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Stability of Ice-Age Ice Caps

by Johannes Weertman

U. S. ARMY COLD REGIONS RESEARCH
AND ENGINEERING LABORATORY
Corps of Engineers
Hanover, New Hampshire

PREFACE

This report was prepared by Dr. Weertman* under personal services contract with U. S. Army Cold Regions Research and Engineering Laboratory. The work was done under task 5010.01139, Mechanics of deformation of snow and ice, for the Research Division, J. A. Bender, chief.

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W. L. NUNGESSER
Colonel, Corps of Engineers
Director

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The stability of large ice caps was investigated by using the present-day theory of the flow of ice in glaciers and ice sheets. The type of instability considered is that first mentioned by Bodvarsson (See SIP 14559). It is concluded that a small Arctic ice cap can become unstable and expand into a large ice-age ice sheet as a result of moderate changes in the regime of the ice cap. A large continental ice sheet can also become unstable and shrink to nothing if the snow accumulation is reduced or the ablation rate increased. The results obtained fit well into the Ewing-Donn theory of ice ages. There is the possibility that the inherent instability of ice-age ice sheets is in itself sufficient to explain both the formation and the disappearance of these ice sheets.

UNCLASSIFIED

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2. Glaciers--Growth--Mathematical analysis

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STABILITY OF ICE-AGE ICE CAPS

by

J. Weertman*

Introduction

In this paper, we consider a problem important to any theory of the ice ages: the stability of continental ice sheets. We wish to show from the mechanics of ice flow that a small ice cap situated in high latitudes on a continental landmass may be unstable in the sense that, if its width exceeds a critical size, the ice cap will grow unchecked until it reaches lower latitudes and is of continental dimensions. Further, it can be shown that once an ice cap reaches this size, another instability may set in if the rate of accumulation decreases or the rate of ablation increases. The ice cap may then shrink to a small size or disappear.

As has been noted many times, an ice cap will grow when the snowfall on it increases or the melting at its edge decreases and will shrink when the snowfall decreases or the melting increases. The new aspect introduced here is the use of the recently developed theory of the mechanics of glaciers and ice caps to calculate the sensitivity of an ice cap to changes in rates of accumulation and ablation.

The type of instability with which we are concerned has been pointed out by Bodvarsson (1955), who came across this behavior while investigating ice sheet and glacier profiles. His work was based on boundary conditions rather different from those which would be used now.†

The instability discussed here is different in nature from that recently analyzed by Nye (1960). Nye considered the problem of the manner in which a glacier or ice sheet approaches a stable steady-state profile. He showed that, in ablation regions, unstable behavior may occur before an equilibrium profile finally is reached. It is implicit in Nye's treatment that there are always stable steady-state profiles which a glacier or ice sheet will approach. In other words, if the accumulation or ablation conditions change slightly, a glacier or ice cap will assume a new steady-state shape with a slightly different thickness and width. Nye was not primarily concerned with the problem that a steady-state profile may be in unstable equilibrium.

Qualitative reason for instability in an ice cap

Figure 1 shows half of the profile of an ice cap. An accumulation region exists in the higher elevations of this profile and an ablation region in the lower elevations. The border separating these two regions is often called the firn line or the snowline. If an ice cap is in equilibrium, this border must divide the areas in such a manner that the total accumulation is exactly equal to the total ablation. If such is not the case, the ice cap must either grow or shrink. In this paper, the elevation above sea level at which the firn line must lie if the ice cap is to be in equilibrium is called the equilibrium firn line and is designated h_f. The elevation of the actual firn or snowline (called hereafter the snowline) is designated h_s.

If the average accumulation and ablation rates are held constant, the larger an ice cap, the higher is the elevation of the equilibrium firn line. Figure 2 shows a schematic plot of this elevation h_f versus the distance R from the center of an ice cap to the equilibrium firn line. In general, the elevation of the snowline h_s will follow a different curve than h_f. In Figure 2, it is shown rising with distance, as would be the case for an ice cap growing from high into low latitudes. An equilibrium ice cap can exist only where these two curves meet. The equation for equilibrium is therefore

$$h_s = h_f \quad \text{(equilibrium condition).} \tag{1}$$

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† As the sliding velocity of an ice cap or glacier over its bed, he used a quantity which is proportional to the stress acting at the bed and inversely proportional to the thickness of ice over the bed. In other words, he assumed $U \propto (dh/dx)$, whereas we shall use $U \propto h^2 (dh/dx)^2$.

