Research Report 207

UNIFIED TREATMENT OF VECTORS AND TENSORS IN n-DIMENSIONAL EUCLIDEAN SPACE

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JUNE 1968

U.S. ARMY MATERIEL COMMAND
COLD REGIONS RESEARCH & ENGINEERING LABORATORY
HANOVER, NEW HAMPSHIRE

DA Task 1T014501B52A02

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PREFACE

This report was prepared by Dr. Shunsuke Takagi, Physical Sciences Branch, Research Division. Work on this subject was done in connection with difficulties which arise in the continuum mechanics pertaining to large deformations of such media as soil and snow.

The author was encouraged by Yoshimura (1957) to initiate this research.

An in-house lecture course on tensor analysis conducted by the author from April 1964 to February 1965 at the U. S. Army Cold Regions Research and Engineering Laboratory contributed to elucidating basic concepts in this paper.

The U. S. Army Mathematical Research Center, University of Wisconsin, helped the author toward finishing the paper. The author is especially grateful to Prof. Ben Noble, MRC and University of Wisconsin, for his encouragement.

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SUMMARY

A unified treatment of vectors, tensors and multivectors in n-dimensional Euclidean space is presented. The unified treatment is so systematized that n-dimensional tensors of arbitrary order are treated similarly to three-dimensional vectors. Work on this subject was done in connection with difficulties which arise in the continuum mechanics pertaining to large deformations of such media as soil and snow.

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INTRODUCTION

The traditional tensor analysis was originally invented for use in the general theory of relativity, whose space is more complicated than a Euclidean space. A Euclidean space is flat, whereas the space for general relativity is curved. In a curved space, differentiation of a vector with regard to one of the coordinates expressing the space yields components that do not belong to the original space.

A flat space is complete, in the sense that all the operations defined in the space change a vector to another vector belonging to the same space. The advantage of completeness should be fully utilized for describing tensors in a Euclidean space.

A vector notation for three-dimensional Euclidean space was originated by Gibbs (1901), and was extended to tensor notation by Hessenberg (1917) and by Wills (1931). Recently this notation was used for the study of large deformation by Yoshimura (1957) and Sedov (1965).

In their notations, vectors and tensors are written as combinations of components and base tensors. For example, a vector **V** is written, with the summation convention applied, as $V^i\mathbf{e}_i = V_i\mathbf{e}^i$, where $\mathbf{e}_i$ and $\mathbf{e}^i$ are dual base vectors, and V^i and V_i are scalars. A second-order tensor **T** is written as $T^{ij}\mathbf{e}_i\mathbf{e}_j = T^i{}_j\mathbf{e}_i\mathbf{e}^j = T_i{}^j\mathbf{e}^i\mathbf{e}_j = T_{ij}\mathbf{e}^i\mathbf{e}^j$, where $\mathbf{e}_i\mathbf{e}_j$, $\mathbf{e}_i\mathbf{e}^j$, $\mathbf{e}^i\mathbf{e}_j$, $\mathbf{e}^i\mathbf{e}^j$ are juxtaposed base vectors, called dyads, and T^{ij}, $T^i{}_j$, $T_i{}^j$, T_{ij} are scalars. Similarly a third-order tensor is written as a linear combination of triads $\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$, and so on. Base vectors, dyads, triads, and so on, are collectively called base tensors in this paper.

This tensor notation will be called tentatively the Gibbs tensor notation. When base tensors are disregarded and only scalar components are written out, the Gibbs tensor analysis yields the traditional tensor analysis.

The Gibbs tensor notation has never been used for $\underline{n}$-dimensional tensor analysis. This is partly because the vector product was defined only in three-dimensional space and could not be extended to $\underline{n}$-dimensional space.

Another system of vector calculus was invented by Grassmann (1844). He invented a special product, now called the exterior product or Grassmann product. Although the exterior product has been used since Cartan (1899) in many fields of pure mathematics, it has never been incorporated into the Gibbs tensor notation.

The exterior product of vectors, called a multivector, is given a Gibbs tensor expression in this paper. The use of this expression allows us to define the vector product of $\underline{n}$-dimensional vectors and to construct an $\underline{n}$-dimensional Gibbs tensor analysis.

An exterior product of more than three vectors represents a higher-than-three-dimensional parallelepiped, and is treated similarly to a vector representing a line segment. The use of multivectors allows us to treat $\underline{n}$-dimensional Euclidean geometry as a simple extension of two- or three-dimensional Euclidean geometry. As an example, some geometric properties of an $\underline{r}$-dimensional manifold embedded in $\underline{n}$-dimensional Euclidean space ($r \leq n$) are discussed.

It is proved in this paper that Green's integrals are valid even for a tensor of any order defined in an r-dimensional manifold embedded in n-dimensional Euclidean space ($r \leqq n$). Stokes' integral is a special case of Green's integrals. The validity of these integrals is proved as one theorem by using the geometry of manifolds embedded in n-dimensional Euclidean space.

The Gibbs tensor notation appears to be the most convenient expression of physical and geometric quantities in a Euclidean space.

CURVILINEAR COORDINATES AND VECTORS

The vector notations developed for three-dimensional curvilinear coordinates by Green and Zerna (1954), Yoshimura (1957), and Sedov (1965) offer a basis for unified treatment of vectors and tensors. Their notations are summarized in this section. Most of their notations can be simply extended, as shown here, to vector notations in n-dimensional curvilinear coordinates. The vector product, however, cannot be simply extended, and will be considered later.

A particular set $X^1, \ldots, X^n$ of Cartesian coordinates is assumed to be fixed in n-dimensional Euclidean space. The set of Cartesian coordinates X^a ($a = 1, \ldots, n$) will be called the standard Cartesian coordinates, which will be used as the standard for representing vectors, tensors, and transformations among them.

A set of n-dimensional curvilinear coordinates x^i ($i = 1, \ldots, n$) is defined by transformation

$$x^i = x^i(X^1, \ldots, X^n). \tag{1}$$

Functions x^i are assumed to be (1) functionally independent of each other and (2) at least three-times continuously differentiable. It is proved by the inverse function theorem that the inverse of eq 1

$$X^a = X^a(x^1, \ldots, x^n) \tag{2}$$

exists and satisfies conditions 1 and 2.

Base vectors $\mathbf{e}_i$($i = 1, \ldots, n$) are defined for curvilinear coordinates x^i by

$$d\mathbf{r} = \mathbf{e}_i dx^i \tag{3}$$

where $d\mathbf{r}$ is a vector of infinitesimal length. The summation convention is applied in eq 3 and in the following equations. Use of the basis $\mathbf{c}_a$ ($a=1, \ldots, n$) for the standard Cartesian coordinates X^a yields

$$d\mathbf{r} = \mathbf{c}_a dX^a. \tag{4}$$

Therefore base vectors $\mathbf{e}_i$ may be determined by

$$\mathbf{e}_i = \mathbf{c}_a \frac{\partial X^a}{\partial x^i}. \tag{5}$$

These base vectors $\mathbf{c}_a$ will be called the standard Cartesian base vectors.

The differential distance ds is given by a scalar product

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r}. \tag{6}$$

Substitution of eq 3 changes eq 6 to

$$ds^i = g_{ij}\, dx^i\, dx^j \tag{7}$$

where

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{8}$$

Use of base vectors allows us to use instead of eq 7 its linearized form (eq. 3).

Vector $\mathbf{e}_i$ is in the direction in which x^i does increase but $x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n$ do not change. Vector $\mathbf{e}_i$ is a function of location. When considered at a point where a set of $\mathbf{e}_i$ is defined, vectors $\mathbf{e}_i$ span a skew linear coordinate space, called tangent space.

Vectors $\mathbf{e}^i$ (j=1, . . . , n) dual to $\mathbf{e}_i$ are introduced by

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j \tag{9}$$

where δ_i^j is a Kronecker delta. Because vectors $\mathbf{e}_i$ are independent of each other, eq. 9 determines $\mathbf{e}^j$ uniquely. Geometrically, $\mathbf{e}^j$ is normal to $\mathbf{e}_1, \ldots, \mathbf{e}_{j-1}, \mathbf{e}_{j+1}, \ldots, \mathbf{e}_n$, as shown by eq. 9, and makes an acute angle with $\mathbf{e}_j$. Vectors $\mathbf{e}_i$ and $\mathbf{e}^j$ are called subindexed and superindexed base vectors, respectively, in this paper.* A set of g^{ij} is defined by

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j. \tag{10}$$

Since vectors $\mathbf{e}^j$ are introduced, a set of dx_j (j=1, . . . , n) may be defined to make

$$\mathbf{e}_i\, dx^i = \mathbf{e}^j\, dx_j\, . \tag{11}$$

Dotting $\mathbf{e}_k$ into† both sides of eq 11 determines dx_k,

$$dx_k = g_{ki}\, dx^i \quad . \tag{12}$$

The right-hand side of eq 12 is not always integrable, and x_i is not always defined.

Equation 12 is integrable if and only if the g_{ij}'s are constant, as proved in Theorem 7 below. The condition is fulfilled for skew linear coordinates, in which $\mathbf{e}_i$ and $\mathbf{e}^i$ are constant vectors. Both x^i and x_i are components of a position vector $\mathbf{r}$

$$\mathbf{r} = x^i e_i = x_j e^j \tag{13}$$

in skew linear coordinates.

In general curvilinear coordinates dx^i and dx_i are respectively components of $d\mathbf{r}$ in the directions of $\mathbf{e}_i$ and $\mathbf{e}^j$ in the tangent space considered.

Theorem 1

The unit tensor **1** defined by

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* Accepted terms are covariant base vector for $\mathbf{e}_i$ and contravariant base vector for $\mathbf{e}^j$. This terminology, however, I feel, is too sophisticated.

† For brevity, "dotting **a** into **b**" is used for the process of introducing **a** and taking the dot product of **a** and **b**.

$$\mathbf{1} = \mathbf{e}_i\mathbf{e}^i \tag{14}$$

is the unit for dot operation; that is, for any vector $\mathbf{V}$, the relation

$$\mathbf{V} \cdot \mathbf{1} = \mathbf{1} \cdot \mathbf{V} = \mathbf{V} \tag{15}$$

is valid. The tensor $\mathbf{1}$ is the idemfactor introduced by Gibbs (1901). It can be written in various forms,

$$\mathbf{1} = \mathbf{e}_i\mathbf{e}^i = \mathbf{e}^i\mathbf{e}_i = g_{ij}\,\mathbf{e}^i\mathbf{e}^j = g^{ij}\,\mathbf{e}_i\mathbf{e}_j\,. \tag{16}$$

It is invariant under transformations of curvilinear coordinates; that is, for another set of dual bases $\mathbf{f}_p$, $\mathbf{f}^p$, the tensor $\mathbf{1}$ has the same form,

$$\mathbf{1} = \mathbf{f}_p\mathbf{f}^p\,. \tag{17}$$

Proof. Let

$$\mathbf{V} = V^i\mathbf{e}_i = V_i\mathbf{e}^i \tag{18}$$

be an arbitrary vector in a tangent space. Dotting $\mathbf{e}^k$ and $\mathbf{e}_k$ respectively into the second and third members of eq 18 yields

$$V^k = \mathbf{V} \cdot \mathbf{e}^k \tag{19}$$

and

$$V_k = \mathbf{V} \cdot \mathbf{e}_k\,. \tag{20}$$

Substituting eq 19 and 20 into eq 18 yields

$$\mathbf{V} = \mathbf{V} \cdot \mathbf{e}^i\mathbf{e}_i = \mathbf{V} \cdot \mathbf{e}_i\mathbf{e}^i \tag{21}$$

where $\mathbf{e}^i\mathbf{e}_i$ and $\mathbf{e}_i\mathbf{e}^i$ are juxtapositions of two vectors. Equation 21 shows that both $\mathbf{e}^i\mathbf{e}_i$ and $\mathbf{e}_i\mathbf{e}^i$ are the units for dotting from the right.

Substituting modified forms of eq 19 and eq 20,

$$V^k = \mathbf{e}^k \cdot \mathbf{V} \tag{22}$$

and

$$V_k = \mathbf{e}_k \cdot \mathbf{V} \tag{23}$$

into a modified form of eq 18,

$$\mathbf{V} = \mathbf{e}_i V^i = \mathbf{e}^i V_i \tag{24}$$

yields that $\mathbf{e}_i\mathbf{e}^i$ and $\mathbf{e}^i\mathbf{e}_i$ are also the units for dotting from the left.

Juxtaposing $\mathbf{e}^j$ from the right with eq 8 yields $g_{ij}\mathbf{e}^j = \mathbf{e}_i \cdot \mathbf{e}_j\mathbf{e}^j$, which becomes

$$\mathbf{e}_i = g_{ij}\mathbf{e}^j \tag{25}$$

by use of the property of $\mathbf{e}_j\mathbf{e}^j$ just established. Similarly eq 10 may be changed to

$$\mathbf{e}^i = g^{ij}\,\mathbf{e}_j. \tag{26}$$

Use of eq 25 and 26 shows that both $\mathbf{e}_i\mathbf{e}^i$ and $\mathbf{e}^i\mathbf{e}_i$ transform to $g_{ij}\, \mathbf{e}^i\mathbf{e}^j$ or $g^{ij}\, \mathbf{e}_i\mathbf{e}_j$, proving eq 16.

Consider a transformation

$$x^i = x^i(y^1, \ldots, y^n), \quad (i=1, \ldots, n) . \tag{27}$$

Base vectors $\mathbf{e}_i$ for x^i and $\mathbf{f}_p$ for y^p are defined respectively by

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i} \tag{28}$$

and

$$\mathbf{f}_p = \frac{\partial \mathbf{r}}{\partial y^p} . \tag{29}$$

Vectors $\mathbf{e}_i$ are transformed to $\mathbf{f}_p$ by

$$\mathbf{e}_i = \mathbf{f}_p \frac{\partial y^p}{\partial x^i} . \tag{30}$$

The transformation of base vectors $\mathbf{e}^i$ to $\mathbf{f}^p$ is given by

$$\mathbf{e}^i = \mathbf{f}^p \frac{\partial x^i}{\partial y^p} . \tag{31}$$

Equation 31 is valid because this is compatible with eq 30 when $\mathbf{e}^i$ and $\mathbf{f}^p$ are defined respectively by

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^{\,j}$$

and

$$\mathbf{f}_p \cdot \mathbf{f}^q = \delta_p^{\,q} .$$

Substituting eq 30 and 31 into eq 14 yields

$$\mathbf{1} = \mathbf{f}_p \frac{\partial y^p}{\partial x^i} \mathbf{f}^q \frac{\partial x^i}{\partial y^q} = \mathbf{f}_p\mathbf{f}^p$$

proving eq 17.

For the standard Cartesian base vectors $\mathbf{c}_a$, $\mathbf{1}$ is

$$\mathbf{1} = \mathbf{c}_a\mathbf{c}_a \tag{32}$$

because $\mathbf{c}_a = \mathbf{c}^a$. The right-hand side of eq 32 is the idemfactor introduced by Gibbs (1901). Q.E.D.

In an Euclidean space $\partial \mathbf{e}_i/\partial x^j$ is a vector belonging to the same space; thus,

$$\frac{\partial \mathbf{e}_i}{\partial x^j} = \Gamma_{ij}^{\,h}\, \mathbf{e}_h \tag{33}$$

where Γ_{ij}^h is a component of vector $\partial \mathbf{e}_i/\partial x^j$ along the direction of $\mathbf{e}_h$. Components

Γ^h_{ij} are called Christoffel symbols or a set of affine connection in Riemannian geometry.

Proposition 1

Components Γ^h_{ij} in eq 33 are symmetric with regard to suffixes i, j:

$$\Gamma^h_{ij} = \Gamma^h_{ji} \; . \tag{34}$$

Proof. The order of differentiations in $\partial^2 \mathbf{r}/\partial x^i \partial x^j$ may be changed to obtain

$$\frac{\partial^2 \mathbf{r}}{\partial x^i \partial x^j} = \frac{\partial^2 \mathbf{r}}{\partial x^j \partial x^i} \; . \tag{35}$$

The validity of eq 35 becomes evident when $\mathbf{r}$ is expressed in the standard Cartesian coordinates,

$$\mathbf{r} = X^a \mathbf{c}_a \; . \tag{36}$$

Vectors $\mathbf{c}_a$ are constant vectors fixed to the space, and components X^a only are differentiated when $\mathbf{r}$ in eq 36 is differentiated.

Use of eq 28 changes eq 35 to

$$\frac{\partial \mathbf{e}_j}{\partial x^i} = \frac{\partial \mathbf{e}_i}{\partial x^j} . \tag{37}$$

Rewriting eq 37 by use of eq 33 yields eq 34.

Proposition 2

$$\frac{\partial \mathbf{e}^i}{\partial x^j} = - \Gamma^i_{jk} \, \mathbf{e}^k \; . \tag{38}$$

Proof. Differentiating eq 9 with regard to x^k yields

$$\mathbf{e}_i \cdot \frac{\partial \mathbf{e}^j}{\partial x^k} = - \Gamma^j_{ik} \tag{39}$$

where eq 33 is used. Juxtaposing $\mathbf{e}^i$ from the left and using the property eq 15 of $\mathbf{I}$ changes eq 39 to eq 38. Q.E.D.

