarly the reference melting rate is

$$v_{y, \text{ ref}} = \frac{k\Delta T_{\text{ref}}}{\rho L a_0} \left(\frac{c}{9ax} \right)^{\frac{1}{3}} \propto \frac{k\Delta T_{\text{ref}}}{\rho L a_0} \left(\frac{1}{a_0} \right)^{\frac{1}{3}}. \quad (32)$$

Figure 3. Relationship between dimensionless temperature distribution and X as a function of β .

Figure 4. Relationship between ψ , θ , and β .

The ratio of the two melting rates is

$$\frac{v_y}{v_{y, \text{ref}}} = \left(\frac{\Delta T}{\Delta T_{\text{ref}}} \right) \left(\frac{a_0}{a_N} \right)^{\frac{4}{3}} = \left(\frac{\Delta T}{\Delta T_{\text{ref}}} \right) \left(\frac{\psi}{\psi_{\text{ref}}} \right)^{\frac{4}{3}}. \quad (33)$$

Take $\Delta T_{\text{ref}} = 0.01 \frac{L\rho}{k}$, since $\phi = \frac{k\Delta T}{\rho L}$, $\phi_{\text{ref}} = 0.01$, also $\frac{\Delta T}{\Delta T_{\text{ref}}} = \frac{\phi}{\phi_{\text{ref}}}$. Table II illustrates the calculation of the melting velocity ratio as function of $\frac{\Delta T}{\Delta T_{\text{ref}}}$.

Figure 5 clearly indicates the effect of the temperature-difference ratio on the melting-velocity ratio.

For the case of flow over a flat plate with no pressure gradient, the local Nusselt number is given as (Schlichting, 1955)

$$\text{Nu}_x = 0.332 \text{Pr}^{\frac{1}{3}} \text{Re}_x^{\frac{1}{2}} \quad (34)$$

where

Pr = Prandtl number

Re_x = local Reynolds number.

The Nusselt number for a similar problem with melting taking place can be estimated as

$$\text{Nu}_x = 0.332 \left(\frac{a_0}{a_N} \right)^{\frac{4}{3}} (\text{Pr})^{\frac{1}{3}} (\text{Re}_x)^{\frac{1}{2}}. \quad (35)$$

Appendix D shows the calculation of the melting rate under flow conditions.

Since the Leveque solution is known to be valid for the entrance region of a circular pipe, the present solution can also be considered as an asymptotic solution to the modified Graetz-Nusselt problem for mass transfer if small values of ℓ/D (D represents the diameter of a circular conduit) are used and the appropriate expression for $V_y|_0$ is used in place of eq 4.

Table II. Calculation of melting-velocity ratio as function of temperature-difference ratio.

$\frac{\Delta T}{\Delta T_{\text{ref}}} = \frac{\phi}{\phi_{\text{ref}}}$	ϕ	ψ	$\frac{\psi}{\psi_{\text{ref}}}$	$\left(\frac{\psi}{\psi_{\text{ref}}} \right)^{\frac{4}{3}}$	$\frac{v_y}{v_{y, \text{ref}}}$
5	0.05	0.973	0.9789	0.9716	4.858
10	0.1	0.949	0.9547	0.9409	9.409
20	0.2	0.9020	0.9074	0.8784	17.568
40	0.4	0.8230	0.8280	0.7775	31.100
60	0.6	0.759	0.7640	0.6983	41.898
80	0.8	0.7076	0.7119	0.6357	50.856
100	1.0	0.6634	0.6674	0.5833	58.330

$$\psi_{\text{ref}} = \frac{0.893}{0.8984} = 0.994$$

Figure 5. Relationship between melting-velocity ratio and ratio of temperature difference.

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APPENDIX A.

Convergence of the sequence (a_n) .

In equation 20 of the report, the temperature distribution is given in terms of the limiting value of the sequence (a_n) as $n \rightarrow \infty$. The sequence is defined as

$$\begin{aligned} a_0 &= \int_0^{\infty} \exp(-y^3) dy \\ a_1 &= \int_0^{\infty} \exp(-y^3 + \frac{\beta}{a_0} y) dy \\ a_2 &= \int_0^{\infty} \exp(-y^3 + \frac{\beta}{a_1} y) dy \\ &\vdots \\ a_n &= \int_0^{\infty} \exp(-y^3 + \frac{\beta}{a_{n-1}} y) dy. \end{aligned}$$