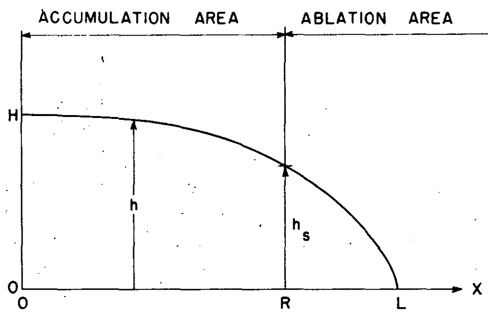

Figure 1. Cross section of half of an ice cap.

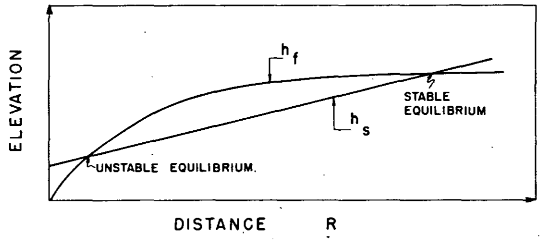

Figure 2. Schematic plot of the elevation of the snowline h_s and the equilibrium firn line h_f of an equilibrium ice cap vs distance R from the center of an ice cap to the boundary between the accumulation and ablation areas.

Whether this equilibrium is stable or unstable can be determined by considering an ice cap slightly larger or slightly smaller than the equilibrium ice cap. In order to be stable, an ice cap which is slightly larger than the profile for which eq 1 is valid must tend to shrink, and one slightly smaller must tend to grow. If h_s is less than h_f, an ice cap will grow and if h_s is greater than h_f the cap will shrink. Hence, a stable ice cap fulfills the condition that, when $h_s = h_f$,

$$dh_s/dR < dh_f/dR \qquad \text{(stable equilibrium).} \tag{2}$$

In Figure 2, the right-hand intersection point represents a stable ice cap.

An ice cap is in unstable equilibrium if a slightly smaller ice cap would shrink and a slightly larger one would grow. Hence, in this situation, we find the condition that, when $h_s = h_f$,

$$dh_s/dR > dh_f/dR \qquad \text{(unstable equilibrium).} \tag{3}$$

The left-hand intersection point of Figure 2 represents an ice cap in unstable equilibrium.

If the size of an ice cap influences the average rate of accumulation and ablation, h_f will depart from the curve calculated on the basis of these quantities remaining constant. Equilibrium ice caps can occur where this new curve meets the curve for h_s. (It should be noted that h_s itself may be influenced by the size of the ice cap.)

Review of the calculation of an ice cap profile

In order to determine whether or not an ice cap is in a state of equilibrium, and, if it is, whether this state is stable or unstable, it is necessary to know the height h_f of the equilibrium firn line as a function of the size of the ice cap. This height can be found from a calculation of the profile of an equilibrium ice cap. The solution of this problem was worked out by Nye (1959), using the principle of conservation of mass. It is possible to calculate the profiles of circular ice caps and of two-dimensional ice caps (i.e., ice caps with one axis very much longer than the other and, therefore, with profiles that are essentially independent of distance measured along the long axis). Since the profile of the two cases is approximately the same (under the same accumulation conditions, the circular ice cap has a thickness about 10% less than the two-dimensional ice cap, Nye, 1959), we shall discuss only the two-dimensional ice cap throughout this paper. The two-dimensional solution is probably just as good an approximation to the ice-age ice sheets as is that of the circular ice cap.