Equations 33 and 38 yield the so-called covariant differentiation. For example, the differentiation of $\mathbf{V} = V^i \mathbf{e}_i$ is

$$\frac{\partial \mathbf{V}}{\partial x^k} = \frac{\partial V^i}{\partial x^k} \mathbf{e}_i + V^i \Gamma^j_{ik} \mathbf{e}_j = \mathbf{e}_i V^i{}_{,k} \tag{40}$$

where

$$V^i{}_{,k} = \frac{\partial V^i}{\partial x^k} + V^j \Gamma^i_{jk} \; . \tag{41}$$

Similarly the differentiation of $\mathbf{V} = V_i \mathbf{e}^i$ is

$$\frac{\partial \mathbf{V}}{\partial x^k} = \mathbf{e}^i V_{i,k} \tag{42}$$

where

$$V_{i,k} = \frac{\partial V_i}{\partial x^k} - V_j \Gamma^j_{ki} . \tag{43}$$

The covariant orders in the components V^i and V_i are increased by one as a result of differentiation with regard to x^k. This fact, however, cannot be a reason for calling the operation the covariant differentiation, because, if x_i, instead of x^i, is chosen at the outset for designating curvilinear coordinates, the contravariant orders, instead of covariant orders, of the components V^i and V_i will be increased by one. In fact, x_i may be used instead of x^i in skew linear coordinates. A better term for the differentiation from the standpoint of this paper may be component differentiation.

Use of base vectors facilitates proofs of relationships between tensor components as shown below.

Example 1 $\quad g_{ij}\, g^{jk} = \delta_i^{\ k}$. (44)

Proof. $\quad g_{ij}\, g^{jk} = (\mathbf{e}_i \cdot \mathbf{e}_j)(\mathbf{e}^j \cdot \mathbf{e}^k)$ (a)

$$= \mathbf{e}_i \cdot \mathbf{e}_j \mathbf{e}^j \cdot \mathbf{e}^k \tag{b}$$

$$= \mathbf{e}_i \cdot \mathbf{e}^k \tag{c}$$

$$= \delta_i^{\ k} . \tag{d}$$

Equation a is obtained by use of definitions 8 and 10. The meaning of the operations in eq b is clear even though the parentheses in eq a are deleted to give the juxtaposition of $\mathbf{e}_j$ and $\mathbf{e}^j$. Equation b becomes eq c by use of the property (eq 15) of 1.

Example 2 $\quad \Gamma_{hij}$ defined by

$$\Gamma_{hij} = g_{kh}\, \Gamma^h_{ij} \tag{45}$$

satisfies the relationship

$$2\Gamma_{hij} = \frac{\partial g_{hi}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^h} . \tag{46}$$

Proof. $\quad \Gamma_{hij}$ defined by eq 45 satisfies

$$\Gamma^{\ell}_{ij} = g^{\ell h}\, \Gamma_{hij} . \tag{47}$$

Equation 47 is proved by substituting eq 45 into the right-hand side of eq 47 and using eq 44.

Substituting eq 28 and 47 into the left- and right-hand sides of eq 33, respectively, and using eq 26, yields

$$\frac{\partial^2 \mathbf{r}}{\partial x^i \partial x^j} = \Gamma_{hij}\, \mathbf{e}^h . \tag{48}$$

Dotting $\mathbf{e}_k$ on both sides of eq 48 yields

$$\Gamma_{kij} = \frac{\partial \mathbf{r}}{\partial x^k} \cdot \frac{\partial^2 \mathbf{r}}{\partial x^i \partial x^j} . \tag{49}$$

The right-hand side of eq 49 is transformed:

$$2 \frac{\partial \mathbf{r}}{\partial x^k} \cdot \frac{\partial^2 \mathbf{r}}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^i} \left(\frac{\partial \mathbf{r}}{\partial x^k} \cdot \frac{\partial \mathbf{r}}{\partial x^j}\right) + \frac{\partial}{\partial x^j} \left(\frac{\partial \mathbf{r}}{\partial x^k} \cdot \frac{\partial \mathbf{r}}{\partial x^i}\right) - \frac{\partial}{\partial x^k} \left(\frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^j}\right)$$
$$= \frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} . \tag{50}$$

Combining eq 49 and 50 proves eq 46. Q.E.D.

DEFINITION OF TENSORS

Tensors in curvilinear coordinates are defined in tangent spaces in the curvilinear coordinates considered. The formulas derived in the following for tensors in curvilinear coordinates, when only components are written out, yield formulas of tensor analysis in Riemannian geometry.

A base tensor is an arrangement of juxtaposed r base vectors, subindexed or superindexed. Base vectors in a base tensor are non-commutative.

The order of suffixes, lower or upper, in a base tensor determines the type of tensor.

Let $i \ldots h$ be a set of r numbers arbitrarily chosen from $1 \ldots n$. Given a set of n^r base-tensors $\mathbf{e}_i \ldots \mathbf{e}_h$ as a tensor basis, a set of n^r real numbers $T^{i \ldots h}$ defines a tensor $\mathbf{T}$ of order r,

$$\mathbf{T} = T^{i \ldots h} \mathbf{e}_i \ldots \mathbf{e}_h . \tag{51}$$

Real components only are considered here for simplicity.

Components of a different type of the same tensor, for example, $T_{i \cdot k \cdot \ldots \cdot}^{\cdot j \cdot \ell \ldots h}$, are found to make

$$\mathbf{T} = T_{i \cdot k \cdot \ldots \cdot}^{\cdot j \cdot \ell \ldots h} \mathbf{e}^i \mathbf{e}_j \mathbf{e}^k \mathbf{e}_\ell \ldots \mathbf{e}_h . \tag{52}$$

A set of base tensors, such as in eq 52, whose type is different from those in eq 51 may be used to define a tensor. A different choice of tensor basis using the same set of real numbers as components defines a different tensor.

Tensor $\mathbf{T}$ defined by eq 51 has different components $S^{a \ldots b}$ when other vectors $\mathbf{f}_a$ are used as base vectors forming base tensors. The new components $S^{a \ldots b}$ are found to make

$$\mathbf{T} = S^{a \ldots b} \mathbf{f}_a \ldots \mathbf{f}_b . \tag{53}$$

Transformation of coordinates is a transformation of base vectors, and one has

$$\mathbf{e}_i = \xi_i^a \mathbf{f}_a . \tag{54}$$

Substituting eq 54 into eq 51 and comparing the results with eq 53 determines $S^{a\ldots b}$ in terms of $T^{i\ldots h}$.

Base tensors $\mathbf{e}_{i_1}\mathbf{e}_{i_2}\ldots\mathbf{e}_{i_s}$ are called, following Gibbs (1901), dyads, triads, tetrads, ..., r-ads respectively for s=2, 3, 4, ..., r. In an r-ad, $\mathbf{e}_{i_1}\mathbf{e}_{i_2}\ldots\mathbf{e}_{i_r}$, vectors $\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots$, and $\mathbf{e}_{i_r}$ are respectively called the first, second, ..., and rth members. Base vectors in a base tensor are not commutative.

Juxtaposing a tensor basis $\mathbf{e}_i\ldots\mathbf{e}_j$ from the left or the right of another tensor basis $\mathbf{e}_h\ldots\mathbf{e}_k$ means juxtaposing two tensor bases to compose $\mathbf{e}_i\ldots\mathbf{e}_j\mathbf{e}_h\ldots\mathbf{e}_k$ or $\mathbf{e}_h\ldots\mathbf{e}_k\mathbf{e}_i\ldots\mathbf{e}_j$ respectively. A juxtaposed tensor basis is a tensor product of component tensor bases. If necessary, the vectors in the two bases may be arranged in a new order to define another tensor product.

Tensor products are ruled by a multilinear non-commutative algebra of base vectors, as illustrated by use of dyads in the following:

rule (1) $\quad \mathbf{e}_1(\lambda\mathbf{e}_1 + \mu\mathbf{e}_2) = \lambda\mathbf{e}_1\mathbf{e}_1 + \mu\mathbf{e}_1\mathbf{e}_2$

and $\quad (\lambda\mathbf{e}_1 + \mu\mathbf{e}_2)\mathbf{e}_1 = \lambda\mathbf{e}_1\mathbf{e}_1 + \mu\mathbf{e}_2\mathbf{e}_1$

and

rule (2) $\quad \mathbf{e}_1\mathbf{e}_2 \neq \mathbf{e}_2\mathbf{e}_1.$

Because of the multilinear property, the differentiation of a tensor product is executed factor by factor; for example,

$$\frac{\partial\mathbf{e}^1\mathbf{e}^2}{\partial x^i} = \frac{\partial\mathbf{e}^1}{\partial x^i}\mathbf{e}^2 + \mathbf{e}^1\frac{\partial\mathbf{e}^2}{\partial x^i}.$$

Taking the scalar product of two multiads is executed by taking the dot product of pairs of vectors appropriately chosen from the two multiads. For example, pairs for dotting an r-ad $\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_r}$ into an s-ad $\mathbf{e}_{j_1}\ldots\mathbf{e}_{j_s}$ are indicated by arrows with numbers at their bases:

$$\begin{array}{l}\underset{\uparrow_1}{\mathbf{e}_{i_1}}\ldots\underset{\uparrow_t}{\mathbf{e}_{i_t}}\ldots\mathbf{e}_{i_r}\cdot\underset{\uparrow_1}{\mathbf{e}_{j_1}}\ldots\underset{\uparrow_t}{\mathbf{e}_{j_t}}\ldots\mathbf{e}_{j_s}\\ = (\mathbf{e}_{i_1}\cdot\mathbf{e}_{j_1})\ldots(\mathbf{e}_{i_t}\cdot\mathbf{e}_{j_t})\mathbf{e}_{i_{t+1}}\ldots\mathbf{e}_{i_r}\mathbf{e}_{j_{t+1}}\ldots\mathbf{e}_{j_s}.\end{array} \tag{55}$$

Base vectors that are not dotted are juxtaposed according to the rule of forming a tensor product, as shown in eq 55. A different choice of pairs for dotting yields a different scalar product.

Forming the type of scalar product, as in eq 55, in which each pair for dotting is a pair of base vectors that, when counted in the respective base tensors from left to right, are labeled by the same cardinal number will be called dotting from the front.

Another type of scalar product, such as

$$\mathbf{e}_a\cdots\mathbf{e}_b\underset{\substack{\uparrow\\1}}{\mathbf{e}_{i_1}}\cdots\underset{\substack{\uparrow\\t}}{\mathbf{e}_{i_t}}\cdot\mathbf{e}_p\cdots\mathbf{e}_q\underset{\substack{\uparrow\\1}}{\mathbf{e}_{j_1}}\cdots\underset{\substack{\uparrow\\t}}{\mathbf{e}_{j_t}}$$

$$= \mathbf{e}_a\cdots\mathbf{e}_b\mathbf{e}_p\cdots\mathbf{e}_q(\mathbf{e}_{i_1}\cdot\mathbf{e}_{j_1})\cdots(\mathbf{e}_{i_t}\cdot\mathbf{e}_{j_t}) \tag{56}$$

will be called dotting from the rear, in which each pair for dotting is a pair of base vectors that, when counted from right to left, are labeled by the same cardinal number. Dotting from the front and dotting from the rear are both dotting in natural arrangement.

If the scalar product of two base tensors is obtained by dotting all the base vectors in one of the base tensors, the process is called dotting in full. If the orders of two tensors are equal, dotting from the front and dotting from the rear are the same type of scalar product, called dotting in full in natural arrangement.

An arbitrary set of Cartesian base vectors is obtained by rotating the standard Cartesian base vectors. A Cartesian expression of tensor **T** is a form of **T** with a set of Cartesian base vectors forming base tensors. The standard Cartesian expression of **T** is a particular Cartesian expression in which the standard Cartesian base vectors $\mathbf{c}_a$ are used for forming base tensors.

Theorem 2

Let **T** be a tensor of any order defined in n-dimensional Euclidean space. The components of **T** in the standard Cartesian expression are assumed to be differentiable at least twice. Then with whatever base vectors **T** may be expressed, the order of differentiation in a second-order derivative of **T** can be exchanged:

$$\frac{\partial^2 \mathbf{T}}{\partial x^i \partial x^j} = \frac{\partial^2 \mathbf{T}}{\partial x^i \partial x^i} . \tag{57}$$

Proof. Equation 57 is true for **T** in the standard Cartesian expression. Then, eq 57 must be true for any expression of **T**, because **T** is invariant under coordinate transformations. Q.E.D.

EXTERIOR PRODUCT OF VECTORS

An exterior product* of r vectors is an algebraic expression of a geometric quantity formed by an ordered set of r vectors. It is defined in the following by induction.

An exterior product $\mathbf{a}\wedge\mathbf{b}$ of vectors **a** and **b** is formed by use of the wedge product $\wedge$, called wedging, defined by rules (1) and (2) below:

rule (1): $\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}$

rule (2): $\mathbf{a}\wedge(\lambda\mathbf{b}+\mu\mathbf{c}) = \lambda\mathbf{a}\wedge\mathbf{b}+\mu\mathbf{a}\wedge\mathbf{c}$

$(\lambda\mathbf{a}+\mu\mathbf{b})\wedge\mathbf{c} = \lambda\mathbf{a}\wedge\mathbf{c}+\mu\mathbf{b}\wedge\mathbf{c}$

where λ and μ are scalars.

Assume that all the exterior products of up to r-1 vectors, where $r \geqq 3$, are defined. Then an exterior product

$$\mathbf{R} = \mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_r \tag{58}$$

*Flanders (1963), Fleming (1965), Willmore (1959) and others explain exterior products. Only elemental knowledge of finite-dimensional Euclidean spaces is used in this paper for introducing exterior products of vectors.

of r vectors is defined if the product $\wedge$ satisfies rule (3) below.

For any s satisfying $1 \leqq s \leqq r-1$, let $\mathbf{T}_s$ and $\mathbf{U}_s$ be

$$\mathbf{T}_s = \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_s$$

and

$$\mathbf{U}_s = \mathbf{a}_{s+1} \wedge \cdots \wedge \mathbf{a}_r .$$

Then,

rule (3): $\mathbf{T}_s \wedge \mathbf{U}_s$ is unique.

The exterior product **R** in eq 58 is defined by $\mathbf{R} = \mathbf{T}_s \wedge \mathbf{U}_s$.

Rule (3) is necessary because an exterior product of vectors represents a geometric quantity that is independent of the process of construction.

The consistency of algebraic rules (1), (2), and (3) is shown by Theorem 3 below.

<u>Theorem 3</u>

R in eq 58 may be identified with the determinant,

$$\mathbf{R} = \begin{vmatrix} \mathbf{a}_1 \ldots \mathbf{a}_r \\ \cdots\cdots \\ \mathbf{a}_1 \ldots \mathbf{a}_r \end{vmatrix} \tag{59}$$

which is calculated as a sum of signed products in the usual way, with the convention that the first, ..., rth member of the r-ads that appear on developing the determinant must be elements from the first, ..., $\underline{r}$th row, respectively, of the determinant under consideration.

<u>Proof.</u> Use of rules (1), (2), and (3) shows that the transformation

$$\mathbf{a}_i = \xi_i^{\,j}\, \mathbf{b}_j \ (i,\ j = 1, \ldots, r) \tag{60}$$

changes **R** in eq 58 to

$$\mathbf{R} = \begin{vmatrix} \xi_1^1 \ldots \xi_r^1 \\ \cdots\cdots \\ \xi_1^r \ldots \xi_r^r \end{vmatrix} \mathbf{S}, \tag{61}$$

where

$$\mathbf{S} = \mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_r .$$

The determinant forms of R and **S**, D(**R**) and D(**S**) say, transform in exactly the same way as given in terms of R and S in eq 61. This fact allows us to write $\mathbf{R} = \rho D(\mathbf{R})$, where ρ is a constant. ρ is arbitrarily made equal to 1 in eq 59. Q.E.D.

The convention mentioned in Theorem 3 for developing the determinant (eq 59) gives a tensor expression of **R**

$$\mathbf{R} = \overset{\circ}{\epsilon}{}^{i\ldots h}\, \mathbf{a}_i \ldots \mathbf{a}_h \tag{63}$$

as the development of eq 59, where $i \ldots h$ is a permutation of $1 \ldots r$ and $\overset{\circ}{\epsilon}{}^{i \ldots h}$ is a permutation symbol.

Exterior products of two, three, four, ..., r vectors are called bivectors, trivectors, ..., and r-vectors. They are called multivectors as a whole. Their geometric meaning is shown in the next theorem.

Theorem 4

It is consistent to interpret R in eq 59 as an algebraic expression of the oriented $\underline{r}$-dimensional parallelepiped spanned by r vectors, $\mathbf{a}_1, \ldots, \mathbf{a}_r$.

Proof. Choose Cartesian base vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ in the $\underline{r}$-dimensional subspace spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_r$. Transformation

$$\mathbf{a}_i = a_i{}^j \, \mathbf{u}_j \tag{64}$$

changes $\mathbf{R}$ in eq 59 to

$$R = V\mathbf{O}_r \tag{65}$$

where

$$\mathbf{O}_r = \begin{vmatrix} \mathbf{u}_1 \ldots \mathbf{u}_r \\ \ldots\ldots\ldots \\ \mathbf{u}_1 \ldots \mathbf{u}_r \end{vmatrix} \tag{66}$$

$\underline{V}$ in eq 65 is the determinant of the matrix $(a_i{}^j)$, that is, the "signed volume" of an oriented parallelepiped spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_r$.

Equation 65 shows the consistency of the interpretation that **R** and **O** are respectively algebraic expressions of the oriented $\underline{r}$-dimensional parallelepiped spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_r$ and of the oriented unit cube spanned by $\mathbf{u}_1, \ldots, \mathbf{u}_r$. Q.E.D.

The $\mathbf{O}_r$ in eq 66 defines the orientation of the subspace spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_r$.

Corollary

An $\underline{r}$-vector is an $\underline{r}$th order tensor whose basis is an orientation of the $\underline{r}$-vector and whose component is a scalar equal to the volume of the $\underline{r}$-dimensional parallelepiped spanned by the $\underline{r}$-vector.

The scalar product of multivectors is obtained as shown below.