The sequence is convergent for any given positive value of β . Since

$$\exp(\frac{\beta}{a_0} y) > 1,$$

therefore

$$a_1 > a_0, \text{ also } \frac{1}{a_1} < \frac{1}{a_0}$$

and

$$\exp(\frac{\beta}{a_1} y) < \exp(\frac{\beta}{a_0} y).$$

However

$$\exp(\frac{\beta}{a_1} y) > 1,$$

hence

$$a_0 < a_2 < a_1.$$

Similarly we can see

$$\frac{1}{a_1} < \frac{1}{a_2} < \frac{1}{a_1}.$$

Therefore

$$a_2 < a_3 < a_1,$$

but

$$a_2 < a_4 < a_3.$$

From this demonstration, it is clear that the magnitude of a_n oscillates and diminishes its amplitude and approaches to a limiting value a_N , as $n \rightarrow \infty$. The following sketch represents this sequence:

APPENDIX B.

Numerical evaluation of the integral $\int_0^{\infty} \exp(-y^3 + wy) dy$.

For $0 \leq w \leq 1$, the following holds:

$$\int_0^{\infty} \exp(-y^3 + wy) dy \geq \int_0^{\infty} \exp(-y^3) dy = \frac{\Gamma(\frac{1}{3})}{3}.$$

Consider the following:

$$\begin{aligned} \int_0^{\infty} \exp(-y^3 + wy) dy &= \int_0^a \exp(-y^3 + wy) dy + \int_a^b \exp(-y^3 + wy) dy + \\ &+ \int_b^{\infty} \exp(-y^3 + wy) dy. \end{aligned} \quad (B1)$$

The value of b was evaluated under such a condition that the third integral in eq B1 gives less than the desired accuracy.

Let

$$\begin{aligned} A &= \int_b^{\infty} \exp(-y^3 + wy) dy \\ &\leq \int_b^{\infty} \exp(-y^3 + y) dy. \end{aligned}$$

Assume $b \geq 2$

$$-y^3 + y = -y(y^2 - 1) < -y^2 \text{ for } y \geq 2.$$

Therefore

$$\begin{aligned} A &< \int_b^{\infty} \exp(-y^2) dy \\ &< \frac{\sqrt{\pi}}{2} [1 - \operatorname{erf}(b)]. \end{aligned}$$

It is desired to make this integral less than 0.0002. From tabulated data on error function, for $b = 2.7$, $\operatorname{erf}(2.7) = 0.999866$. Therefore

$$\begin{aligned} A &< \frac{\sqrt{\pi}}{2} (1 - 0.999866) \\ &< 0.000119. \end{aligned}$$

Thus eq B1 can be written as

$$\int_0^{\infty} \exp(-y^3 + wy) dy = \int_0^a \exp(-y^3 + wy) dy + \int_a^b \exp(-y^3 + wy) dy + \int_{b'}^{2.7} \exp(-y^3 + wy) dy \quad (B2)$$

Consider the third integral of eq B2:

$$I_c = \int_{b'}^{2.7} \exp(-y^3 + wy) dy = I(y) \exp(wy) \Big|_{b'}^{2.7} - \int_{b'}^{2.7} I(y) \exp(wy) dy$$

where

$$I(y) = \int_0^{\infty} \exp(-y^3) dy.$$

Since

$$I(b') [\exp(2.7w) - \exp(b'w)] < \int_{b'}^{2.7} I(y) w \exp(wy) dy < I(2.7) [\exp(2.7w) - \exp(b'w)],$$

the difference between the lower and upper limit is given as

$$\Delta = [I(2.7) - I(b')] [\exp(2.7w) - \exp(b'w)].$$

When $w = 0$, $\Delta = 0$, and maximum Δ occurs at maximum value of w . For maximum $w = 1$, therefore:

$$(\Delta)_{\max} = [I(2.7) - I(b')] [\exp(2.7) - \exp(b')].$$

For $b' = 2.0$

$$(\Delta)_{\max} = (0.8929795 - 0.8929535) (14.879732 - 7.389056) = 0.000195 = 0.0002.$$

The integration of eq B2 can thus be written as

$$\int_0^{\infty} \exp(-y^3 + wy) dy = \int_0^2 \exp(-y^3 + wy) dy + \int_{2.0}^{2.7} \exp(-y^3 + wy) dy. \quad (B3)$$

The last part of the integral in eq B3, as shown above, can be represented by the following expression within the desired accuracy:

$$I_c = \int_2^{2.7} \exp(-y^3 + wy) dy = \exp(2.7w) [I(2.7) - I(2.0)] = 0.0000260 \exp(2.7w). \quad (B4)$$