In Figure 1, let U represent the average velocity of ice passing through a cross section at distance x from the center. The total volume of ice passing through this cross section (for a unit length of the ice cap) is Uh, where h is the height of the profile above the bed, which we assume to be a flat surface. If the ice cap is in a steady-state, the principle of conservation of mass requires that this volume of ice equal the total accumulation of ice between the center of the ice cap and the point x. Let A represent the accumulation or ablation (A is negative when there is ablation) at any point on the upper surface. Then,

$$Uh = \int A\,dx. \qquad (4)$$

Nye assumes that the average velocity U is given by the equation

$$U = B\tau^m \qquad (5)$$

where B is a constant, m is a constant whose value is of the order of 2 to 2.5, and τ is the shear stress* acting at the bed of an ice cap. The value of τ is given by

$$\tau = -\rho g h\,(dh/dx) \qquad (6)$$

where ρ is the density of ice, g the gravitational acceleration, and dh/dx is the slope of the upper ice surface. It is reasonable to use the velocity given by eq 5, since a velocity of this form is to be expected both from an analysis of the differential flow of ice within an ice mass (Nye, 1959) and from a theoretical study of the sliding of ice over a glacier or ice-cap bed (Weertman, 1957). Since the value of m in eq 5 is approximately 2, we shall assign this value to it now in order to avoid some cumbersome expressions in later sections of this paper. When eq 5 and 6 are substituted into eq 4 and m is set equal to 2,

$$B(\rho g)^2 h^3 (dh/dx)^2 = \int_0^x A\,dx. \qquad (7)$$

The solution of this equation is the profile of an equilibrium ice cap.

As an example of a profile obtained from eq 7, consider the case of an ice sheet which entirely covers the landmass on which it is resting, as in the Antarctic today, and which has a constant rate of accumulation. We shall use this result later in the paper. At the edge of the ice cap, there is an infinite rate of ablation since the ice flows into the sea. In this situation, eq 7 reduces to

$$h^{3/2}(dh/dx) = \left(\frac{Ax}{B}\right)^{1/2} \rho g. \qquad (8)$$

The boundary condition is that the ice thickness must equal zero at the edge of the ice cap. The equation satisfying this condition is

$$h^{5/2} = H^{5/2} - (c/A)(Ax)^{3/2} \qquad (9)$$

where H is the ice thickness at the center of the ice cap and is given by the equation

$$L = A^{-1}(A/c)^{2/3} H^{5/3} \qquad (10)$$

L is the distance from the center to the edge, and constant c in these equations is given by

$$1/c = 3B^{1/2}\rho(g/5). \qquad (11)$$

The profile of the ice cap given by these last equations is one of stable equilibrium because the value of L is fixed and ablation occurs only at the position L. Moreover, the height H increases with increasing accumulation rate and decreases when A is decreased. Hence, a new equilibrium profile can always be approached if the accumulation rate is changed. In the next section, we consider equilibrium ice cap profiles which are unstable.

Snowline of fixed elevation

The profile calculated in the last section applies to an unusual situation in which all the ablation occurs at the very edge of the ice cap. Now consider the case where there is an appreciable ablation area existing on an ice cap whose edge does not reach

* Nye's analysis assumes that the shear stress is the dominant stress and that longitudinal stresses are unimportant. It is possible (Weertman, 1961a) to calculate ice cap profiles when longitudinal stresses also are large. Since the profiles so obtained (Weertman, 1961a) are almost the same as those found by Nye, there is no need to use this more complicated theory in the above analysis.

the sea. Eq 7 determines the profile once the accumulation and ablation are known as a function of distance. An approximation to this profile can be obtained by using an average accumulation rate a and an average ablation rate $\bar{a}$ which are defined by

$$a = R^{-1} \int_0^R A\,dx \tag{12}$$

$$\bar{a} = -(L - R)^{-1} \int_R^L A\,dx$$

where, as before, R is the distance at which both accumulation and ablation are zero. The term $\bar{a}$ is defined so that it is a positive quantity. The profile obtained by using these average rates should be a good approximation to the profile derived from the actual values of A, since Nye's theory is valid only under conditions where the variables h, dh/dx, and A are slowly varying functions of the distance x. If A varies only slowly* with x (and in the field this is usually the case), average values of accumulation and ablation must be of the same order of magnitude as actual values over most of the ice cap.

If eq 12 is substituted into eq 7 and if the requirement for an ice cap to be in equilibrium

$$aR - \bar{a}(L - R) = 0 \tag{13}$$

is satisfied, the following equations are obtained for the equilibrium profile (assuming that the snowline elevation h_s does not depend on distance x).