Proposition 3

Dotting

$$\mathbf{R} = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_r \tag{67}$$

into

$$\mathbf{S} = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \ldots \wedge \mathbf{b}_r \tag{68}$$

in full in natural arrangement yields

$$\mathbf{R}\cdot\mathbf{S} = r!\begin{vmatrix} \mathbf{a}_1\cdot\mathbf{b}_1 \ldots \mathbf{a}_r\cdot\mathbf{b}_1 \\ \ldots\ldots\ldots\ldots \\ \mathbf{a}_1\cdot\mathbf{b}_r \ldots \mathbf{a}_r\cdot\mathbf{b}_r \end{vmatrix} . \tag{69}$$

<u>Proposition 4</u>

Dotting

$$\mathbf{R} = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_r \tag{70}$$

into

$$\mathbf{S} = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_{s+1} \wedge \cdots \wedge \mathbf{b}_{s+r} \tag{71}$$

in full from the rear yields

$$\mathbf{R}\cdot\mathbf{S} = \sum_{\substack{k_1<\ldots<k_s \\ k_{s+1}<\ldots<k_{s+r}}} \mathring{\epsilon}^{k_1\ldots k_s k_{s+1}\ldots k_{s+r}} \left[(\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_r)\cdot(\mathbf{b}_{k_{s+1}}\wedge\ldots\wedge\mathbf{b}_{k_{s+r}})\right] \mathbf{b}_{k_1}\wedge\ldots\wedge\mathbf{b}_{k_s}, \tag{72}$$

where the dotting in the brackets is in full in natural arrangement.

<u>Proof.</u> Both propositions 3 and 4 are proved in the following.

Scalar products of multivectors are calculated by dotting the developments of determinant forms of multivectors. Thus, dotting R in eq 70 into S in eq 71 from the rear yields

$$\mathbf{R}\cdot\mathbf{S} = \epsilon^{i_1\ldots i_r}\,\underset{\uparrow_1}{\mathbf{a}_{i_1}}\ldots\underset{\uparrow_r}{\mathbf{a}_{i_r}} \cdot \mathring{\epsilon}^{j_1\ldots j_{s+1}\ldots j_{s+r}}\,\mathbf{b}_{j_1}\ldots\underset{\uparrow_1}{\mathbf{b}_{j_{s+1}}}\ldots\underset{\uparrow_r}{\mathbf{b}_{j_{s+r}}} \tag{a}$$

$$= \mathring{\epsilon}^{i_1\ldots i_r}\,\mathring{\epsilon}^{j_1\ldots j_{s+1}\ldots j_{s+r}}\,(\mathbf{a}_{i_1}\cdot\mathbf{b}_{j_{s+1}})\ldots(\mathbf{a}_{i_r}\cdot\mathbf{b}_{j_{s+r}})\mathbf{b}_{j_1}\ldots\mathbf{b}_{j_s} \tag{b}$$

$$= \mathring{\epsilon}^{j_1\ldots j_{s+1}\ldots j_{s+r}}\begin{vmatrix} \mathbf{a}_1\cdot\mathbf{b}_{j_{s+1}} \ldots \mathbf{a}_1\cdot\mathbf{b}_{j_{s+r}} \\ \ldots\ldots\ldots\ldots \\ \mathbf{a}_r\cdot\mathbf{b}_{j_{s+1}} \ldots \mathbf{a}_r\cdot\mathbf{b}_{j_{s+r}} \end{vmatrix}\mathbf{b}_{j_1}\ldots\mathbf{b}_{j_s} \tag{c}$$

$$= r!\sum_{k_{s+1}<\ldots<k_{s+r}} \mathring{\epsilon}^{j_1\ldots j_s k_{s+1}\ldots k_{s+r}}\begin{vmatrix} \mathbf{a}_1\cdot\mathbf{b}_{k_{s+1}} \ldots \mathbf{a}_1\cdot\mathbf{b}_{k_{s+r}} \\ \ldots\ldots\ldots\ldots \\ \mathbf{a}_r\cdot\mathbf{b}_{k_{s+1}} \ldots \mathbf{a}_r\cdot\mathbf{b}_{k_{s+r}} \end{vmatrix}\mathbf{b}_{j_1}\ldots\mathbf{b}_{j_s} \tag{d}$$

$$= r! \sum_{\substack{k_1<\ldots<k_s \\ k_{s+1}<\ldots<k_{s+r}}} \overset{\circ}{\epsilon}{}^{k_1\ldots k_s k_{s+1}\ldots k_{s+r}} \begin{vmatrix} a_1\cdot b_{k_{s+1}} \ldots a_1\cdot b_{k_{s+r}} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_r\cdot b_{k_{s+1}} \ldots a_r\cdot b_{k_{s+r}} \end{vmatrix} b_{k_1} \wedge \ldots \wedge b_{k_s} \qquad \text{(e)}$$

Equation a shows that the dotting is in full from the rear. Dotting in eq a gives eq b. Summing over $i_1, \ldots, i_r$ in eq b yields eq c. Summing over $j_{s+1}, \ldots, j_{s+r}$ in eq c yields eq d, where $k_{s+1}, \ldots, k_{s+r}$ are in increasing order. Summing over $j_1, \ldots, j_s$ in eq d yields eq e, where $k_1, \ldots, k_s$ are in increasing order.

For a special case in which s=r, the above results yield

$$(a_1 \wedge \ldots \wedge a_r)\cdot(b_1 \wedge \ldots \wedge b_r) = r! \begin{vmatrix} a_1\cdot b_1 \ldots a_r\cdot b_1 \\ \\ a_1\cdot b_r \ldots a_r\cdot b_r \end{vmatrix} \qquad \text{(f)}$$

which proves Proposition 3. Using Proposition 3 changes eq e to eq 72. Q.E.D.

Proposition 5

Dotting

$$R = a_1 \wedge a_2 \wedge \ldots \wedge a_r \qquad (73)$$

into

$$S = b_1 \wedge b_2 \wedge \ldots \wedge b_2 \wedge \ldots \wedge b_{r+s} \qquad (74)$$

in full from the front yields

$$R \cdot S = \sum_{\substack{k_1<\ldots<k_r \\ k_{r+1}<\ldots<k_{r+s}}} \overset{\circ}{\epsilon}{}^{k_1\ldots k_r k_{r+1}\ldots k_{r+s}} \left[(a_1 \wedge \ldots \wedge a_r)\cdot(b_{k_1} \wedge \ldots \wedge b_{k_r})\right] b_{k_{r+1}} \wedge \ldots \wedge b_{k_{r+s}} \qquad (75)$$

where the dotting in the brackets is in full in natural arrangement.

Proof.

$$R \cdot S = \overset{\circ}{\epsilon}{}^{i_1\ldots i_r} \underset{\uparrow\ \ldots\ \uparrow \atop 1 \quad\ \ r}{a_{i_1}\ldots a_{i_r}} \cdot \overset{\circ}{\epsilon}{}^{j_1\ldots j_r\ldots j_{r+s}} \underset{\uparrow\ \ldots\ \uparrow \atop 1 \quad\ \ r}{b_{j_1}\ldots b_{j_r}}\ldots b_{j_{r+s}} \qquad \text{(a)}$$

$$= \overset{\circ}{\epsilon}{}^{i_1\ldots i_r} \overset{\circ}{\epsilon}{}^{j_1\ldots j_r\ldots j_{r+s}} (a_{i_1}\cdot b_{j_1})\ldots(a_{i_r}\cdot b_{j_r})\, b_{j_{r+1}}\ldots b_{j_{r+s}} \qquad \text{(b)}$$

$$= \overset{\circ}{\epsilon}{}^{j_1\ldots j_r\ldots j_{r+s}} \begin{vmatrix} a_1\cdot b_{j_1} \ldots a_1\cdot b_{j_r} \\ \ldots\ldots\ldots\ldots \\ a_r\cdot b_{j_1} \ldots a_r\, b_{j_r} \end{vmatrix} b_{j_{r+1}}\ldots b_{j_{r+s}} \qquad \text{(c)}$$

$$= r! \sum_{k_1<\ldots<k_r} \overset{\circ}{\epsilon}^{k_1 \ldots k_r j_{r+1} \ldots j_{r+s}} \begin{vmatrix} a_1 \cdot b_{k_1} \ldots a_2 \cdot b_{k_r} \\ \ldots\ldots\ldots\ldots \\ a_r \cdot b_{k_1} \ldots a_r \cdot b_{k_r} \end{vmatrix} b_{j_{r+1}} \ldots b_{j_{r+s}} \qquad \text{(d)}$$

$$= r! \sum_{\substack{k_1<\ldots<k_r \\ k_{r+1}<\ldots<k_{r+s}}} \overset{\circ}{\epsilon}^{k_1 \ldots k_r k_{r+1} \ldots k_{r+s}} \begin{vmatrix} a_1 \cdot b_{k_1} \ldots a_1 \cdot b_{k_r} \\ \ldots\ldots\ldots\ldots \\ a_r \cdot b_{k_1} \ldots a_1 \cdot b_{k_r} \end{vmatrix} b_{k_{r+1}} \ \ldots \ b_{k_{r+s}} \qquad \text{(e)}$$

$$= (75).$$

The calculation process is the same as in Proposition 4. Q.E.D.

Corollary

Let $i, \ldots, j$ and $h, \ldots, k$ be two sets of r integers chosen from $1, \ldots, n$. Then,

$$\mathbf{e}_i \wedge \ldots \wedge \mathbf{e}_j \cdot \mathbf{e}^h \wedge \ldots \wedge \mathbf{e}^k = r! \ \delta^{h \ldots k}_{i \ldots j} \qquad (76)$$

where $\delta^{h \ldots k}_{i \ldots j}$ is a generalized Kronecker delta given by Veblen (1927, p. 3). The dotting in eq 76 is in full in natural arrangement.

Proof.

$$\mathbf{e}_i \wedge \ldots \wedge \mathbf{e}_j \cdot \mathbf{e}^h \wedge \ldots \wedge \mathbf{e}^k \qquad \text{(a)}$$

$$= r! \begin{vmatrix} \delta_i^h \ldots \delta_j^k \\ \ldots\ldots\ldots \\ \delta_i^k \ldots \delta_j^k \end{vmatrix} \qquad \text{(b)}$$

$$= r! \ \delta^{h \ldots k}_{i \ldots j}. \qquad \text{(c)}$$

Equation a becomes eq b by use of eq 69. The determinant in eq b is equal to zero if $h \ldots k$ is not a permutation of $i \ldots j$. If $h \ldots k$ is a permutation of $i \ldots j$, rearranging $h \ldots k$ into $i \ldots j$ yields the determinant of the unit matrix. Therefore the value of the determinant in eq b is equal to a generalized Kronecker delta $\delta^{h \ldots k}_{i \ldots j}$. Q.E.D.

The epsilon tensor $\boldsymbol{\epsilon}$ is the exterior product of the standard Cartesian base vectors $\mathbf{c}_a (a=1, \ldots, n)$:

$$\boldsymbol{\epsilon} = \mathbf{c}_1 \wedge \mathbf{c}_2 \wedge \ldots \wedge \mathbf{c}_n. \qquad (77)$$

It may be written in a tensor form

$$\boldsymbol{\epsilon} = \overset{\circ}{\epsilon}^{ab \ldots c} \mathbf{c}_a \mathbf{c}_b \ldots \mathbf{c}_c, \qquad (78)$$

where $ab \ldots c$ is a permutation of $12 \ldots n$, and $\overset{\circ}{\epsilon}^{ab \ldots c}$ is a permutation symbol. $\boldsymbol{\epsilon}$ defines the orientation of n-dimensional Euclidean space.

The cube spanned by the standard Cartesian base vectors $\mathbf{c}_a$ will be called the standard unit cube.

Proposition 6

All the n-dimensional unit cubes are equal to either $+\epsilon$ or $-\epsilon$.

Proof. Use of eq 65 immediately proves the proposition. Q.E.D.

Proposition 7

ϵ is expressed in a skew coordinate system spanned by $\mathbf{e}_i$, $\mathbf{e}^i$ by

$$\epsilon = \frac{1}{V}\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n \tag{79}$$

$$= V\,\mathbf{e}^1 \wedge \ldots \wedge \mathbf{e}^n \tag{80}$$

where V is the signed volume of the oriented parallelepiped spanned by the ordered set $\mathbf{e}_1, \ldots, \mathbf{e}_n$. The reciprocal $1/V$ is the signed volume of the oriented parallelepiped spanned by the ordered set $\mathbf{e}^1, \ldots, \mathbf{e}^n$.

Proof. Use of eq 65 yields

$$\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n = V\,\mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_n \tag{81}$$

which proves eq 79. The equation

$$\mathbf{e}^1 \wedge \ldots \wedge \mathbf{e}^n = \frac{1}{V}\,\mathbf{c}_1 \wedge \ldots \wedge \mathbf{c}_n \tag{82}$$

is also true, as may be shown by dotting eq 82 into eq 81 in full in natural arrangement. Q.E.D.

Proposition 8

ϵ is the tensor expression of the unit for measuring n-dimensional volume.

Proof. Proposition 8 is evidently true when Propositions 6 and 7 are considered respectively for Cartesian coordinates and for skew linear coordinates. Q.E.D.

Proposition 9

The orientation O_r of an r-vector $\mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_r$ is an epsilon tensor defined in an r-dimensional space spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_r$.

It is customary to write

$$V = \sqrt{g} \tag{83}$$

on the assumption that the unit cube $\mathbf{c}_1 \wedge \ldots \wedge \mathbf{c}_n$ and the parallelepiped $\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n$ have the same orientation. Then eq 79 and eq 80 respectively yield

$$\epsilon = \frac{1}{\sqrt{g}}\,\mathring{\epsilon}^{i\ldots j}\,\mathbf{e}_i \ldots \mathbf{e}_j \tag{84}$$

$$= \sqrt{g}\,\mathring{\epsilon}_{i\ldots j}\,\mathbf{e}^i \ldots \mathbf{e}^j \tag{85}$$

where $i\ldots j$ is a permutation of $1\ldots n$ and both $\mathring{\epsilon}^{i\ldots j}$ and $\mathring{\epsilon}_{i\ldots j}$ are permutation symbols. Equations 84 and 85 determine the components $\epsilon^{i\ldots j}$ and $\epsilon_{i\ldots j}$ of the ϵ in the skew coordinate system:

$$\epsilon^{i\ldots j} = \frac{1}{\sqrt{g}} \overset{\circ}{\epsilon}{}^{i\ldots j} \tag{86}$$

and

$$\epsilon_{i\ldots j} = \sqrt{g}\, \overset{\circ}{\epsilon}_{i\ldots j} \,. \tag{87}$$

<u>Proposition 10</u>

The $\underline{r}$-dimensional faces of the standard unit cube are the base $\underline{r}$-vectors for the standard Cartesian expression of an $\underline{r}$-vector.

<u>Proof.</u> Substituting

$$\mathbf{a}_i = \xi_i^{\,a}\, \mathbf{c}_a \tag{88}$$

into **R** of eq 67 yields

$$\mathbf{R} = \xi_1^{\,a} \xi_2^{\,b} \ldots \xi_r^{\,c}\, \mathbf{c}_a \wedge \mathbf{c}_b \wedge \ldots \wedge \mathbf{c}_c \quad , \tag{89}$$

where ab...c is a set of r numbers chosen from 1...n. Numbers a, b, ..., c are all different because they are suffixes of an exterior product.

Summing up all the exterior products whose suffixes are derangements of ab ...c changes eq 89 to

$$\mathbf{R} = \sum_{(ab\ldots c)} \begin{vmatrix} \xi_1^a \; \xi_1^b \ldots \xi_1^c \\ \ldots\ldots\ldots \\ \xi_r^a \; \xi_r^b \ldots \xi_r^c \end{vmatrix} \quad \mathbf{c}_a \wedge \mathbf{c}_b \wedge \ldots \wedge \mathbf{c}_c \quad , \tag{90}$$

where the summation on the right-hand side, indicated by (ab...c), is over all the different combinations of r different numbers chosen from 1, ..., n. The $\underline{r}$-vector $\mathbf{c}_a \wedge \mathbf{c}_b \wedge \ldots \wedge \mathbf{c}_c$ on the right-hand side is an $\underline{r}$-dimensional face of the standard unit cube. Equation 90 is the standard Cartesian expression of **R**. Q.E.D.

Equation 90 is a breakdown of **R** into the base $\underline{r}$-vectors. The square of the magnitude of **R** is given by $\mathbf{R}\cdot\mathbf{R}$. The sign of the magnitude of **R** is determined by the orientation of **R**.

The orientation $\mathbf{O}_r$ of an $\underline{r}$-dimensional subspace S_r is said to have the same sign as, or the opposite sign to, the orientation $\mathbf{O}_{n-r}$ of the subspace S_{n-r} complementary to S_r, when the condition

$$\mathbf{O}_{n-r} \wedge \mathbf{O}_r = k\boldsymbol{\epsilon} \tag{91}$$

or

$$\mathbf{O}_{n-r} \wedge \mathbf{O}_r = -k\boldsymbol{\epsilon} \tag{92}$$

is satisfied, where k is a positive number.

<u>Proposition 11</u>

An (n-1)-dimensional face

$$\mathbf{S} = \mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_{s-1} \wedge \mathbf{c}_{s+1} \wedge \cdots \wedge \mathbf{c}_n \tag{93}$$

of the standard unit cube has the orientation of the sign $(-1)^{s-1}$.

Proof. Transferring $\mathbf{c}_s$ in $\boldsymbol{\epsilon} = \mathbf{c}_1 \wedge \ldots \wedge \mathbf{c}_n$ to the extreme left of $\boldsymbol{\epsilon}$ yields

$$\mathbf{c}_s \wedge \mathbf{S} = (-1)^{s-1} \boldsymbol{\epsilon}. \tag{94}$$

$\mathbf{c}_s$ in eq 94 is in the direction of dx^s and is positively oriented. Then $(-1)^{s-1}\,\mathbf{S}$ is the positive orientation. Q.E.D.