Values of I_c calculated from eq B4 as a function of w are given in Table B1. The integral of

$$\int_0^2 \exp(-y^3 + wy) dy$$

can be obtained by expanding $\exp(-y^3 + wy)$ into a series and carrying out the integration term by term. The number of terms required will depend on the accuracy desired. Since $\exp(-y^3)$ decreases rapidly as y increases, the number of terms determined for the condition of

$$\int_0^2 \exp(y) dy$$

will be more than sufficient for the above integral. From standard integration tables, the definite integral of

$$\int_0^2 \exp(y) dy = e^2 - 1 = 6.38906.$$

If $\exp(y)$ is expanded in an exponential series,

$$\exp(y) = 1 + y + \frac{1}{2!} y^2 + \frac{1}{3!} y^3 + \frac{1}{4!} y^4 + \dots$$

Therefore

$$\begin{aligned} \int_0^2 \exp(y) dy = & 2 + \frac{1}{2} 2^2 + \frac{1}{2!3} 2^3 + \frac{1}{3!4} 2^4 + \frac{1}{4!5} 2^5 + \frac{1}{5!6} 2^6 + \frac{1}{6!7} 2^7 + \dots + \\ & + \frac{2^{n+1}}{n!(n+1)}. \end{aligned}$$

If seven terms are taken, the accuracy will be equal to

$$\frac{6.38906 - 6.38079}{6.38906} \times 100 = 0.13\%.$$

The first integral of eq B3 is further divided as the summation of

$$\int_0^1 \exp(-y^3 + wy) dy + \int_1^2 \exp(-y^3 + wy) dy.$$

The value of $\int_0^1 \exp(-y^3 + wy) dy$ can be approximated to the desired accuracy by the

following expression

$$\begin{aligned} I_a &= \int_0^1 \exp(-y^3 + wy) dy \\ &= y \left(-\frac{1}{4} y^4 - \frac{1}{2} wy^2 \right) + \frac{1}{2} \left(\frac{1}{7} y^7 - \frac{2}{5} wy^5 + \frac{1}{3} w^2 y^3 \right) \\ &\quad - \frac{1}{6} \left(\frac{1}{10} y^{10} - \frac{3}{8} wy^8 + \frac{1}{2} w^2 y^6 - \frac{1}{4} w^3 y^4 \right) \\ &\quad + \frac{1}{24} \left(\frac{1}{13} y^{13} - \frac{4}{11} wy^{11} + \frac{2}{3} w^2 y^9 - \frac{4}{7} w^3 y^7 + \frac{1}{5} w^4 y^5 \right) \\ &\quad - \frac{1}{120} \left(\frac{1}{16} y^{16} - \frac{5}{14} wy^{14} + \frac{10}{12} w^2 y^{12} - w^3 y^{10} + \frac{5}{8} w^4 y^8 - \frac{1}{6} w^5 y^6 \right) \\ &\quad + \dots \end{aligned}$$

I_a values calculated from the above expression are accurate to four digits after the decimal point; they are tabulated in Table B1. The integration of

$$\int_1^2 \exp(-y^3 + wy) dy$$

Table BI. Numerical approximation of the integral

$$\int_0^{\infty} \exp(-y^3 + wy) dy \text{ as function of } \underline{w}.$$

w	$I_a = \int_0^1 \exp(-y^3 + wy) dy$	$I_b = \int_1^2 \exp(-y^3 + wy) dy$	$I_c = \int_2^{2.7} \exp(-y^3 + wy) dy$	$I_a + I_b + I_c = a_n$
0.0	0.80745	0.08575	0.000026	0.89323
0.1	0.84367	0.09660	0.000030	0.94030
0.2	0.88189	0.10869	0.000040	0.99062
0.3	0.92266	0.12277	0.000060	1.04549
0.4	0.96601	0.13792	0.000080	1.10401
0.6	1.06132	0.17520	0.000130	1.23665
0.8	1.16963	0.22284	0.000230	1.39270
1.0	1.29265	0.28374	0.000370	1.57678

is approximated by the Euler-Maclaurin formula which is

$$\begin{aligned}
 I_b = \frac{1}{\Delta y} \int_a^{a+r\Delta y} f(y) dy &= \left(\frac{1}{2} f_0 + f_1 + f_2 + f_3 + \dots + f_{r-1} + \frac{1}{2} f_r \right) \\
 &\quad - \frac{\Delta y}{12} [f'(a+r\Delta y) - f'(a)] \\
 &\quad + \frac{(\Delta y)^3}{720} [f'''(a+r\Delta y) - f'''(a)] \\
 &\quad - \frac{(\Delta y)^5}{30240} [f^{(5)}(a+r\Delta y) - f^{(5)}(a)] \\
 &\quad + \dots
 \end{aligned}$$