Region for which $h > h_s$ and $0 \leq x < R$

$$h^{5/2} = H^{5/2} - (c/a)(ax)^{3/2}. \tag{14a}$$

At $h = h_s$ and $x = R$

$$h = h_f = h_s. \tag{14b}$$

Region for which $h < h_s$ and $R < x \leq L$

$$h^{5/2} = h_s^{5/2} + (c/\bar{a})\left\{[aR - \bar{a}(x - R)]^{3/2} - [aR]^{3/2}\right\} \tag{14c}$$

The distance R is given by

$$R = a^{-1}(\bar{a}/c)^{2/3} h_s^{5/3} \tag{15a}$$

the thickness H at the center by

$$H = (1 + \bar{a}/a)^{2/5} h_s \tag{15b}$$

and the distance L from center to edge by

$$L = (1 + \bar{a}/a)^{1/3}(\bar{a}c^2)^{-1/3} H^{5/3} = (\bar{a})^{-1}(1 + \bar{a}/a)(\bar{a}/c)^{2/3} h_s^{5/3}$$

$$= (1 + \bar{a}/a)R. \tag{16}$$

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* The example of the last section violates this condition. No real difficulty is caused by the existence of an infinite ablation rate at the edge of the ice cap. The very edge of an ice cap must be excluded from Nye's analysis at any rate because in this region the rate of change of ice thickness is no longer a slowly varying function. An objection may be raised to the use of the average accumulation and ablation rates given by eq 12 in the region of the firn line since there is a discontinuity at R. This discontinuity, which results in a discontinuous change in the slope of the upper surface at R, would not occur in nature. This difficulty in the profile is confined, however, to a region of the order of the thickness of the ice cap on either side of the firn line and, hence, is of only minor extent.

The equilibrium profile defined by eq 13 to 16 obviously is in unstable equilibrium. Let us consider an ice cap whose profile is given by this equation, with arbitrary values of $\underline{a}$, $\bar{a}$, and $\underline{h_s}$. Now suppose that the snowline is lowered slightly. From eq 13 through 16 it can be seen that the new equilibrium profile has a smaller half width $\underline{L}$ and thickness $\underline{H}$. However, the snowline elevation now is lower than the elevation of the equilibrium firn line and, hence, the ice cap will grow, since total accumulation exceeds total ablation. The equilibrium profile cannot be approached and the ice cap will grow if the average accumulation and ablation remained unchanged. On the other hand, if $\underline{h_s}$ is increased slightly, the ice cap will shrink until it disappears.

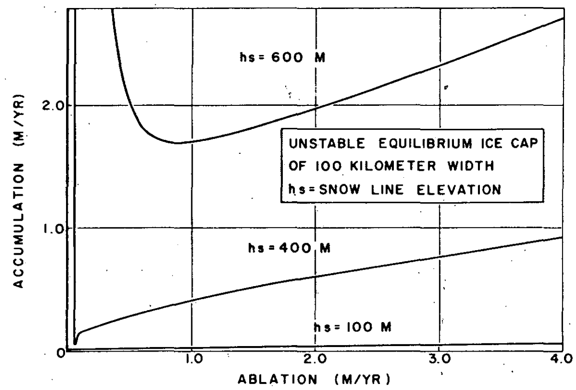

Figure 3. Curves of accumulation vs ablation for an unstable equilibrium ice cap 100 km wide for three different snowline elevations.

Eq 13 to 16 predict that the lower the elevation of the snowline, the smaller the width (width = 2L) of the equilibrium ice cap. Since this ice cap is unstable, one would expect that the lower the snowline, the easier it would be for a small ice cap existing in the Arctic to become "critical" and start growing to continental size. The width of the unstable equilibrium ice cap given by eq 13 to 16 also depends sensitively on the average accumulation and ablation rates. Figures 3 and 4 show plots of the values of $\underline{a}$ vs $\bar{a}$ which satisfy these equations for various elevations of the snowline when the width (2L) is 100 km and when it is 3000 km. A value of $c = 2\ (\text{m-yr})^{1/2}$ was used in making the calculations. This value of $\underline{c}$ is obtained from a comparison of theoretical ice cap profiles with measured profiles (Weertman, 1961a). Values of $\underline{a}$ and $\bar{a}$ which lie above a given curve would lead to the ice cap growing indefinitely in size. If $\underline{a}$ and $\bar{a}$ are below the curve, then the ice cap will shrink away to nothing. From Figure 3, it can be seen that very small rates of accumulation would lead to the unstable growth of an ice cap 100 km wide, if the snowline is as low as 100 m. On the other hand, if the snowline is at 600 m, a very high rate of accumulation (~ 2 m/yr) would be required to maintain the ice cap or cause it to grow. (At present, the snowline at high latitudes in the Arctic appears to be around 100 to 400 m elevation, Matthes, 1942.)