The orientation given in Proposition 11 conforms with the orientation defined in combinatory topology for an (n-1)-dimensional face of an n-dimensional simplex (see, for example, Cairns, 1961, p. 92).

Lemma 1

$$\sqrt{g} = \frac{1}{n!} (\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n) \cdot (\mathbf{c}_1 \wedge \ldots \wedge \mathbf{c}_n) . \tag{95}$$

Proof. Dotting $\mathbf{c}_1 \wedge \ldots \wedge \mathbf{c}_n$ into both sides of eq 81 in full in natural arrangement yields eq 95, where use is made of eq 69 for executing dotting and of the relation $V = \sqrt{g}$. Q.E.D.

Lemma 2

$$\frac{\partial \sqrt{g}}{\partial x^i} = \Gamma^p_{pi} \sqrt{g} . \tag{96}$$

Proof. Differentiating eq 95 with regard to x^i yields

$$\frac{\partial \sqrt{g}}{\partial x^i} = \frac{1}{n!} \sum_{j=1}^{n} (\mathbf{e}_1 \wedge \ldots \wedge \frac{\partial \mathbf{e}_j}{\partial x^i} \wedge \ldots \wedge \mathbf{e}_n) \cdot (\mathbf{c}_1 \wedge \ldots \wedge \mathbf{c}_n) \tag{a}$$

$$= \frac{1}{n!} \sum_{j=1}^{n} (\mathbf{e}_1 \wedge \ldots \wedge \Gamma^k_{ji}\, \mathbf{e}_k \wedge \ldots \wedge \mathbf{e}_n) \cdot (\mathbf{c}_1 \wedge \ldots \wedge \mathbf{c}_n) \tag{b}$$

$$= \frac{1}{n!} \Gamma^j_{ji} (\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n) \cdot (\mathbf{c}_1 \wedge \ldots \wedge \mathbf{c}_n) \tag{c}$$

$$= \Gamma^p_{pi} \sqrt{g}. \tag{d}$$

Equation a is obtained by differentiating $\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n$ in eq 95 factor by factor. Equation a becomes eq b by use of eq 33. $\Gamma^k_{ji}\,\mathbf{e}_k$ in eq b, where k is a summation index, is a sum of n terms, but only one term in which k = j survives, because an exterior product in which a vector is repeated at least twice is equal to zero. The sum of the survivors in eq b yields eq c. Use of eq 95 changes eq c to eq d. Q.E.D.

Theorem 5

The unit tensor $\mathbf{1}$ and the epsilon tensor $\boldsymbol{\epsilon}$ are constant tensors whose derivatives with regard to a space coordinate are all zero:

$$\frac{\partial \mathbf{1}}{\partial x^i} = 0 \tag{97}$$

and

$$\frac{\partial \boldsymbol{\epsilon}}{\partial x^i} = 0. \tag{98}$$

Proof. The proposition is evidently true, because **1** and $\boldsymbol{\epsilon}$ can be given the standard Cartesian expressions even in a tangent space spanned to a curvilinear coordinate system.

The proposition may be directly proved by use of the formulas developed so far for curvilinear coordinates:

$$\frac{\partial \mathbf{1}}{\partial x^i} = \frac{\partial \mathbf{e}_i \mathbf{e}^i}{\partial x^j} \tag{a}$$

$$= \frac{\partial \mathbf{e}_i}{\partial x^j}\, \mathbf{e}^i + \mathbf{e}_i\, \frac{\partial \mathbf{e}^i}{\partial x^j} \tag{b}$$

$$= \Gamma_{ij}^{k}\, \mathbf{e}_k \mathbf{e}^i + \mathbf{e}_i\, (-\Gamma_{jk}^{i}\, \mathbf{e}^k) \tag{c}$$

$$= \Gamma_{ij}^{k}\, \mathbf{e}_k \mathbf{e}^i - \Gamma_{ji}^{k}\, \mathbf{e}_k \mathbf{e}^i \tag{d}$$

$$= 0.$$

Equation b becomes eq c by use of eq 33 and 38. Equation c becomes eq d by exchanging summation indexes i and k in the second term. Thus eq 97 is proved.

$$\frac{\partial \boldsymbol{\epsilon}}{\partial x^h} \tag{e}$$

$$= \frac{\partial}{\partial x^h} (\sqrt{g}\, \mathbf{e}^1 \wedge \ldots \wedge \mathbf{e}^n) \tag{f}$$

$$= \frac{\partial \sqrt{g}}{\partial x^h}\, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n + \sqrt{g} \sum_{\nu=1}^{n} \mathbf{e}^1 \wedge \cdots \wedge \frac{\partial \mathbf{e}^\nu}{\partial x^h} \wedge \cdots \wedge \mathbf{e}^n \tag{g}$$

$$= \sqrt{g}\, \Gamma_{ph}^{p}\, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n - \sqrt{g} \sum_{\nu=1}^{n} \mathbf{e}^1 \wedge \cdots \wedge \Gamma_{hp}^{\nu}\, \mathbf{e}^p \wedge \cdots \wedge \mathbf{e}^n \tag{h}$$

$$= \sqrt{g}\, \Gamma_{ph}^{p}\, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n - \sqrt{g}\, \Gamma_{hp}^{p}\, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n \tag{i}$$

$$= 0. \tag{j}$$

Equation e becomes eq f by expressing $\boldsymbol{\epsilon}$ by use of eq 80 with $V = \sqrt{g}$. Differentiating eq f factor by factor yields eq g. Equation g becomes eq h by use of eq 96 and 38. $\Gamma_{hp}^{\nu}\, \mathbf{e}^p$ in eq h, where p is a summation index, is a sum of n terms; but only one term in which $p = \nu$ survives because an exterior product in which a vector is repeated at least twice is equal to zero. The sum of all the survivors is the second term of eq i, which is equal to the first term of eq i because of eq 34. Thus eq 98 is proved.

Theorem 6

An alternating tensor of order r is a tensor expressible in the form of a linear combination of r-vectors.

Proof. An alternating tensor is defined as a tensor in which (1) any derangement of suffixes of a component yields another component of the same tensor and (2) a component $T^{i..j..k..h}$ changes sign when its two arbitrary suffixes j, k are exchanged

$$T^{i..k..j..h} = -T^{i..j..k..h}.$$

The sum of all the base tensors whose components are obtained by permuting i..j..k..h in $T^{i..j..k..h}$ yields an exterior product of base vectors. An alternating tensor of order r, therefore, can be expressed as a linear combination of r-vectors.

Conversely, a tensor **S** defined by

$$\mathbf{S} = \sum_{(i...h)} S^{i...h}\, \mathbf{e}_i \wedge \ldots \wedge \mathbf{e}_h \tag{99}$$

is an alternating tensor, where the summation is over the combinations (i..h) or r different letters chosen from 1, . . . , n.

To prove the last proposition, expand the r-vectors in eq 99 as sums of tensor products:

$$\mathbf{S} = S^{i..h}\, \delta^{\lambda..\omega}_{i..h}\, \mathbf{e}_\lambda..\mathbf{e}_\omega \tag{100}$$

where $\delta^{\lambda..\omega}_{\alpha..h}$ is a generalized Kornecker delta. Then components

$$T^{\lambda..\mu..\nu..\omega} = S^{i..j..k..h}\, \delta^{\lambda..\mu..\nu..\omega}_{i..j..k..h} \tag{101}$$

are alternating, because

$$\begin{aligned} T^{\lambda..\nu..\lambda..\omega} &= S^{i..j..k..h}\, \delta^{\lambda..\nu..\mu..\omega}_{i..j..k..h} \\ &= S^{i..j..k..h}\, \delta^{\lambda..\mu..\nu..\omega}_{i..j..k..h} \\ &= -T^{\lambda..\mu..\nu..\omega}. \end{aligned}$$

Theorem 7

Subindexed coordinates x_k are defined only for skew-linear coordinates.

Proof. Equation 12 becomes integrable if and only if equation

$$\frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{kj}}{\partial x^i} = 0 \tag{102}$$

is true for any three numbers i, j, k chosen from 1, . . . , n.

The left-hand side of eq 102 is transformed:

$$\frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{kj}}{\partial x^i} \tag{a}$$

$$= \frac{\partial \mathbf{e}_k \cdot \mathbf{e}_i}{\partial x^j} - \frac{\partial \mathbf{e}_k \cdot \mathbf{e}_j}{\partial x^i} \tag{b}$$

$$= \left\{\frac{\partial \mathbf{e}_k}{\partial x^j} \cdot \mathbf{e}_i + \mathbf{e}_k \cdot \frac{\partial \mathbf{e}_i}{\partial x^j}\right\} - \left\{\frac{\partial \mathbf{e}_k}{\partial x^i} \cdot \mathbf{e}_j + \mathbf{e}_k \cdot \frac{\partial \mathbf{e}_j}{\partial x^i}\right\} \tag{c}$$

$$= \frac{\partial \mathbf{e}_j}{\partial x^k} \cdot \mathbf{e}_i - \frac{\partial \mathbf{e}_i}{\partial x^k} \cdot \mathbf{e}_j \,. \tag{d}$$

Equation c becomes eq d by use of eq 37.

A transformation of eq 102,

$$\mathbf{e}_i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} = \frac{\partial \mathbf{e}_i}{\partial x^k} \cdot \mathbf{e}_j \,, \tag{e}$$

is juxtaposed with $\mathbf{e}^i$ on the left on both sides to yield

$$\frac{\partial \mathbf{e}_j}{\partial x^k} = \mathbf{e}^i \frac{\partial \mathbf{e}_i}{\partial x^k} \cdot \mathbf{e}_j \,. \tag{f}$$

Juxtaposing $\mathbf{e}^j$ on the right on both sides of eq f yields

$$\frac{\partial \mathbf{e}_j}{\partial x^k} \mathbf{e}^j = \mathbf{e}^i \frac{\partial \mathbf{e}_i}{\partial x^k} \,. \tag{g}$$

Changing the summation index j to i, eq g becomes

$$\mathbf{e}^i \wedge \frac{\partial \mathbf{e}_i}{\partial x^k} = 0 \,. \tag{h}$$

Use of eq 37 changes eq h to

$$\mathbf{e}^i \wedge \frac{\partial \mathbf{e}_k}{\partial x^i} = 0 \,. \tag{i}$$

Substituting $\mathbf{e}_k = g_{kh}\mathbf{e}^h$, eq i becomes

$$\mathbf{e}^i \wedge \mathbf{e}^h \frac{\partial g_{kh}}{\partial x^i} = 0 \tag{j}$$

where the formula

$$\mathbf{e}^i \wedge \frac{\partial \mathbf{e}^h}{\partial x^i} = 0 \tag{103}$$

is used. Equation 103 is evidently true because

$$\mathbf{e}^i \wedge \frac{\partial \mathbf{e}^h}{\partial x^i} = \mathbf{e}^i \wedge (- \Gamma^h_{ij}\, \mathbf{e}^j) \tag{k}$$

$$= 0 \tag{l}$$

where eq k becomes zero because of eq 34. Equation j shows that for $i \neq h$

$$\frac{\partial g_{kh}}{\partial x^i} = 0 \,. \tag{m}$$

Therefore g_{kh} is a function of x^h only.

Further use of eq 102 yields that for $i \neq j$

$$g_{ij} = \text{const.} \tag{n}$$

To show this, set $k = j = 1$ in eq 102. Then

$$\frac{\partial g_{1i}}{\partial x^1} = \frac{\partial g_{11}}{\partial x^i} . \tag{o}$$

The right-hand side of eq o is equal to zero, if $i \neq 1$, because of eq m. Therefore g_{1i}, where $i \neq 1$, does not contain x^1, proving eq n.

If, as shown above, $g_{11}, \ldots, g_{nn}$ are respectively functions of $x^1, \ldots, x^n$ only, then use of eq 8 shows that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are respectively functions of $x^1, \ldots, x^n$ only. Then g_{ij} is a function of x^i and x^j, which, if $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are not orthogonal to each other, is contrary to eq n. Therefore, if $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are not orthogonal to each other, only a skew linear coordinate system is possible.

To discuss the case of orthogonal curvilinear coordinates, suffix 1 will again be used. Use of eq 33 and 47 yields

$$\frac{\partial \mathbf{e}_1}{\partial x^i} = \mathbf{\Gamma}_{j1i}\, \mathbf{e}^j . \tag{p}$$

Use of eq 46 shows that, under the present assumption, only Γ_{111} is not zero. Therefore eq p becomes

$$\left.\begin{aligned} &\frac{\partial \mathbf{e}_1}{\partial x^1} = \Gamma_{111}\, \mathbf{e}^1 \\ &\text{and} \\ &\frac{\partial \mathbf{e}_1}{\partial x^i} = 0 \qquad \text{for } i \neq 1 \end{aligned}\right\} \tag{q}$$

where

$$2\Gamma_{111} = \frac{\partial g_{11}}{\partial x^1} . \tag{r}$$

Changing $\mathbf{e}^1$ to $\mathbf{e}_1$ on the right-hand side of the first equation in (q) yields

$$\frac{\partial \mathbf{e}_1}{\partial x^1} = \Gamma_{111}\, g^{11}\, \mathbf{e}_1 . \tag{s}$$

Because $\Gamma_{111}\, g^{11}$ is a function of x^1 only, eq s is integrated to

$$\mathbf{e}_1 = \mathbf{u}_1 \exp\left(\int \Gamma_{111}\, g^{11}\, dx^1 \right) \tag{t}$$

where $\mathbf{u}_1$ is a constant vector. Use of $\mathbf{u}_1$ instead of $\mathbf{e}_1$ gives an orthogonal linear coordinate system. Therefore, the latter case is a special skew linear coordinate system. Q. E. D.

CROSS PRODUCT

The vector product defined in three-dimensional Euclidean space is extended here to an operation in n-dimensional Euclidean space.

Let **S** be an exterior product of a finite number of vectors. The cross product of an ordered set of t exterior products $\mathbf{S}_1, \ldots, \mathbf{S}_t$, denoted by $\mathbf{S}_1 \times \ldots \times \mathbf{S}_t$, is defined by

$$\mathbf{S}_1 \times \ldots \times \mathbf{S}_t = \frac{1}{r!} \epsilon \cdot (\mathbf{S}_1 \wedge \cdots \wedge \mathbf{S}_t) \tag{104}$$

where r is the number of independent vectors contained in the set $\mathbf{S}_1, \ldots, \mathbf{S}_t$. The dotting on the right-hand side of eq 104 is, if not otherwise stated, executed in full from the rear. If all the vectors in the exterior products $\mathbf{S}_1, \ldots, \mathbf{S}_t$ are not linearly independent, $\mathbf{S}_1 \times \ldots \times \mathbf{S}_t$ is equal to zero.

Theorem 8

All the vectors, r in all, in the exterior products $\mathbf{S}_1, \ldots, \mathbf{S}_t$ in eq 104 are assumed to be linearly independent of each other. Let V and O_M be respectively the absolute volume and the orientation of the parallelepiped spanned by the r independent vectors in $\mathbf{S}_1, \ldots, \mathbf{S}_t$. Let O_N be the orientation of a set of normals to all the vectors in $\mathbf{S}_1, \ldots, \mathbf{S}_t$, satisfying

$$\epsilon = O_N \wedge O_M \tag{105}$$

where ϵ is the orientation of the whole space. Then, the cross product defined by eq 104 is an s-vector VO_N,

$$\frac{1}{r!} \epsilon \cdot (\mathbf{S}_1 \wedge \cdots \wedge \mathbf{S}_t) = VO_N. \tag{106}$$

The dotting in eq 106 is in full from the rear.

Proof. Let the r independent vectors in $\mathbf{S}_1, \ldots, \mathbf{S}_t$ be $\mathbf{e}_{s+1}, \ldots, \mathbf{e}_{s+r}$, where $r+s = n$. The orientation of $\mathbf{e}_{s+1} \wedge \cdots \wedge \mathbf{e}_{s+r}$ is O_M.

A set of Cartesian base vectors $u_1, \ldots, u_s$ spanning the subspace normal to $\mathbf{e}_{s+1}, \ldots, \mathbf{e}_{s+r}$ is found as follows. Find independent vectors $\mathbf{e}_1, \ldots, \mathbf{e}_s$ forming a complete set of subindexed base vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Determine a set of superindexed base vectors $\mathbf{e}^1, \ldots, \mathbf{e}^n$. Vectors $\mathbf{e}^1, \ldots, \mathbf{e}^s$ are normals to $\mathbf{e}_{s+1}, \ldots, \mathbf{e}_n$. A set of Cartesian vectors spanning the space of $\mathbf{e}^1, \ldots, \mathbf{e}^s$ is a set $u_1, \ldots, u_s$.

Let the set $u_1, \ldots, u_s, \mathbf{e}^{s+1}, \ldots, \mathbf{e}^n$ have the same orientation as ϵ by changing the order or reversing directions, if necessary, of $u_1, \ldots, u_s$. Let V be the volume of the parallelepiped spanned by the base vectors dual to $u_1, \ldots, u_s, \mathbf{e}^{s+1}, \ldots, \mathbf{e}^n$. Then,

$$\epsilon = Vu_1 \wedge \ldots \wedge u_s \wedge \mathbf{e}^{s+1} \wedge \ldots \wedge \mathbf{e}^n. \tag{107}$$

Because

$$O_N = u_1 \wedge \cdots \wedge u_s \tag{108}$$

we find that V is the absolute volume of $\mathbf{e}_{s+1} \wedge \cdots \wedge \mathbf{e}_n$.