In this case, $a = 1$, Δy is taken as 0.1, $r = 10$, $f(y) = \exp(-y^3 + wy)$, $f'(a+r\Delta y) = f'(2) = (-12 + w) \exp(-8 + 2w)$, $f'(a) = f'(1) = (-3 + w) \exp(-1 + w)$. The terms containing derivatives higher than first are neglected. The integration of

$$\begin{aligned}
 I_b = \int_1^2 \exp(-y^3 + wy) dy &= 0.1 \left[\frac{1}{2} \exp(-1 + w) + \exp[(-1.1)^3 + 1.1w] \right. \\
 &\quad + \exp[(-1.2)^3 + 1.2w] + \exp[(-1.3)^3 + 1.3w] + \exp[(-1.4)^3 + 1.4w] \\
 &\quad + \exp[(-1.5)^3 + 1.5w] + \exp[(-1.6)^3 + 1.6w] + \exp[(-1.7)^3 + 1.7w] \\
 &\quad + \exp[(-1.8)^3 + 1.8w] + \exp[(-1.9)^3 + 1.9w] + \frac{1}{2} \exp[(-2.0)^3 + 2.0w] \\
 &\quad \left. - \frac{0.01}{12} (-12 + w) \exp(8 + 2w) - (-3 + w) \exp(-1 + w) \right].
 \end{aligned}$$

Numerical values of I_b are also included in Table BI. The sum of $I_a + I_b + I_c$ gives the desired value of the integral $\int_0^{\infty} \exp(-y^3 + wy) dy$. Figure 2 shows the relationship

between a_n and w , and is used for the evaluation of the limiting values a_N of the sequence (a_n) .

APPENDIX C.

Evaluation of the limiting value a_N of the sequence (a_n) .

The limiting values a_N are evaluated as a function of β in Figure 2. Table CI summarizes this iterative process in evaluating a_N . It is noted from this table that the values of a_n oscillate and diminish in amplitude as n increases, as discussed in Appendix A. The limiting values for each corresponding a_N are underlined.

Table CI. Evaluation of the limiting value a_N as function of β .

β	a_0	w_0	a_1	w_1	a_2	w_2	a_3	w_3	a_4	w_4	a_5	w_5	a_6	w_6	a_7	w_7	a_8	w_8	a_9
0.0	<u>0.893</u>																		
0.1	0.8930	0.1120	0.9440	0.1059	0.9405	0.1063	<u>0.941</u>												
0.2	0.8930	0.2240	1.0025	0.1995	0.9900	0.202	0.990												
0.3	0.8930	0.3359	1.066	0.2814	1.0350	0.2899	1.0395	0.2886	<u>1.0388</u>										
0.4	0.8930	0.4479	1.135	0.3534	1.0762	0.3717	1.0865	0.3682	1.0845	0.3688	<u>1.085</u>								
0.5	1.000	0.5000	1.170	0.4274	1.121	0.4460	1.134	0.4409	1.1302	0.4424	1.132	0.4417	<u>1.1310</u>						
0.6	1.000	0.6000	1.237	0.4850	1.160	0.5172	1.181	0.5080	1.1743	0.5109	1.1763	0.5100	<u>1.176</u>						
0.7	1.000	0.7000	1.313	0.5331	1.191	0.5877	1.228	0.5700	1.214	0.5766	1.220	0.5738	<u>1.219</u>						
0.8	1.000	0.8000	1.392	0.5747	1.219	0.6563	1.279	0.6255	1.255	0.6375	1.264	0.6329	1.261	0.6344	<u>1.262</u>				
0.9	1.000	0.9000	1.480	0.6081	1.242	0.7246	1.332	0.6757	1.294	0.6955	1.3090	0.6875	1.303	0.6907	1.305	0.6897	<u>1.305</u>		
1.0	1.000	1.000	1.5762	0.6344	1.2623	0.7922	1.386	0.7215	1.329	0.7524	1.354	0.7386	1.3423	0.7450	1.3488	0.7418	1.345	0.7435	<u>1.346</u>

APPENDIX D.

Typical calculation on the melting rate.