Profile of an ice cap with constant accumulation and ablation rates and a variable snowline

In this section, we calculate the profile of an ice cap which has a uniform rate of accumulation and ablation, but whose snowline is a function of latitude. One knows, of course, that the snowline lies at low elevations in the Arctic and that it rises to elevations of the order of 3 to 4 km in temperate latitudes (Matthes, 1942). For mathematical convenience, let us assume that this rise proceeds linearly, so that we can write

$$h_s = \bar{h} + sx \qquad (17)$$

where $\underline{s}$ is a constant of the order of 10^{-3} and $\bar{h}$ is the value of $\underline{h_s}$ in the far North, which seems to lie in the range from 100 to 400 m above sea level (Matthes, 1942). We shall assume in this section that the center of an ice cap does not change its position as the ice cap changes in size. The next section will consider the more realistic case where the center moves as the ice cap grows or shrinks.

The analysis of the previous section is valid with the exception of eq 15a, which, in order to take eq 17 into account, should be replaced by

$$h_s \equiv \bar{h} + sR = a^{3/5}\ (c/\bar{a})^{2/5} R^{3/5} \equiv h_f\ . \qquad (18)$$

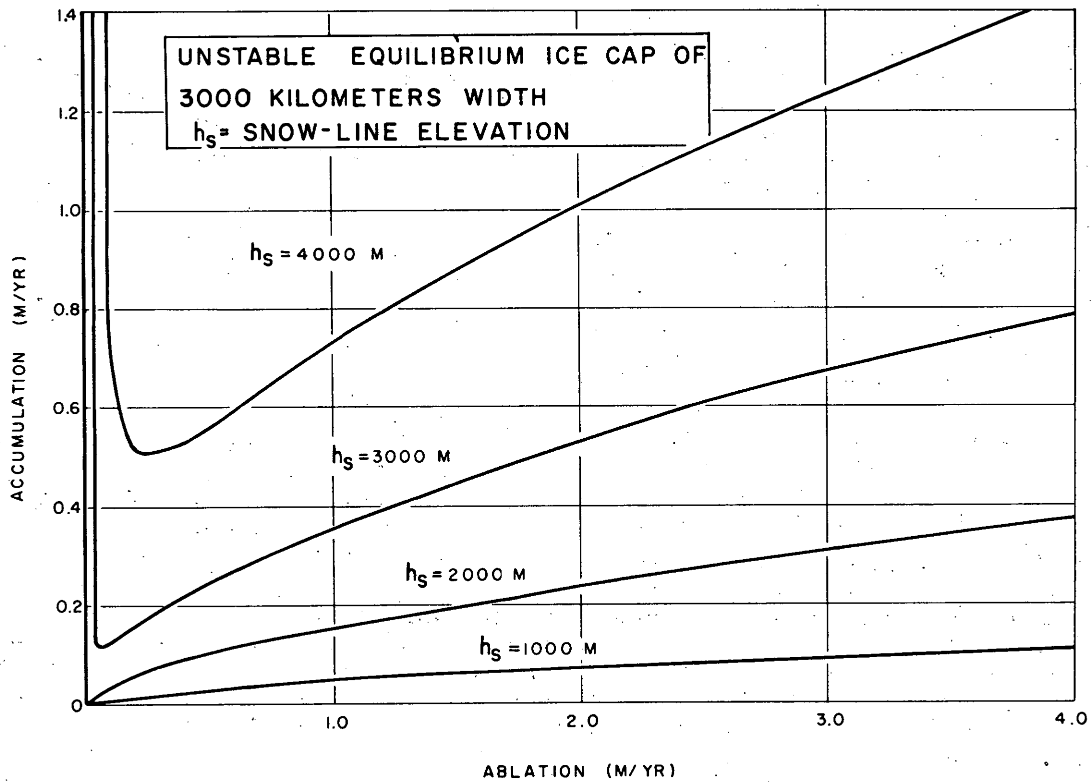

Figure 4. Curves of accumulation vs ablation for an unstable equilibrium ice cap 3000 km wide for four different snowline elevations.