Dotting eq 107 into the exterior product

$$\mathbf{S}_1 \wedge \cdots \wedge \mathbf{S}_t = \mathbf{e}_{s+1} \wedge \cdots \wedge \mathbf{e}_n \tag{109}$$

in full from the rear yields eq 106.

The validity of eq 105 becomes manifest when the equation

$$\mathbf{O}_M = V\,\mathbf{e}^{s+1} \wedge \cdots \wedge \mathbf{e}^m \tag{110}$$

is substituted into eq 107. Q.E.D.

Defining eq 105 on a plane means, as explained below, defining the positive direction of $\mathbf{u}_\theta$ of measuring angle θ on the plane. The orientation ϵ of a plane is defined by introducing a set of right-handed Cartesian base vectors $\mathbf{c}_x$, $\mathbf{c}_y$. An outward normal to a circle about origin O is in the direction $\mathbf{u}_r$ of radius $\overrightarrow{OP}$, where P is a point on the circle. Then, the direction $\mathbf{u}_\theta$ on the circle in which θ increases is chosen to make

$$\mathbf{u}_r \wedge \mathbf{u}_\theta = \mathbf{c}_x \wedge \mathbf{c}_y . \tag{111}$$

NABLA OPERATOR

Nabla operator $\boldsymbol{\nabla}$, called del by Gibbs, is defined in a tangent space by

$$\boldsymbol{\nabla} = \mathbf{e}^i \frac{\partial}{\partial x^i} . \tag{112}$$

Theorem 9

Nabla is invariant under transformations of curvilinear coordinates.

Proof. Because the operator $\partial/\partial x^i$ transforms similarly to $\mathbf{e}_i$, the proof for the invariance of $\mathbf{1}$ (Theorem 1) applies to this case. Q.E.D.

In skew linear coordinates where x_i is defined as well as x^i, $\boldsymbol{\nabla}$ may be given the form

$$\boldsymbol{\nabla} = \mathbf{e}_i \frac{\partial}{\partial x_i} . \tag{113}$$

Nabla $\boldsymbol{\nabla}$, when combined with the algebraic operations defined for base vectors, expresses four operations – gradient, divergence, curl, and exterior curl:

$$\text{grad } \mathbf{T} = \mathbf{e}^i \frac{\partial \mathbf{T}}{\partial x^i} \tag{114}$$

$$\text{div } \mathbf{T} = \mathbf{e}^i \cdot \frac{\partial \mathbf{T}}{\partial x^i} \tag{115}$$

$$\text{curl } \mathbf{S} = \mathbf{e}^i \times \frac{\partial \mathbf{S}}{\partial x^i} \tag{116}$$

and

$$\text{exterior curl } \mathbf{T} = \mathbf{e}^i \wedge \frac{\partial \mathbf{T}}{\partial x^i} \qquad (117)$$

where **T** is an n-dimensional tensor of any order and **S** is an n-dimensional alternating tensor. The cross product in eq 116 is defined by eq 104. In these operations, differentiations of T and S with regard to x^i must be executed first, and then the results should be combined with $\mathbf{e}^i$.

Operand tensor T in eq 114 is of order zero or higher; and T in eq 115, S in eq 116, and T in eq 117 are of order one or higher.

If T or S in eq 114 through 117 is a tensor of order two or higher, indication should be given of the location of a base vector in the tensor bases of $\partial \mathbf{T}/\partial x^i$ or $\partial \mathbf{S}/\partial x^i$ with which $\mathbf{e}^i$ should be combined. It is the custom that, if no indication is given in the operations of eq 114 or 115, $\mathbf{e}^i$ is put at the extreme left of the tensor bases of $\partial \mathbf{T}/\partial x^i$ to form grad T or dotted into the first member of the tensor bases of $\partial \mathbf{T}/\partial x^i$ to form div **T**. The exterior products of $\mathbf{e}^i$ and $\partial \mathbf{S}/\partial x^i$ in eq 116 or of $\mathbf{e}^i$ and $\partial \mathbf{T}/\partial x^i$ in eq 117 are customarily formed by wedging $\mathbf{e}^i$ with the extreme left member in the tensor bases of $\partial \mathbf{S}/\partial x^i$ or $\partial \mathbf{T}/\partial x^i$, respectively. If T is not alternating, $\mathbf{e}^i$, thus added, and the first member of $\partial \mathbf{T}/\partial x^i$ form a bivector.

The curl operation may be defined even for a tensor **T** which is not necessarily alternating, with the understanding that dotting ϵ into $\mathbf{e}^i \wedge \partial \mathbf{T}/\partial x^i$ is executed only for an alternating part of the base tensors.

The exterior differentiation introduced by Cartan (1899) is given in a tensor form* $(dx^j \mathbf{e}_j) \cdot (\mathbf{e}^i \wedge \partial/\partial x^i)$, where the wedging in the second set of parentheses must be executed prior to dotting between the two sets of parentheses.

The operand in the exterior differentiation is customarily assumed to be alternating (See, for example, Flanders, 1963; Fleming, 1965; Wilmore, 1959). This is not assumed in this paper.

Theorem 10

If a formula written in invariant form is true in a coordinate system, it is true in all coordinate systems.

Proof. The theorem is evidently true because the formula is invariant under coordinate transformations. Q.E.D.

Corollary

All the three-dimensional vector formulas developed by use of the three-dimensional standard Cartesian coordinate system are true even in a three-dimensional curvilinear coordinate system, if the formulas are written in invariant forms.

Furthermore, it can be proved that many three-dimensional vector formulas written in invariant forms by applying the conventions developed so far in this paper are true even when three-dimensional vectors are replaced with n-dimensional tensors of any order.

In the following, a proof of a formula containing differential operators valid for an n-dimensional tensor of any order is shown as an example. It is proved by using curvilinear coordinates. We use curvilinear coordinates because they are the most general coordinates.

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* See Fleming (1965) for a similar notation. Fleming's vector approach is completed by the method in this paper.

Example 3 The formula

$$\operatorname{div}\operatorname{curl}\mathbf{T} = 0 \tag{118}$$

is valid in n-dimensional Euclidean space for a tensor $\mathbf{T}$ of any order, which may not necessarily be alternating. Equation 118 is valid even if any definition of dotting $\boldsymbol{\epsilon}$ may be used for defining curl.

Proof.

$$\operatorname{div}\operatorname{curl}\mathbf{T} \tag{a}$$

$$= \mathbf{e}^i \cdot \frac{\partial}{\partial x^i}\left[\boldsymbol{\epsilon}\cdot\left(\mathbf{e}^j \wedge \frac{\partial \mathbf{T}}{\partial x^j}\right)\right] \tag{b}$$

$$= \mathbf{e}^i \cdot \frac{\partial}{\partial x^i}\left[\boldsymbol{\epsilon}\cdot\left(\mathbf{e}^j \wedge \frac{\partial \mathbf{e}^k\mathbf{e}_k\cdot\mathbf{T}}{\partial x^i}\right)\right] \tag{c}$$

$$= \mathbf{e}^i \cdot \frac{\partial}{\partial x^i}\left[\boldsymbol{\epsilon}\cdot\left(\mathbf{e}^j \wedge \mathbf{e}^k\,\frac{\partial \mathbf{e}_k\cdot\mathbf{T}}{\partial x^j}\right)\right] \tag{d}$$

$$= \mathbf{e}^i \cdot \boldsymbol{\epsilon}\cdot\frac{\partial}{\partial x^i}\left(\mathbf{e}^j \wedge \mathbf{e}^k\,\frac{\partial \mathbf{e}_k\cdot\mathbf{T}}{\partial x^i}\right) \tag{e}$$

$$= \mathbf{e}^i \cdot \boldsymbol{\epsilon}\cdot\left[\left(-\Gamma^j_{ih}\,\mathbf{e}^h \wedge \mathbf{e}^k - \mathbf{e}^j \wedge \Gamma^j_{ih}\,\mathbf{e}^h\right)\frac{\partial \mathbf{e}_k\cdot\mathbf{T}}{\partial x^j} + \mathbf{e}^j \wedge \mathbf{e}^k\,\frac{\partial^2 \mathbf{e}_k\cdot\mathbf{T}}{\partial x^i \partial x^j}\right] \tag{f}$$

$$= 2\mathbf{e}_s\ldots\mathbf{e}_t\left[\left(-\epsilon^{ihks\ldots t}\,\Gamma^j_{ih} - \epsilon^{ijhs\ldots t}\,\Gamma^k_{ih}\right)\frac{\partial \mathbf{e}_k\cdot\mathbf{T}}{\partial x^j} + \right.$$

$$\left. + \epsilon^{ijks\ldots t}\,\frac{\partial^2 \mathbf{e}_k\cdot\mathbf{T}}{\partial x^i \partial x^j}\right] \tag{g}$$

$$= 0. \tag{h}$$

Equation a becomes eq b by definitions of divergence and curl operations. Equation b becomes eq c by use of the property (eq 15) of $\mathbf{1}$. Equation c becomes eq d by use of eq 103. Use of $\mathbf{1}$ in eq c enables us to write expressly the bivector formed by combining $\mathbf{e}^i$ and the first member of T, as shown in eq d. Equation d becomes eq e because $\boldsymbol{\epsilon}$ is a constant tensor (Theorem 5). Differentiating eq e factor by factor yields eq f. Equation f becomes eq g because

$$\mathbf{e}^i \cdot \boldsymbol{\epsilon}\,(\mathbf{e}^h \wedge \mathbf{e}^k) = \underset{1}{\mathbf{e}^{i}}\cdot\epsilon^{pqrs\ldots t}\,\underset{1}{\mathbf{e}_p}\underset{2}{\mathbf{e}_q}\underset{3}{\mathbf{e}_r}\mathbf{e}_s\ldots\mathbf{e}_t\cdot(\underset{2}{\mathbf{e}^h}\underset{3}{\mathbf{e}^k}-\underset{2}{\mathbf{e}^k}\underset{3}{\mathbf{e}^h}) \tag{i}$$

$$= 2\epsilon^{ihks\ldots t}\,\mathbf{e}_s\ldots\mathbf{e}_t. \tag{j}$$

Equation i shows one of the definitions of dotting into $\boldsymbol{\epsilon}$. Other definitions of dotting into $\boldsymbol{\epsilon}$, if used, yield only a difference in sign in the result, because $\boldsymbol{\epsilon}$ is alternating. Therefore the result (eq h) is obtained for any definition of dotting into $\boldsymbol{\epsilon}$. Equation g vanishes because Γ^j_{ih} and $\partial^2\mathbf{e}_k\cdot\mathbf{T}/\partial x^i\partial x^j$ are symmetric with regard to ih and ij, respectively. The latter is based on the fact that $\mathbf{e}_k\cdot\mathbf{T}$ is a tensor (Theorem 2).

GREEN-STOKES' INTEGRALS

Green's integrals are valid even in n-dimensional Euclidean space for a tensor T of any order under the conditions specified later. The domain V of a volume integral is an r-dimensional region in an r-dimensional manifold M embedded in n-dimensional Euclidean space. The domain S of a surface integral is an (r-1)-dimensional manifold bounding V.

Stokes' integral is a transformation of a divergence integral whose integrand is a vector.

The proof shown below is an extension of a proof in three-dimensional Euclidean space to n-dimensional Euclidean space. The extension is made possible by developing n-dimensional Euclidean geometry analogous to three-dimensional Euclidean geometry.

Equating

$$x^1 = \text{const}, \ldots, x^s = \text{const} \tag{119}$$

in curvilinear coordinates $x^i (i=1, \ldots, s, \ldots, n)$ defines an r-dimensional manifold M, where $n = r+s$. The manifold M is assumed to be orientable.

We assume that equating

$$x^{s+1} = \text{const}, \ldots, x^{s+r} = \text{const} \tag{120}$$

defines an s-dimensional manifold N normal to M.

We will use

indexes $\lambda, \mu, \nu, \ldots$ for $1, \ldots, s$

indexes $a, b, c, \ldots$ for $s+1, \ldots, s+r$

and

indexes $i, j, k, \ldots$ for $1, \ldots, n$.

The manifold M is spanned by curves $x^a (a=s+1, \ldots, s+r)$, along which $x^1, \ldots, x^{a-1}, x^{a+1}, \ldots, x^r, \ldots, x^n$ are constant. Base vectors $\mathbf{e}_a$ are tangent to curves x^a, lying in M; and base vectors $\mathbf{e}^\lambda (\lambda=1, \ldots, s)$ are normal to M. The orientation of $\mathbf{e}_{s+1} \wedge \ldots \wedge \mathbf{e}_{s+r}$ and $\mathbf{e}^{s+1} \wedge \ldots \wedge \mathbf{e}^{s+r}$ is O_M. O_M defines positive orientation in M.

We have

$$g^{a\lambda} = 0 \tag{121}$$

and

$$g_{a\lambda} = 0 \tag{122}$$

because base vectors $\mathbf{e}^a$ lie in M and base vectors $\mathbf{e}_\lambda$ are normal to M.

The orientation O_N of $\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_s$ or $\mathbf{e}^1 \wedge \ldots \wedge \mathbf{e}^s$ is defined to satisfy

$$\epsilon = O_N \wedge O_M \tag{123}$$

by reordering, if necessary, base vectors in N or M. $\mathbf{O}_N$ defines positive orientation in N. If $\mathbf{e}_i$ or $\mathbf{e}^i$ is reordered, x^i must be similarly reordered.

Theorem 11

Differentiating a vector defined in the manifold M with regard to a coordinate of M produces another vector that does not necessarily lie in M.

Proof. Use of eq 33 yields

$$\frac{\partial \mathbf{e}_a}{\partial x^b} = \Gamma_{ab}^{\ i}\,\mathbf{e}_i = \Gamma_{ab}^{\ c}\,\mathbf{e}_c + \Gamma_{ab}^{\ \lambda}\,\mathbf{e}_\lambda \,. \tag{124}$$

The second term on the right-hand side of eq 124 is not zero, because $\Gamma_{ab}^{\ \lambda}$ is not necessarily zero.

To show this, use is made of eq 47 and 121, yielding

$$\Gamma_{ab}^{\ \lambda} = g^{\lambda\mu}\,\Gamma_{\mu ab} \,. \tag{125}$$

Substituting eq 46 and 122 changes eq 125 to

$$\Gamma_{ab}^{\ \lambda} = -\frac{1}{2}\,g^{\lambda\mu}\,\frac{\partial g_{ab}}{\partial x^\mu} \,. \tag{126}$$

The right-hand side of eq 126 is not necessarily zero because g_{ab} can change along the normal $\mathbf{e}_\mu$. Q.E.D.

Theorem 12

Let the orientation and the volume of $\mathbf{e}_{s+1}\wedge\dots\wedge\mathbf{e}_{s+r}$ be $\mathbf{O}_M$ and $\sqrt{g_M}$ respectively; then

$$\mathbf{e}_{s+1}\wedge\dots\wedge\mathbf{e}_{s+r} = \sqrt{g_M}\;\mathbf{O}_M \,. \tag{127}$$

Equation 127 is equivalent to

$$\mathbf{e}^{s+1}\wedge\dots\wedge\mathbf{e}^{s+r} = \frac{1}{\sqrt{g_M}}\;\mathbf{O}_M \,. \tag{128}$$

Orientation $\mathbf{O}_M$ changes continuously. g_M is given by

$$g_M = \frac{1}{r!}\,(\mathbf{e}_{s+1}\wedge\dots\wedge\mathbf{e}_{s+r})\cdot(\mathbf{e}_{s+1}\wedge\dots\wedge\mathbf{e}_{s+r}) \,. \tag{129}$$

The magnitude dV of an r-dimensional volume element

$$dx^{s+1}\,\mathbf{e}_{s+1}\wedge\dots\wedge dx^{s+r}\,\mathbf{e}_{s+r} \tag{130}$$

is

$$dV = \sqrt{g_M}\;dx^{s+1}\dots dx^{s+r} \,. \tag{131}$$

Proof. Dotting eq 128 into eq 127 in full in natural arrangement proves eq 128. Dotting eq 127 into itself in full in natural arrangement yields eq 129. Use is made of eq 69 in executing these dottings. By definition, eq 130 is

$$dx^{s+1}\,\mathbf{e}_{s+1}\wedge\dots\wedge dx^{s+r}\,\mathbf{e}_{s+r} = \mathbf{O}_M\,dV \,. \tag{132}$$

Substituting eq 127 into eq 132 yields eq 131. Q.E.D.

Corollary

$$\frac{\partial\sqrt{g_M}}{\partial x^a} = \Gamma_{ab}^{\;b}\sqrt{g_M}\,. \tag{133}$$

Proof. Differentiation of eq 129 is calculated:

$$\frac{\partial g_M}{\partial x^a} = \frac{2}{r!}\sum_b \left(\mathbf{e}_{s+1}\wedge\cdots\wedge\frac{\partial \mathbf{e}_b}{\partial x^a}\wedge\cdots\wedge\mathbf{e}_{s+r}\right)\cdot\left(\mathbf{e}_{s+1}\wedge\cdots\wedge\mathbf{e}_{s+r}\right) \tag{a}$$

$$= \frac{2}{r!}\sum_b \left(\mathbf{e}_{s+1}\wedge\cdots\wedge\Gamma_{ab}^{\;i}\mathbf{e}_i\wedge\cdots\wedge\mathbf{e}_{s+r}\right)\cdot\left(\mathbf{e}_{s+1}\wedge\cdots\wedge\mathbf{e}_{s+r}\right) \tag{b}$$

$$= \frac{2}{r!}\Gamma_{ab}^{\;b}\left(\mathbf{e}_{s+1}\wedge\cdots\wedge\mathbf{e}_{s+r}\right)\cdot\left(\mathbf{e}_{s+1}\wedge\cdots\wedge\mathbf{e}_{s+r}\right) \tag{c}$$

$$= 2\,\Gamma_{ab}^{\;b}\,g_M\,. \tag{d}$$

Equation a becomes eq b by use of eq 33. $\Gamma_{ab}^{\;i}\mathbf{e}_i$ in eq b is a sum of n terms; however, only one term for which i = b survives. The i for a survivor cannot be equal to $s+1,\ldots,b-1,\ b+1,\ldots,s+r$, because one vector cannot appear twice in an exterior product. The i for a survivor cannot be equal to $1,\ldots,s$, because dotting $\mathbf{e}_\lambda$ into $\mathbf{e}_{s+1}\wedge\cdots\wedge\mathbf{e}_{s+r}$, where λ stands for $1,\ldots,s$, cancels out terms, thus eq b becomes eq c. Equation d proves eq 133. Q.E.D.