Eq 35 is used to evaluate the Nusselt number Nu . Consider the case where water is flowing over a melting ice plate. Assume the water velocity of the main stream v is 1 ft/sec, and consider the point x to be 1 ft from the leading edge of the plate. Let

$$\beta = \frac{c_p (T_\infty - T_m)}{L} = \frac{c_p \Delta T}{L} = 0.1, \text{ in which } c_p \text{ is the specific heat of the water at constant}$$

pressure; T_∞ is the water temperature of the main stream; T_m the melting temperature of the plate; and L the latent heat of fusion (80 cal/g or 144 Btu/lb). Then, ΔT is found to be

$$\Delta T = T_\infty - T_m = T_\infty - 32 = (0.1)(144)(1) = 14.4F$$

or

$$T_\infty = 14.4 + 32 = 46.4F.$$

Eq 35 is expressed in terms of dimensionless groups of local Reynolds numbers and Prandtl numbers. In order to evaluate the Nu_x , these quantities have to be determined. At 46.4F the viscosity of water μ is about 8.4×10^{-4} lb m/ft-sec, where lb m stands for pound mass. Thus the local Reynolds number is

$$Re_x = \frac{vpx}{\mu} = \frac{(1)(62.4)(1)}{8.4 \times 10^{-4}} = 74,285.$$

The thermal conductivity of water k at 46.4F is about 9.72×10^{-5} Btu/ft-sec-F; c_p is 1 Btu/lb m-F. Therefore, the Prandtl number is

$$\frac{c_p \mu}{k} = \frac{(1)(8.4 \times 10^{-4})}{9.72 \times 10^{-5}} = 8.65$$

$$\left(\frac{c_p \mu}{k}\right)^{\frac{1}{3}} = 2.05.$$

From Figure 4, when $\beta = 0.1$

$$\theta = \left(\frac{a_0}{a_N}\right)^{\frac{4}{3}} = 0.949.$$

By substituting these quantities into eq 35, value of Nu_x is found to be

$$Nu_x = \frac{hx}{k} = 0.332 \left(\frac{a_0}{a_N}\right)^{\frac{4}{3}} (Pr)^{\frac{1}{3}} (Re_x)^{\frac{1}{2}}$$

$$= 0.332 (0.949)(2.05) \left(\sqrt{74285}\right) = 176$$

hence

$$h = \frac{Nu_x k}{x} = \frac{(176)(9.72 \times 10^{-5})}{1} = 171 \times 10^{-2} \text{ Btu/ft}^2\text{-F-sec.}$$

The melting rate of ice at $x = 1$ ft is thus found to be

$$v_y = \frac{h \Delta T}{\rho L} = \frac{(171 \times 10^{-2})(144)}{(144)(62.4)} = 0.274 \times 10^{-4} \text{ ft/sec.}$$

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U. S. Army Cold Regions Research and Engineering Laboratory, Army Materiel Command, Hanover, N.H. LAMINAR HEAT TRANSFER OVER A MELTING PLATE - THE MODIFIED LEVEQUE PROBLEM - Yin-Chao Yen and Chi Tien		1. Heat transfer--Mathematical analysis 2. Ice--Melting 3. Models (Ice) I. Yen, Yin-Chao II. Tien, Chi III. U. S. Army Cold Regions Research and Engineering Laboratory	U. S. Army Cold Regions Research and Engineering Laboratory, Army Materiel Command, Hanover, N.H. LAMINAR HEAT TRANSFER OVER A MELTING PLATE - THE MODIFIED LEVEQUE PROBLEM - Yin-Chao Yen and Chi Tien		1. Heat transfer--Mathematical analysis 2. Ice--Melting 3. Models (Ice) I. Yen, Yin-Chao II. Tien, Chi III. U. S. Army Cold Regions Research and Engineering Laboratory
Research Report 125, Jan 1964, 10p-illus-tables-app. DA Task 8X9-27-001-03 Unclassified Report			Research Report 125, Jan 1964, 10p-illus-tables-app. DA Task 8X9-27-001-03 Unclassified Report		
Heat transfer in a system consisting of a fluid flowing over a melting plate composed of the same material as the fluid was investigated. The test model is similar to the heat transfer phenomenon occurring when an iceberg drifts in warm sea water. The flow of a liquid, initially at a uniform temperature, over a flat plate composed of the solid phase of the flowing liquid at constant temperature was assumed. For simplification of subsequent analysis the flow was stipulated as the Leveque type. The temperature distribution in the flowing fluid is determined; melting at the interface is shown to result in a decrease in the Nusselt number; consideration is given to the practical application of the results; and a simple numerical calculation of the melting rate of ice is given.			Heat transfer in a system consisting of a fluid flowing over a melting plate composed of the same material as the fluid was investigated. The test model is similar to the heat transfer phenomenon occurring when an iceberg drifts in warm sea water. The flow of a liquid, initially at a uniform temperature, over a flat plate composed of the solid phase of the flowing liquid at constant temperature was assumed. For simplification of subsequent analysis the flow was stipulated as the Leveque type. The temperature distribution in the flowing fluid is determined; melting at the interface is shown to result in a decrease in the Nusselt number; consideration is given to the practical application of the results; and a simple numerical calculation of the melting rate of ice is given.		

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