There may be one, two, or no values of R which satisfy eq 18. These values of R represent the points of intersection of the curves of h_f and h_s vs R shown in Figure 2. If the two curves in this figure never meet (this occurs when a is very small, or $\bar{a}$ or s is very large) there is no possible equilibrium ice cap and any ice cap already in existence will shrink and disappear. If s is equal to zero, only one intersection point exists. This is the case of the last section, and there we found that the ice cap is not stable. If s is increased from zero to some finite value, two intersections will occur. The intersection with the smaller value of R gives a profile that approximates the case where s is equal to zero and clearly corresponds to an unstable ice cap. Consider the intersection at the larger value of R. Let us take the case where R is so large that $sR > \bar{h}$. In this situation, R is approximately equal to

$$R = \left(\frac{1}{s}\right)^{5/2}\left(\frac{c}{\bar{a}}\right) a^{3/2} \qquad (19)$$

and h_s, H, and L in eq 10 become

$$h_s \cong \left(\frac{1}{s}\right)^{3/2}\left(\frac{c}{\bar{a}}\right) a^{3/2} \qquad (20)$$

$$H \cong \left(\frac{1}{s}\right)^{3/2}\left(1 + \frac{a}{\bar{a}}\right)^{2/5} c\left(\frac{a}{\bar{a}}\right)^{3/5} a^{1/2} \qquad (21)$$

$$L \cong \left(\frac{1}{s}\right)^{5/2}\left(1 + \frac{a}{\bar{a}}\right)\left(\frac{c}{\bar{a}}\right) a^{3/2} \qquad (22)$$

The magnitudes of H and L in these equations increase with increasing a and decreasing $\bar{a}$ and decrease with decreasing a and increasing $\bar{a}$. Hence, the profile obtained is a stable equilibrium one. If the rates of accumulation and ablation are changed, the ice cap will be able to approach a new equilibrium profile.

Effect of change in position of center of ice cap

The center of an ice cap located on a continent in the Northern Hemisphere will shift southward as the ice cap grows. The northern edge of the ice cap will terminate at the Arctic Ocean. The northern half of the ice cap will have essentially no ablation area. Equations 9 and 10, therefore, describe the northern half of the profile. If eq 10 is compared with eq 16, it will be seen that the width of the southern half of the cap is essentially the same as that of the northern half if average ablation is larger than average accumulation in the southern half, an assumption which appears to be reasonable. Hence, the center of the ice cap will be approximately midway between the Arctic Ocean and the southern edge.

Because the center of an ice cap migrates southward, the analysis of the previous section has to be modified slightly. The height of the equilibrium firn line still is

$$h_f = a^{3/5}(c/\bar{a})^{2/5}R^{3/5}$$

where R is measured from the center, but the snowline at R now is given by

$$h_s \equiv \bar{h} + s(L' + R)$$

where L' is the width of the northern half of the ice cap. Since R, L, and L' all have about the same value if a is less than $\bar{a}$, we can use

$$h_s \equiv \bar{h} + 2sR = a^{3/5}(c/\bar{a})^{2/5}R^{3/5} \equiv h_f \quad (23)$$

to replace eq 18 and thus to obtain the ice cap profile. When eq 23 is used for h_s and h_f, eq 19 to 22 become (for $\bar{h} << 2sR$)