Theorem 13

Let O_N and $\sqrt{g_N}$ be the orientation and absolute volume respectively of $\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_s$; then,

$$\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_s = \sqrt{g_N}\,O_N. \tag{134}$$

Equation 134 is equivalent to

$$\mathbf{e}^1\wedge\cdots\wedge\mathbf{e}^s = \frac{1}{\sqrt{g_N}}\,O_N\,. \tag{135}$$

g_N satisfies the equation

$$g = g_N\,g_M\,, \tag{136}$$

where $\sqrt{g}$ is the n-dimensional absolute volume.

Proof. Dotting eq 135 into eq 134 in full in natural arrangement proves eq 135. Dotting $\boldsymbol{\epsilon}$ into $\mathbf{e}_{s+1}\wedge\cdots\wedge\mathbf{e}_{s+r}$ in full from the rear yields

$$\frac{1}{r!}\,\boldsymbol{\epsilon}\cdot\left(\mathbf{e}_{s+1}\wedge\cdots\wedge\mathbf{e}_{s+r}\right) \tag{a}$$

$$= \sqrt{g}\,\mathbf{e}^1\wedge\cdots\wedge\mathbf{e}^s \tag{b}$$

where eq 80 is used with $V = \sqrt{g}$ to express $\boldsymbol{\epsilon}$. Rewriting eq a by use of eq 106 yields

$$\sqrt{g_M}\;O_N = \sqrt{g}\,\mathbf{e}^1\wedge\cdots\wedge\mathbf{e}^s. \tag{c}$$

Substituting eq 135 into eq c yields eq 136. Q.E.D.

Lemma

Let an (r-1)-dimensional surface S be defined in the r-dimensional manifold M by

$$x^a = f(x^{s+1}, \ldots, x^{a-1}; x^{a+1}, \ldots, x^{s+r}) . \tag{137}$$

Assume that all the points in a neighborhood of point P_0 on surface S have unique (r-1)-dimensional tangent planes in r-dimensional tangent spaces to M. Let dS and O_S be the absolute area and positive orientation, respectively, of the parallelogram constructed by taking infinitesimal line segments $dx^{s+1}, \ldots, dx^{a-1}$, $dx^{a+1}, \ldots, dx^{s+r}$ along coordinate axes $x^{s+1}, \ldots, x^{a-1}$, $x^{a+1}, \ldots, x^{s+r}$, respectively, on the (r-1)-dimensional tangent plane to S at P_0. Then,

$$\begin{aligned} O_s\, dS = (-1)^{a-1-s} [& e_{s+1} \wedge \cdots \wedge e_{a-1} \wedge e_{a+1} \wedge \cdots \wedge e_{s+r} + \\ & + \frac{\partial f}{\partial x^{s+1}} e_a \wedge e_{s+2} \wedge \cdots \wedge e_{a-1} \wedge e_{a+1} \wedge \cdots \wedge e_{s+r} + \\ & \cdots\cdots\cdots\cdots \\ & + e_{s+1} \wedge \cdots \wedge e_{a-2} \wedge \frac{\partial f}{\partial x^{a-1}} e_a \wedge e_{a+1} \wedge \cdots \wedge e_{s+r} + \\ & + e_{s+1} \wedge \cdots \wedge e_{a-1} \wedge \frac{\partial f}{\partial x^{a+1}} e_a \wedge e_{a+2} \wedge \cdots \wedge e_{s+r} + \\ & \cdots\cdots\cdots\cdots \\ & + e_{s+1} \wedge \cdots \wedge e_{a-1} \wedge e_{a+1} \wedge \cdots \wedge \frac{\partial f}{\partial x^{s+r}} e_a]\, dx^{s+1} \ldots dx^{a-1} dx^{a+1} \ldots dx^{s+r}. \end{aligned} \tag{138}$$

Proof. Let P_0 on S have coordinates $x^{s+1}, \ldots, x^{a-1}$, $f(x^{s+1}, \ldots, x^{a-1}, x^{a+1}, \ldots, x^{s+r})$, $x^{a+1}, \ldots, x^{s+r}$. Let P_h ($h = s+1, \ldots, a-1, a+1, \ldots, s+r$) be a point whose coordinates are the same as the coordinates of P_0 except x^h which is increased to $x^h + dx^h$.

Vectors $\overrightarrow{P_0 P_h}$ ($h = s+1, \ldots, a-1, a+1, \ldots, s+r$) are expressed by calculating differences in the coordinates of P_0 and P_h:

$$\overrightarrow{P_0 P_{s+1}} = (e_{s+1} + \frac{\partial f}{\partial x^{s+1}} e_a)\, dx^{s+1}$$

$$\cdots\cdots\cdots\cdots$$

$$\overrightarrow{P_0 P_{a-1}} = (e_{a-1} + \frac{\partial f}{\partial x^{a-1}} e_a)\, dx^{a-1}$$

$$\overrightarrow{P_0 P_{a+1}} = (e_{a+1} + \frac{\partial f}{\partial x^{a+1}} e_a)\, dx^{a+1}$$

$$\cdots\cdots\cdots\cdots$$

$$\overrightarrow{P_0 P_{s+r}} = (e_{s+r} + \frac{\partial f}{\partial x^{s+r}} e_a)\, dx^{s+r} .$$

The existence of $\partial f / \partial x^h$ is assured by the existence of a unique tangent plane at P_0.

Let dS be

$$dS = \overrightarrow{P_0 P_{s+1}} \wedge \cdots \wedge \overrightarrow{P_0 P_{a-1}} \wedge \overrightarrow{P_0 P_{a+1}} \wedge \cdots \wedge \overrightarrow{P_0 P_{s+r}} \cdot$$

Wedging $\overrightarrow{P_0P_{s+1}}, \ldots, \overrightarrow{P_0P_{a-1}}, \overrightarrow{P_0P_{a+1}}, \ldots,$ and $\overrightarrow{P_0P_{s+r}}$ yields

$$dS = [\quad]\, dx^{s+1} \ldots dx^{a-1} dx^{a+1} \ldots dx^{s+r} \tag{139}$$

where [ ] stands for the quantity inside the brackets in eq 138.

The orientation of dS is equal to the orientation of $e_{s+1} \wedge \cdots \wedge e_{a-1} \wedge e_{a+1} \wedge \cdots e_{s+r}$, the first term in [ ], because during the change $\overrightarrow{P_0P_{s+1}} \Rightarrow e_{s+1}dx^{s+1}, \ldots, \overrightarrow{P_0P_{a-1}} \Rightarrow e_{a-1}dx^{a-1}, \overrightarrow{P_0P_{a+1}} \Rightarrow e_{a+1}dx^{a+1}, \ldots, \overrightarrow{P_0P_{s+r}} \Rightarrow e_{s+r}dx^{s+r}$, the area of the (r-1)-dimensional parallelogram does not become zero. By proposition 11, the sign of the orientation of $e_{s+1} \wedge \cdots \wedge e_{a-1} \wedge e_{a+1} \wedge \cdots \wedge e_{s+r}$ is equal to the sign of $(-1)^{a-s-1} O_M$. Because O_M is positive by definition, we have

$$dS = (-1)^{a-s-1} O_s \, dS. \tag{140}$$

Combining eq 139 and eq 140 yields eq 138. Q.E.D.

Theorem 14

Let $\mathbf{n}$ be a unit vector drawn normal to surface S, defined in the lemma, making an acute angle with $+e_a$ in the tangent space to M at P_0. $\mathbf{n}$ and O_s make

$$O_M = \mathbf{n} \wedge O_s. \tag{141}$$

Then,

$$(\mathbf{n} \cdot e_a) dS = \sqrt{g_M}\, dx^{s+1} \ldots dx^{a-1} dx^{a+1} \ldots dx^{s+r}. \tag{142}$$

Proof. The orientation $O_N \wedge \mathbf{n}$ of normals to the parallelogram $O_s dS$ in the lemma is obtained by applying eq 106 to eq 138:

$$O_N \wedge \mathbf{n}\, dS = \frac{1}{(r-1)!}\, \epsilon \cdot O_s\, dS. \tag{143}$$

Dotting ϵ into the first term in the brackets of eq 138 yields

$$\epsilon \cdot e_{s+1} \wedge \cdots \wedge e_{a-1} \wedge e_{a+1} \wedge \cdots \wedge e_{s+r} \tag{a}$$

$$= \sqrt{g}\, e^1 \wedge \ldots \wedge \underset{1}{e^{s+2}} \wedge \ldots \wedge \underset{r-1}{e^{s+r}} \cdot \underset{1}{e_{s+1}} \wedge \cdots \wedge \underset{a-1-s}{e_{a-1}} \wedge \underset{a-s}{e_{a+1}} \wedge \cdots \wedge \underset{r-1}{e_{s+r}} \tag{b}$$

$$= (-1)^{a-s-1} \sqrt{g}\, e^1 \wedge \ldots \wedge e^s \wedge e^a \wedge \underset{1}{e^{s+1}} \wedge \ldots \wedge \underset{a-1-s}{e^{a-1}} \wedge \underset{a-s}{e^{a+1}} \wedge \ldots \wedge \underset{r-1}{e^{s+r}} \cdot$$

$$\cdot\, \underset{1}{e_{s+1}} \wedge \cdots \wedge \underset{a-1-s}{e_{a-1}} \wedge \underset{a-s}{e_{a+1}} \wedge \cdots \wedge \underset{r-1}{e_{s+r}} \tag{c}$$

$$= (-1)^{a-s-1} (r-1)!\, \sqrt{g}\, e^1 \wedge \ldots \wedge e^s \wedge e^a. \tag{d}$$

Equation a becomes eq b by use of eq 80 with $V = \sqrt{g}$. Vector e^a in ϵ in eq b is brought between e^s and e^{s+1} to obtain eq c. Executing dotting in eq c yields eq d, where use is made of eq 72.

Dotting ϵ into the (p+1)th term in the brackets of eq 138 is calculated:

$$\epsilon\cdot e_{s+1}\wedge\cdots\wedge e_{s+p-1}\wedge\frac{\partial f}{\partial x^{s+p}}\, e_a\wedge e_{s+p+1}\wedge\cdots\wedge e_{a-1}\wedge e_{a+1}\wedge\cdots\wedge e_{s+r} \tag{e}$$

$$= \sqrt{g}\, e^1\wedge\ldots\wedge \underset{\substack{\uparrow\\ 1}}{e^{s+2}}\wedge\ldots\wedge \underset{\substack{\uparrow\\ r-1}}{e^{s+r}}$$

$$\cdot\ \underset{\substack{\uparrow\\ 1}}{e_{s+1}}\wedge\cdots\wedge \underset{\substack{\uparrow\\ p-1}}{e_{s+p-1}}\wedge\frac{\partial f}{\partial x^{s+p}}\, \underset{\substack{\uparrow\\ p}}{e_a}\wedge \underset{\substack{\uparrow\\ p+1}}{e_{s+p+1}}\wedge\cdots\wedge \underset{\substack{\uparrow\\ a-1-s}}{e_{a-1}}\wedge \underset{\substack{\uparrow\\ a-s}}{e_{a+1}}\wedge\cdots\wedge \underset{\substack{\uparrow\\ r-1}}{e_{s+r}} \tag{f}$$

$$= \sqrt{g}\,\frac{\partial f}{\partial x^{s+p}}\,(-1)^{a-s-1}\, e^1\wedge\ldots\wedge e^s\wedge e^a\wedge \underset{\substack{\uparrow\\ 1}}{e^{s+1}}\wedge\ldots\wedge \underset{\substack{\uparrow\\ a-1}}{e^{a-1}}\wedge \underset{\substack{\uparrow\\ a}}{e^{a+1}}\wedge\ldots\wedge \underset{\substack{\uparrow\\ r-1}}{e^{s+r}}\ .$$

$$\cdot\ \underset{\substack{\uparrow\\ 1}}{e_{s+1}}\wedge\cdots\wedge \underset{\substack{\uparrow\\ p-1}}{e_{s+p-1}}\wedge \underset{\substack{\uparrow\\ p}}{e_a}\wedge \underset{\substack{\uparrow\\ p+1}}{e_{s+p+1}}\wedge\cdots\wedge \underset{\substack{\uparrow\\ a-1}}{e_{a-1}}\wedge \underset{\substack{\uparrow\\ a}}{e_{a+1}}\wedge\cdots\wedge \underset{\substack{\uparrow\\ r-1}}{e_{s+r}} \tag{g}$$

$$= -(-1)^{a-s-1}\sqrt{g}\,\frac{\partial f}{\partial x^{s+p}}\, e^1\wedge\ldots\wedge e^s\wedge e^{s+p}\wedge \underset{\substack{\uparrow\\ 1}}{e^{s+1}}\wedge\ldots\wedge \underset{\substack{\uparrow\\ p-1}}{e^{s+p-1}}\wedge \underset{\substack{\uparrow\\ p}}{e^a}\wedge \underset{\substack{\uparrow\\ p+1}}{e^{s+p+1}}\wedge\ldots\wedge$$

$$\ldots \underset{\substack{\uparrow\\ a-1}}{e^{a-1}}\wedge \underset{\substack{\uparrow\\ a}}{e^{a+1}}\wedge\ldots\wedge \underset{\substack{\uparrow\\ r-1}}{e^{s+r}}\ \cdot\ \underset{\substack{\uparrow\\ 1}}{e_{s+1}}\wedge\ldots\wedge \underset{\substack{\uparrow\\ p-1}}{e_{s+p-1}}\wedge \underset{\substack{\uparrow\\ p}}{e_a}\wedge \underset{\substack{\uparrow\\ p+1}}{e_{s+p+1}}\wedge\ldots\wedge \underset{\substack{\uparrow\\ a-1}}{e_{a-1}}\wedge$$

$$\underset{\substack{\uparrow\\ a}}{e_{a+1}}\wedge\ldots\wedge \underset{\substack{\uparrow\\ r-1}}{e_{s+r}} \tag{h}$$

$$= -(-1)^{a-s-1}\sqrt{g}\,\frac{\partial f}{\partial x^{s+p}}\,(r-1)!\; e^1\wedge\ldots\wedge e^s\wedge e^{s+p}\ . \tag{i}$$

p in this case satisfies $s+1 \leqq s+p \leqq a-1$. Equation e becomes eq f by use of eq 80 with $V = \sqrt{g}$. Equation f becomes eq g by transferring e^a to the place between e^s and e^{s+1} in the first exterior product. Equation g becomes eq h by exchanging e^a and e^{s+p} in the first exterior product, where use is made of the property that exchanging two vectors in an exterior product R results in -R. Executing dotting in eq h yields eq i, where use is made of eq 72.

In this way, dotting ϵ into eq 138 is calculated to yield

$$\epsilon\cdot O_s\, dS$$

$$= (r-1)!\,\sqrt{g}\, e^1\wedge\ldots\wedge e^s\wedge f^a\, dx^{s+1}\ldots dx^{a-1}dx^{a+1}\ldots dx^{s+r} \tag{144}$$

where

$$f^a = e^a - \frac{\partial f}{\partial x^{s+1}}\, e^{s+1} - \ldots - \frac{\partial f}{\partial x^{a-1}}\, e^{a-1} - \frac{\partial f}{\partial x^{a+1}}\, e^{a+1} - \ldots -$$

$$- \frac{\partial f}{\partial x^{s+r}}\, e^{s+r}. \tag{145}$$

Substituting

$$\epsilon = O_N\wedge n\wedge O_s \tag{146}$$

into eq 144 yields

$$O_N \wedge n\, dS \doteq \sqrt{g}\, e^1 \wedge \ldots \wedge e^s \wedge f^a\, dx^{s+1} \ldots dx^{a-1} dx^{a+1} \ldots dx^{s+r} \qquad (147)$$

where use is made of eq 72.

Dotting respectively the left- and right-hand sides of the equation

$$O_N \wedge e_a = \frac{1}{\sqrt{g_N}}\, e_1 \wedge \ldots \wedge e_s \wedge e_a \qquad (148)$$

into the left- and right-hand sides of eq 147 in natural arrangement yields eq 142.

Q.E.D.

Lemma

An (r-1)-dimensional surface S bounds an r-dimensional region V in the r-dimensional manifold M. Surface S may be composed of more than one (r-1)-dimensional region. All the pieces of surface S are assumed to be orientable.

It is assumed that almost all the points P on S are regular. In other words, irregular points Q on S, on which unique tangent (r-1)-dimensional hyperplanes do not exist, constitute a point set whose area measured with an (r-1)-dimensional unit cube is zero.

Let **T** be a tensor of any order, defined on S and V, unique on S and V, and differentiable at least once in V. Let **n** be an outward pointed normal to S lying in the r-dimensional tangent space to M at regular point P on S.

It is assumed that every point P on S where the curve x^a is not tangent to S can be connected with another point P' on S by an arc x^a of finite length passing through V.

Then,

$$\int_S (n \cdot e_a)\, T\, dS = \int_V \frac{\partial T}{\partial x^a}\, dV + \int_V \Gamma^b_{ab}\, T\, dV. \qquad (149)$$

Proof. Draw a bundle of curves x^a through the neighborhood of a point on S. On a curve x^a, only x^a changes and $x^1, \ldots, x^{a-1}, x^{a+1}, \ldots, x^n$ are constant. By assumption, the bundle of curves x^a, cutting surface S at dS_+, cuts surface S again at dS_-. Assume that dS_+ and dS_- are composed of regular points only. The outward pointed normals n_+ at P_+ on dS_+ and n_- at P_- on dS_- drawn exterior to V in the tangent space to M and P_+ and P_- make respectively acute and obtuse angles with $+e_a$. Use of eq 142 yields

$$(n_+ \cdot e_a) dS_+ = \sqrt{g_M}\, dx^{s+1} \ldots dx^{a-1} dx^{a+1} \ldots dx^{s+r} \qquad (150)$$

and

$$(n_- \cdot e_a) dS_- = -\sqrt{g_M}\, dx^{s+1} \ldots dx^{a-1}\, dx^{a+1} \ldots dx^{s+r}\,. \qquad (151)$$

The integration of $\partial T/\partial x^a$ over the bundle of curves x^a, denoted by Bx^a is calculated:

$$\int_{Bx^a} \frac{\partial T}{\partial x^a} dV \qquad (a)$$

$$= \int_{Bx^a} \frac{\partial T}{\partial x^a} \sqrt{g_M} \; dx^{s+1} \ldots dx^{s+r} \tag{b}$$

$$= \int \left\{ \frac{\partial \sqrt{g_M}\, T}{\partial x^a} - \frac{\partial \sqrt{g_M}}{\partial x^a}\, T \right\} dx^{s+1} \ldots dx^{s+r} . \tag{c}$$

Equation a becomes eq b by use of eq 131.

The first term of eq c is integrated:

$$\int_{Bx^a} \frac{\partial \sqrt{g_M}\, T}{\partial x^a} \; dx^{s+1} \ldots dx^{s+r} \tag{d}$$

$$= \Big[\sqrt{g_M}\; T \; dx^{s+1} \ldots dx^{a-1} dx^{a+1} \ldots dx^{s+r} \Big]_{dS_-}^{dS_+} \tag{e}$$

$$= (\mathbf{n}_+ \cdot \mathbf{e}_a)\, T_+ \, dS_+ + (\mathbf{n}_- \cdot \mathbf{e}_a)\, T_- \, dS_- \tag{f}$$

where T_+ and T_- are T on dS_+ and dS_- respectively. Equation e becomes eq f by use of eq 150 and eq 151.

The second term of eq c becomes

$$\int_{Bx^a} \Gamma_{ab}^{\;b}\, T \, dV \tag{g}$$

by use of eq 133.

Thus, one finds

$$(\mathbf{n}_+ \cdot \mathbf{e}_a)\, T_+ \, dS_+ + (\mathbf{n}_- \cdot \mathbf{e}_a)\, T_- \, dS_-$$

$$= \int_{Bx^a} \frac{\partial T}{\partial x^a} \, dV + \int_{Bx^a} \Gamma_{ab}^{\;b}\, T \, dV . \tag{152}$$

Divide the region V into a sum of bundles x^a. Summing up eq 152 over regular bundles whose terminal surfaces dS_+ and dS_- are composed of regular points, one obtains eq 149. Q.E.D.

<u>Theorem 15</u>

Green's integrals

$$\int_V \operatorname{grad} \mathbf{T} \, dV = \int_S \mathbf{n}\mathbf{T} \, dS \tag{153}$$

$$\int_V \operatorname{div} \mathbf{T} \, dV = \int_S \mathbf{n} \cdot \mathbf{T} \, dS \tag{154}$$

$$\int_V \operatorname{curl} \mathbf{T} \, dV = \int_S \mathbf{n} \times \mathbf{T} \, dS \tag{155}$$

$$\int_V \text{exterior curl } \mathbf{T} \, dV = \int_S \mathbf{n} \wedge \mathbf{T} \, dS \tag{156}$$

are valid for V, S, and **T** under the conditions in the previous Lemma. Nabla ∇ in these equations becomes

$$\nabla = \mathbf{e}^a \frac{\partial}{\partial x^a} \tag{157}$$

because **T** is a function of $x^{s+1}, \ldots, x^{s+r}$. ∇ on the left-hand sides and **n** on the right-hand sides operate on the first members of **T**.

Proof. Let all the curves x^a passing through V intersect S twice.

Equations 153, 154, 155 and 156 are proved in almost the same way. For example,

$$\int_S \mathbf{n} \times \mathbf{T}\, dS \tag{a}$$

$$= \int_S (\mathbf{n}\cdot\mathbf{e}_i)\, \mathbf{e}^i \times \mathbf{T}\, dS \tag{b}$$

$$= \int_S (\mathbf{n}\cdot\mathbf{e}_a)\, \mathbf{e}^a \times \mathbf{T}\, dS \tag{c}$$

$$= \int_V \frac{\partial \mathbf{e}^a \times \mathbf{T}}{\partial x^a}\, dV + \int_V \Gamma_{ab}^{\ b}\, \mathbf{e}^a \times \mathbf{T}\, dV \tag{d}$$

$$= \int_V \left\{ -\Gamma_{ab}^{\ b}\, \mathbf{e}^b \times \mathbf{T} + \mathbf{e}^a \times \frac{\partial \mathbf{T}}{\partial x^a} \right\} dV + \int_V \Gamma_{ab}^{\ b}\, \mathbf{e}^a \times \mathbf{T}\, dV \tag{e}$$

$$= \int_V \mathbf{e}^a \times \frac{\partial \mathbf{T}}{\partial x^a}\, dV \tag{f}$$

$$= \int_V \mathbf{e}^i \times \frac{\partial \mathbf{T}}{\partial x^i}\, dV \tag{g}$$

$$= \int_V \operatorname{curl} \mathbf{T}\, dV. \tag{h}$$

Equation a becomes eq b by use of the property (eq 15) of **1**. Equation b becomes eq c because $\mathbf{n} \cdot \mathbf{e}_\lambda = 0$, where λ stands for $1, \ldots, s$. Equation c becomes eq d by use of eq 149. Equation d becomes eq e by executing the differentiation in the integrand, where use is made of eq 38. The first integrand in eq e is equal to the third integrand in eq e because summation indexes a and b may be exchanged. Equation f becomes eq g because **T** does not have x^λ as variables. Thus eq 155 is proved.

Changing the cross product in eq a to a tensor product, a dot product, and an exterior product similarly proves eq 153, 154 and 156. Q. E. D.

A Green's integral usually contains a volume integral over an n-dimensional region. In this paper, a Green's integral contains a volume integral over an r-dimensional manifold, where $r \leq n$. Equation 156 is not usually included in Green's integrals.

Stokes' integral is a transformation, as proved in the following, of a particular case of divergence integral (eq 154) with integrand T changed to a vector.

Lemma 1

$$\underset{\phi\;\underset{1}{\uparrow}\cdots\underset{r-1}{\uparrow}}{O_M}\;\cdot\;\underset{*\;\underset{1}{\uparrow}\cdots\underset{r-1}{\uparrow}}{O_M} = (r-1)!\;\underset{\phi}{e_a}\underset{*}{e^a}\,. \tag{158}$$

Proof.

$$\underset{\underset{1}{\uparrow}\cdots\cdots\underset{r-1}{\uparrow}}{e_{s+1}\wedge e_{s+2}\wedge\cdots\wedge e_{s+r}}\;\cdot\;\underset{\underset{1}{\uparrow}\cdots\cdots\underset{r-1}{\uparrow}}{e^{s+1}\wedge e^{s+2}\wedge\cdots\wedge e^{s+r}} \tag{a}$$

$$= \mathring{\epsilon}^{a_1a_2\ldots a_r}\,\underset{\underset{1}{\uparrow}\cdots\underset{r-1}{\uparrow}}{e_{a_1}e_{a_2}\cdots e_{a_r}}\;\cdot\;\mathring{\epsilon}_{b_1b_2\ldots b_r}\,\underset{\underset{1}{\uparrow}\cdots\underset{r-1}{\uparrow}}{e^{b_1}e^{b_2}\cdots e^{b_r}} \tag{b}$$

$$= \mathring{\epsilon}^{a_1a_2\ldots a_r}\,\mathring{\epsilon}_{b_1b_2\ldots b_r}\,e_{a_1}e^{b_1}\,\delta_{a_2}{}^{b_2}\cdots\delta_{a_r}{}^{b_r} \tag{c}$$

$$= \mathring{\epsilon}^{aa_2\ldots a_r}\,\mathring{\epsilon}_{ba_2\ldots a_r}\,e_a e^b \tag{d}$$

$$= (r-1)!\;\delta^a{}_b\;e_a e^b \tag{e}$$

$$= (r-1)!\;e_a e^a\,. \tag{f}$$

The left-hand side of eq 158 becomes eq a by expressing O_M through use of eq 127 and eq 128. $\mathring{\epsilon}^{a_1\ldots a_r}$ and $\mathring{\epsilon}_{b_1\ldots b_r}$ in eq b are permutation symbols. Executing dotting in eq b yields eq c, which in turn yields eq d. Equation d becomes eq e because $aa_2\ldots a_r$ and $ba_2\ldots a_r$ are both derangements of $12\ldots r$. Suffixes $a_2\ldots a_r$ in eq d are summation indexes. Equation e becomes eq f by summing over b. Q.E.D.

Lemma 2

Let (r-1)-vector **A** be defined by means of vector **v** belonging to M,

$$\underset{\underset{1}{\uparrow}\cdots\underset{r-1}{\uparrow}}{\mathbf{A}} = \underset{\ddagger}{\mathbf{v}}\cdot\;\ddagger\;\underset{\underset{1}{\uparrow}\cdots\underset{r-1}{\uparrow}}{O_M}\,. \tag{159}$$

Then, **v** is given reciprocally by

$$\underset{\ddagger}{\mathbf{v}} = \frac{1}{(r-1)!}\;\ddagger\;\underset{\underset{1}{\uparrow}\cdots\underset{r-1}{\uparrow}}{O_M}\;\cdot\;\underset{\underset{1}{\uparrow}\cdots\underset{r-1}{\uparrow}}{\mathbf{A}}\,. \tag{160}$$

Proof.

$$\underset{\phi}{\mathbf{v}} = \underset{\phi}{e_i}\underset{\ddagger}{e^i}\cdot\underset{\ddagger}{\mathbf{v}} \tag{a}$$

$$= \underset{\phi}{e_a}\underset{\ddagger}{e^a}\cdot\underset{\ddagger}{\mathbf{v}} \tag{b}$$

$$= \frac{1}{(r-1)!}\;\phi\;\underset{\underset{1}{\uparrow}\cdots\underset{r-1}{\uparrow}}{O_M}\;\cdot\;\ddagger\;\underset{\underset{1}{\uparrow}\cdots\underset{r-1}{\uparrow}}{O_M}\;\cdot\;\underset{\ddagger}{\mathbf{v}} \tag{c}$$

$$= \frac{1}{(r-1)!} \underset{\substack{\phi\uparrow\cdots\uparrow\\ 1\quad\ r-1}}{O_M} \cdot \underset{\substack{\uparrow\cdots\uparrow\\ 1\quad r-1}}{\mathbf{A}} \cdot \tag{d}$$

Equation a becomes eq b because $\mathbf{e}^\lambda \cdot \mathbf{v} = 0$, where $\mathbf{e}^\lambda$ is normal to M. Use of eq 158 changes eq b to eq c. Equation c becomes eq d by use of eq 159. Q.E.D.

Lemma 3

Let S be an (r-1)-dimensional surface bounding an r-dimensional region V in M. Let **n** be a unit vector drawn in M normal to S and externally to V at a point where a unique tangent plane exists. Let O_S be defined by eq 141.

Then,

$$\mathbf{n} \cdot \mathbf{v} = \frac{1}{(r-1)!} \underset{\substack{\uparrow\cdots\uparrow\\ 1\quad r-1}}{O_S} \cdot \underset{\substack{\uparrow\cdots\uparrow\\ 1\quad r-1}}{\mathbf{A}} \cdot \tag{161}$$

Proof.

$$\mathbf{n} \cdot \mathbf{v} = \frac{1}{(r-1)!} \underset{\phi}{\mathbf{n}} \cdot \Big(\underset{\substack{\phi\,\uparrow\cdots\uparrow\\ 1\quad r-1}}{O_M} \cdot \underset{\substack{\uparrow\cdots\uparrow\\ 1\quad r-1}}{\mathbf{A}} \Big) \tag{a}$$

$$= \frac{1}{(r-1)!} \underset{\phi}{\mathbf{n}} \cdot \Big(\underset{\phi}{\mathbf{n}} \wedge \underset{\substack{\uparrow\cdots\uparrow\\ 1\quad r-1}}{O_S} \cdot \underset{\substack{\uparrow\cdots\uparrow\\ 1\quad r-1}}{\mathbf{A}} \Big) . \tag{b}$$

Use of eq 160 yields eq a. Equation a becomes eq b by use of eq 141. Executing the dotting indicated by $\uparrow$ in eq b yields eq 161, where use is made of eq 75.
Q.E.D.

Lemma 4

Let **A** be an (r-1)-vector belonging to M; then

$$\mathbf{e}^a \cdot \frac{\partial}{\partial x^a} (O_M \cdot \mathbf{A}) = \mathbf{e}^a \cdot O_M \cdot \frac{\partial \mathbf{A}}{\partial x^a} . \tag{162}$$

Proof.

$$\mathbf{e}^a \cdot \frac{\partial}{\partial x^a} (O_M \cdot \mathbf{A}) \tag{a}$$

$$= \mathbf{e}^a \cdot \frac{\partial}{\partial x^a} \Big(\frac{1}{\sqrt{g_M}} \mathbf{e}_{s+1} \wedge \cdots \wedge \mathbf{e}_{s+r} \cdot \mathbf{A} \Big) \tag{b}$$

$$= \mathbf{e}^a \cdot \Big[-(\sqrt{g_M})^{-2}\, \Gamma^b_{ab} \sqrt{g_M}\, \mathbf{e}_{s+1} \wedge \cdots \wedge \mathbf{e}_{s+r} \cdot \mathbf{A} +$$

$$+ \frac{1}{\sqrt{g_M}} \sum \mathbf{e}_{s+1} \wedge \cdots \wedge \mathbf{e}_{b-1} \wedge \Gamma^i_{ab}\, \mathbf{e}_j \wedge \mathbf{e}_{b+1} \wedge \cdots \wedge \mathbf{e}_{s+r} \cdot \mathbf{A} +$$

$$+ O_M \cdot \frac{\partial \mathbf{A}}{\partial x^a} \Big] \tag{c}$$

$$= \mathbf{e}^a \cdot O_M \cdot \frac{\partial \mathbf{A}}{\partial x^a} . \tag{d}$$

Equation a becomes eq b by expressing O_M through use of eq 127. Differentiating eq b factor by factor yields eq c, where use is made of eq 133 and 33. $\Gamma_{ab}^{\ i}\,\mathbf{e}_i$ in the second term of eq c is a sum of n vectors, of which only $\mathbf{e}_b$ survives, as explained in the following: The survivor must be one of $\mathbf{e}_1, \ldots, \mathbf{e}_s, \mathbf{e}_b$ because other vectors already exist in the r-vector considered. The survivor must be $\mathbf{e}_b$, because the only summand that does not vanish when $\mathbf{A}$ and $\mathbf{e}^a$ are dotted from the right and left respectively is the one that has $\mathbf{e}_b$. The first and second terms in the brackets of eq c cancel to yield eq d.

Lemma 5

Let $\mathbf{A}$ be given by eq 159; then

$$\operatorname{div} \mathbf{v} = \frac{1}{s!}\, O_N \cdot \operatorname{curl} \mathbf{A} \tag{163}$$

$$= \frac{1}{r!}\, \underset{\uparrow s+1 \,\cdots\, \uparrow s+r}{O_M} \cdot \underset{\uparrow s+1}{\mathbf{e}^a} \wedge \underset{\uparrow s+2 \,\cdots\, \uparrow s+r}{\frac{\partial \mathbf{A}}{\partial x^a}} \tag{164}$$

where the curl in eq 163 is defined by

$$\frac{1}{s!}\, O_N \cdot \operatorname{curl} \mathbf{A} = \frac{1}{r!s!}\, \underset{\uparrow 1 \,\cdots\, \uparrow s}{O_N} \cdot \underset{\uparrow 1 \cdots \uparrow s \;\; \uparrow s+1 \cdots \uparrow s+r}{\epsilon} \cdot \underset{\uparrow s+1}{\mathbf{e}^a} \wedge \underset{\uparrow s+2 \,\cdots\, \uparrow s+r}{\frac{\partial \mathbf{A}}{\partial x^a}} \; . \tag{165}$$

Proof. Let

$$\mathbf{A} = \sum_{b=s+1}^{s+r} A_b\, \mathbf{e}^{s+1} \wedge \ldots \wedge \mathbf{e}^{b-1} \wedge \mathbf{e}^{b+1} \wedge \ldots \wedge \mathbf{e}^{s+r} \; . \tag{166}$$

div $\mathbf{v}$ is transformed:

$$\operatorname{div} \mathbf{v} = \mathbf{e}^i \cdot \frac{\partial \mathbf{v}}{\partial x^i} \tag{a}$$

$$= \mathbf{e}^a \cdot \frac{\partial \mathbf{v}}{\partial x^a} \tag{b}$$

$$= \frac{1}{(r-1)!}\, \mathbf{e}^a \cdot \frac{\partial}{\partial x^a}\,(O_M \cdot \mathbf{A}) \tag{c}$$

$$= \frac{1}{(r-1)!}\, \mathbf{e}^a \cdot O_M \cdot \frac{\partial \mathbf{A}}{\partial x^a} \tag{d}$$

$$= \frac{1}{(r-1)!}\, \underset{\uparrow \phi}{\mathbf{e}^a} \cdot \frac{1}{\sqrt{g_M}}\, \underset{\uparrow \phi}{\mathbf{e}_{s+1}} \wedge \underset{\uparrow 1}{\mathbf{e}_{s+2}} \wedge \ldots \wedge \underset{\uparrow r-1}{\mathbf{e}_{s+r}} \cdot \frac{\partial}{\partial x^a} \left\{ \sum_b A_b\, \underset{\uparrow 1}{\mathbf{e}^{s+1}} \wedge \ldots \wedge \underset{\uparrow b-s-1}{\mathbf{e}^{b-1}} \wedge \underset{\uparrow b-s}{\mathbf{e}^{b+1}} \wedge \ldots \wedge \underset{\uparrow r-1}{\mathbf{e}^{s+r}} \right\} \tag{e}$$

$$= \frac{1}{(r-1)!} \sum_a \frac{(-1)^{a-1-s}}{\sqrt{g_M}}\, \mathbf{e}_{s+1} \wedge \cdots \wedge \mathbf{e}_{a-1} \wedge \mathbf{e}_{a+1} \wedge \cdots \wedge \mathbf{e}_{s+r} \; \cdot$$

$$\cdot\sum_{b}\left\{\frac{\partial A_b}{\partial x^a}\, e^{s+1}\wedge\ldots\wedge e^{b-1}\wedge e^{b+1}\wedge\ldots\wedge e^{s+r}\, -\right.$$

$$\left. -\sum_{\substack{c\\ c\neq b}} A_b\, e^{s+1}\wedge\ldots\wedge e^{c-1}\wedge \Gamma^c_{ai}\, e^i\wedge e^{c+1}\wedge\ldots\wedge e^{b-1}\wedge e^{b+1}\wedge\ldots\wedge e^{s+r}\right\} \tag{f}$$

$$= \sum \frac{(-1)^{a-1-s}}{\sqrt{g_M}} \sum_{b}\left\{\frac{\partial A_b}{\partial x^a}\,\delta^{(s+1)\ldots(b-1)(b+1)\ldots(s+r)}_{(s+1)\ldots(a-1)(a+1)\ldots(s+r)}\, -\right.$$

$$\left. -\sum_{\substack{c\\ c\neq b}} A_b\,\Gamma^c_{ai}\,\delta^{(s+1)\ldots(c-1)i(c+1)\ldots(b-1)(b-1)\ldots(s+r)}_{(s+1)\ldots\ldots\ldots(a-1)(a+1)\ldots\ldots\ldots(s+r)}\right\} \tag{g}$$

$$= \sum_{a}\frac{(-1)^{a-1-s}}{\sqrt{g_M}}\left\{\frac{\partial A_a}{\partial x^a} - A_a\sum_{\substack{c\\ c\neq a}}\Gamma^c_{ac} - \sum_{\substack{b\\ b\neq a}}(-1)^{b-1-a}\, A_b\,\Gamma^a_{ab}\right\} \tag{h}$$

$$= \sum_{a}\frac{(-1)^{a-1-s}}{\sqrt{g_M}}\,\frac{\partial A_a}{\partial x^a}\,. \tag{i}$$

Equation a is obtained by the definition of divergence. Equation a becomes eq b because $\mathbf{v}$ does not contain x^λ. Equation b becomes eq c by use of eq 160. Equation c becomes eq d by use of eq 162. Equation d becomes eq e by expressing O_M and A through use of eq 127 and 166 respectively. Dotting $\mathbf{e}^a$ in eq e is executed by transferring $\mathbf{e}_a$ in $\mathbf{e}_{s+1}\wedge\cdots\wedge\mathbf{e}_{s+r}$ to the extreme left, where use is made of eq 75 for executing dotting from the front. Dotting in eq f is executed by use of eq 76. In the first term of eq g, only one kind of summand – the one for which b=a – survives under the summation with regard to eq b, because (s+1)...(b-1)(b+1)...(s+r) must be a derangement of (s+1)...(a-1)(a+1)...(s+r). In the second term of eq g, suffix i is a summation index changing from 1 to n. Two kinds of summands survive under the summation with regard to b, c, and i, because (s+1)...(c-1)i(c+1)... (b-1)(b+1)...(s+r) must be a derangement of (s+1)...(a-1)(a+1)...(s+r). The one has suffixes i=c and b=a, and the other has suffixes i=b and c=a. They are shown in eq h. To show that eq h becomes eq i, the third term of eq h is transformed:

$$-\sum_{a}\frac{(-1)^{a-1-s}}{\sqrt{g_M}}\sum_{\substack{b\\ b\neq a}}(-1)^{b-1-a}\, A_b\,\Gamma^a_{ab} \tag{j}$$

$$= \sum_{a}\frac{1}{\sqrt{g_M}}\sum_{\substack{b\\ b\neq a}}(-1)^{b-s-1}\, A_b\,\Gamma^a_{ab} \tag{k}$$

$$= \sum_b \frac{(-1)^{b-s-1}}{\sqrt{g_M}} A_b \sum_{\substack{a \\ a\neq b}} \Gamma^{a}_{ab} \tag{ℓ}$$

$$= \sum_a \frac{(-1)^{a-s-1}}{\sqrt{g_M}} A_a \sum_{\substack{c \\ c\neq a}} \Gamma^{c}_{ca} \, . \tag{m}$$

Equation k becomes eq ℓ by changing the order of summation. Equation ℓ becomes eq m by changing suffixes b and a to a and c, respectively. Equation m is the second term of eq h, showing that the second and third terms of eq h cancel each other out.

Second, $O_N \cdot$ curl **A** is transformed. Executing dotting O_N into ϵ in eq 165 yields eq 164, where use is made of eq 123 to express ϵ and of eq 75 to execute dotting from the front. Equation 164 transforms:

$$\frac{1}{r!} O_M \cdot e^a \wedge \frac{\partial A}{\partial x^a}$$

$$= \frac{1}{r!} O_M \cdot e^a \wedge \frac{\partial}{\partial x^a} \left\{ \sum_b A_b \, e^{s+1} \wedge \ldots \wedge e^{b-1} \wedge e^{b+1} \wedge \ldots \wedge e^{s+r} \right\} \tag{n}$$

$$= \frac{1}{r!} O_M \cdot e^a \wedge \left\{ \sum_b \frac{\partial A_b}{\partial x^a} e^{s+1} \wedge \ldots \wedge e^{b-1} \wedge e^{b+1} \wedge \ldots \wedge e^{s+r} - \right.$$

$$\left. - \sum_{\substack{c \\ c\neq b}} A_b \, e^{s+1} \wedge \ldots \wedge e^{c-1} \wedge \Gamma^{c}_{ai} e^i \wedge e^{c+1} \wedge \ldots \wedge e^{b-1} \wedge e^{b+1} \wedge \ldots \wedge e^{s+r} \right\} \tag{o}$$

$$= \frac{1}{r!} O_M \cdot \sum_a \sum_b \frac{\partial A_b}{\partial x^a} e^a \wedge e^{s+1} \wedge \ldots \wedge e^{b-1} \wedge e^{b+1} \wedge \ldots \wedge e^{s+r} \tag{p}$$

$$= \frac{1}{\sqrt{g_M}} \sum_a \sum_b \frac{\partial A_b}{\partial x^a} \delta^{a(s+1)\ldots(b-1)(b+1)\ldots(s+r)}_{(s+1)(s+2)\ldots\ldots\ldots\ldots(s+r)} \tag{q}$$

$$= \sum_a \frac{(-1)^{a-1-s}}{\sqrt{g_M}} \frac{\partial A_a}{\partial x^a} \, . \tag{r}$$

Equation 164 becomes eq n by substituting eq 166. Executing the differentiation in eq n yields eq o. The second term in eq o is equal to zero, because it contains $e^a \wedge \Gamma^{c}_{ai} e^i$. Substituting for O_M from eq 127 and executing the dotting in eq p yields eq q, where use is made of eq 76. In the summation with regard to b in eq q, only one term – the one for which b=a – survives, because a(s+1)...(b-1)(b+1)...(s+r) must be a derangement of (s+1)(s+2)...(s+r). Therefore eq q becomes eq r.

Comparing eq m with eq r proves the lemma. Q.E.D.

Theorem 16

Let V be an r-dimensional region in r-dimensional manifold M. V is bounded by an (r-1)-dimensional manifold S. Let A be an (r-1)-vector, defined on S and V, unique on S and V, and differentiable at least once in V. Then under the assumptions for V and S made in Lemma to Theorem 15, the formula

$$\frac{1}{s!}\int_V O_N \cdot \operatorname{curl} \mathbf{A}\, dV = \frac{1}{(r-1)!}\int_S O_S \cdot \mathbf{A}\, dS \tag{167}$$

is valid, where $O_N \cdot \operatorname{curl} \mathbf{A}$ is defined by eq 165. Equation 167 is equivalent to

$$\frac{1}{r!}\int_V O_M \cdot \mathbf{e}^a \wedge \frac{\partial A}{\partial x^a}\, dV = \frac{1}{(r-1)!}\int O_S \cdot \mathbf{A}\, dS\,. \tag{168}$$

Proof. Substituting eq 163 and 161 into

$$\int_V \operatorname{div} \mathbf{v}\, dV = \int_S \mathbf{n} \cdot \mathbf{v}\, dS \tag{169}$$

yields eq 167. Substituting eq 164 and 161 into eq 169 yields eq 168. Equation 169 is obtained from eq 154 by making $\mathbf{T} = \mathbf{v}$. Q.E.D.

Equation 167 and 168 are Stokes' integrals. Factorials in the denominators of eq 167 and 168 cancel the factorials that appear, as shown in eq 69 by r!, in the results of dotting multivectors. Particularly when n=3, r=2, eq 167 takes the familiar form of Stokes' integral.

If, as done by Fleming (1965, p. 217), dotting r-vectors is defined by dividing the right-hand side of eq 69 by r!, all the factorials in eq 167 and 168 disappear. Fleming's definition, however, is not used in this paper in order to show the relation to dotting tensors.

The integrand on the left-hand side of eq 168 is, so to speak, curl **A** defined in M, because O_M is in M as ϵ is in n-dimensional Euclidean space.

Stokes' formula is expressed by use of r-form in the following.

Corollary 1

Equation 168 transforms to

$$\int_V \sum_{a=s+1}^{s+r} (-1)^{a-1-s} \frac{\partial A_a}{\partial x^a}\, dx^{s+1} \dots dx^{s+r}$$

$$= \int_S \sum_{a=s+1}^{s+r} (-1)^{a-1-s} A_a\, dx^{s+1} \dots dx^{a-1} dx^{a+1} \dots dx^{s+r} \tag{170}$$

Proof. It is proved in Lemma 5 to Theorem 16 that the left-hand side of eq 168 transforms to the left-hand side of eq 170.

The right-hand side of eq 168 is transformed:

$$\frac{1}{(r-1)!}\int_S O_S \cdot \mathbf{A}\, dS \tag{a}$$

$$= \frac{1}{(r-1)!} \int_S O_s \cdot \sum_{a=s+1}^{s+r} A_a \, e^{s+1} \wedge \ldots \wedge e^{a-1} \wedge e^{a+1} \wedge \ldots \wedge e^{s+r} \, dS \tag{b}$$

$$= \frac{1}{(r-1)!} \sum_{a=s+1}^{s+r} \int_S (A_a \, e^{s+1} \wedge \ldots \wedge e^{a-1} \wedge e^{a+1} \wedge \ldots \wedge e^{s+r}) \cdot O_s \, dS \tag{c}$$

$$= \sum_{a=s+1}^{s+r} \int_S (-1)^{a-1-s} A_a \, dx^{s+1} \ldots dx^{a-1} dx^{a+1} \ldots dx^{s+r} \, . \tag{d}$$

Equation a becomes eq b by substituting eq 166. In eq b a summation is integrated, but in eq c, an integration is summed. Substituting eq 138 for $O_s dS$ in eq c, where a is a summation index, yields eq d, which is the right-hand side of eq 170. Q. E. D.

An r-form $dx^1 \wedge \ldots \wedge dx^r$ is defined [see, for example, Flanders (1963) and Fleming (1965)] by applying to $dx^1 \ldots dx^r$ the exterior algebra defined by means of wedge product. In this notation, the volume element spanned by $e_1 dx^1, \ldots, e_r dx^r$ is expressed by $dx^1 \wedge \ldots \wedge dx^r$. A linear combination of a finite number of r-forms is also called an r-form. r-forms are devised to use properties of multivectors without mentioning base-vectors.

Interpreting the integrands on the left- and right-hand sides of eq 170 as an r-form and an (r-1)-form respectively permits us to transform eq 170 to 174 below.

Corollary 2

Let $\mathbf{w}$ be an (r-1)-form

$$\mathbf{w} = \sum_{a=s+1}^{s+r} A_a \, dx^{s+1} \wedge \ldots \wedge dx^{a-1} \wedge dx^{a+1} \wedge \ldots \wedge dx^{s+r} \, . \tag{171}$$

Let signed be defined to mean

$$\text{signed } (dx^{s+1} \wedge \cdots \wedge dx^{a-1} \wedge dx^{a+1} \wedge \cdots \wedge dx^{s+r}) =$$

$$= (-1)^{a-1-s} \, dx^{s+1} \wedge \ldots \wedge dx^{a-1} \wedge dx^{a+1} \wedge \ldots \wedge dx^{s+r} \tag{172}$$

and

$$\text{signed } \mathbf{w} = \sum_{a=s+1}^{s+r} A_a \text{ signed } (dx^{s+1} \wedge \ldots \wedge dx^{a-1} \wedge dx^{a+1} \wedge \ldots \wedge dx^{s+r}). \tag{173}$$

Then eq 170 is simplified to

$$\int_V dx^a \wedge \frac{\partial \mathbf{w}}{\partial x^a} = \int_S \text{signed } \mathbf{w}. \tag{174}$$

Proof.

$$\sum_{a=s+1}^{s+r} (-1)^{a-1-s} \frac{\partial A_a}{\partial x^a} dx^{s+1} \wedge \ldots \wedge dx^r \tag{a}$$

$$= \sum_{a=s+1}^{s+r} \frac{\partial A_a}{\partial x^a} dx^a \wedge dx^{s+1} \wedge \ldots \wedge dx^{a-1} \wedge \ldots \wedge dx^{a+1} \wedge \ldots \wedge dx^{s+r} \tag{b}$$

$$= \sum_{a=s+1}^{s+r} dx^a \wedge \sum_{b=s+1}^{s+r} \frac{\partial A_b}{\partial x^a} dx^{s+1} \wedge \ldots \wedge dx^{b-1} \wedge dx^{b+1} \wedge \ldots \wedge dx^{s+r} \tag{c}$$

$$= \sum_{a=s+1}^{s+r} dx^a \wedge \frac{\partial}{\partial x^a} \left\{ \sum_{b=s+1}^{s+r} A_b \, dx^{s+1} \wedge \ldots \wedge dx^{b-1} \wedge dx^{b+1} \wedge \ldots \wedge dx^{s+r} \right\} \tag{d}$$

$$= dx^a \wedge \frac{\partial w}{\partial x^a} . \tag{e}$$

Equation a becomes eq b by a property of exterior product. Equation b becomes eq c because in eq c only one summand – the one for which b=a – survives under the summation with regard to b. Equation c becomes eq d because $dx^{s+1}, \ldots, dx^{a-1}$, $dx^{a+1}, \ldots, dx^{s+r}$ are independent of x^a. Equation d becomes eq c by use of eq 171.

The right-hand side of eq 174 is immediately obtained by changing the right-hand side of eq 170 to an (r-1)-form. Q. E. D.

CONCLUSION

Tensor analysis in this paper is an algebra formed by three operations on vector and scalar functions: (1) tensor product, (2) scalar product, and (3) linear combination with scalar function coefficients. These operations yield tensors whose orders are respectively higher, lower, and the same as the operand tensors. Scalar functions are at least three-times continuously differentiable.

An r-dimensional volume in n-dimensional Euclidean space, where $o<r\leqq n$, is an alternating tensor of r th order. The algebra of tensor analysis allows us to treat n-dimensional Euclidean geometry as a simple extension of two- or three-dimensional Euclidean geometry.

An r-dimensional manifold is discussed on the assumption that it is embedded in n-dimensional Euclidean space, where $o<r\leqq n$, using n-dimensional curvilinear coordinate systems.

The tensor analysis thus developed appears to be the most convenient for geometry and physics in Euclidean spaces.

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Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)
U.S. Army Cold Regions Research and Engineering Laboratory, Hanover, N.H.

2a. REPORT SECURITY CLASSIFICATION
Unclassified

2b. GROUP

3. REPORT TITLE
UNIFIED TREATMENT OF VECTORS AND TENSORS IN n-DIMENSIONAL EUCLIDEAN SPACE

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Research Report

5. AUTHOR(S) (First name, middle initial, last name)
Shunsuke Takagi

6. REPORT DATE
June 1968

7a. TOTAL NO. OF PAGES
48

7b. NO. OF REFS
13

8a. CONTRACT OR GRANT NO.

b. PROJECT NO.

c. DA Task 1T014501B52A02

d.

9a. ORIGINATOR'S REPORT NUMBER(S)
Research Report 207

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT
This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY
U.S. Army Cold Regions Research and Engineering Laboratory

13. ABSTRACT
A unified treatment of vectors, tensors and multivectors in n-dimensional Euclidean space is presented. The unified treatment is so systematized that n-dimensional tensors of arbitrary order are treated similarly to three-dimensional vectors. Work on this subject was done in connection with difficulties which arise in the continuum mechanics pertaining to large deformations of such media as soil and snow.

DD FORM 1 NOV 65 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

Unclassified
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Tensors Vectors Euclidean space						