$$R = (1/2s)^{5/2}(c/\bar{a})a^{3/2} \quad (24)$$

$$h_s = (1/2s)^{3/2}(c/\bar{a})a^{3/2} \quad (25)$$

$$H = (1/2s)^{3/2}(1 + a/\bar{a})^{2/5}c(a/\bar{a})^{3/5}a^{1/2} \quad (26)$$

$$L = (1/2s)^{5/2}(1 + a/\bar{a})(c/\bar{a})a^{3/2}. \quad (27)$$

To show how sensitive the width of a stable equilibrium ice cap is to accumulation and ablation conditions, we have plotted curves of accumulation versus ablation for ice caps of specific widths (Fig. 5, 6). We have set $s = 10^{-3}$ and chosen values for $\bar{h}$ of 100 and 400 m. The width in these figures is assumed to equal twice R, and R is determined from eq 23. For a fixed value of $\bar{h}$, there is a minimum width below which it is impossible for an ice cap to exist in either a stable or unstable equilibrium. (This minimum width occurs when the two curves of Figure 2 are exactly tangent to one another.) The region in the accumulation versus ablation diagram where equilibrium ice sheets cannot exist is labeled the forbidden region. Regardless of its initial size, a non-equilibrium ice cap whose accumulation and ablation rates lie in this region will disappear in time. It should be noted that small changes in the rate of accumulation can change the width of a stable equilibrium ice cap by relatively large amounts and can even make it impossible for an equilibrium ice cap to exist.

The width of a stable ice cap is particularly sensitive to the rate of change, s, of the elevation of the snowline. To illustrate this sensitivity, we have plotted curves of accumulation vs width of equilibrium ice caps (both stable and unstable), for a fixed value of ablation (Fig. 7). Two different values of s were used to calculate these curves from eq 23. On each curve, points to the right of the minimum represent stable equilibrium widths and points to the left unstable widths. The curve corresponding to the smaller value of s (the lower rate of change of snowline elevation with distance) has a

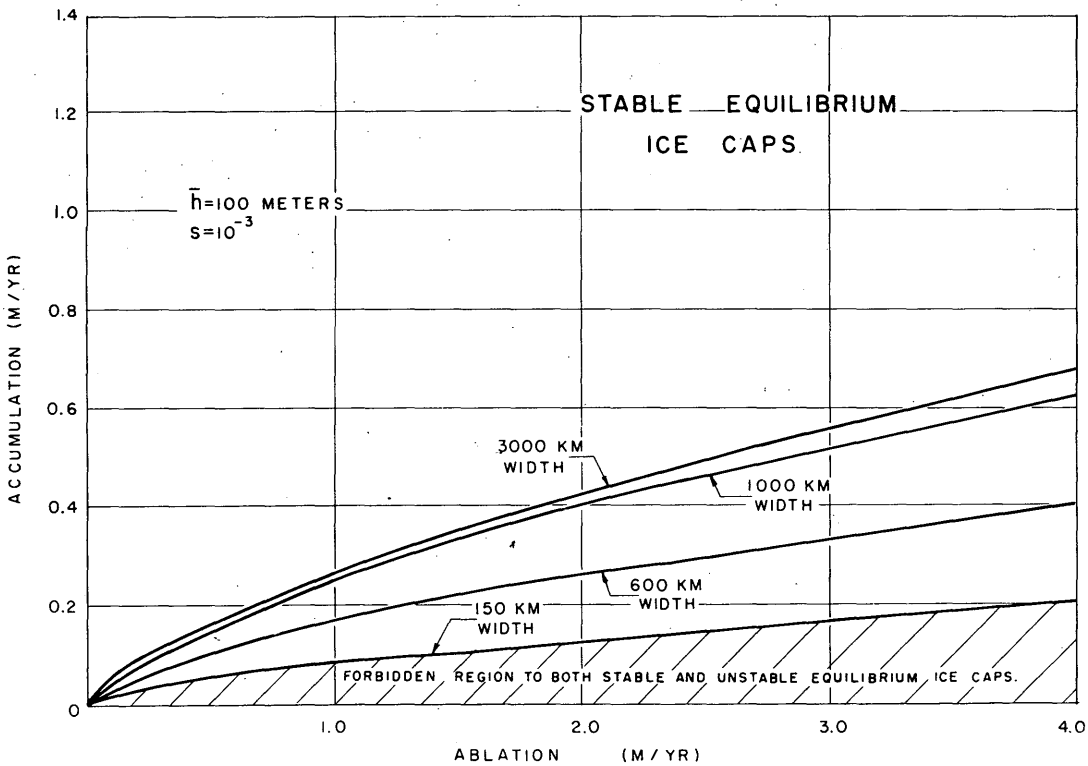

Figure 5. Curves of accumulation vs ablation for stable equilibrium ice caps of four different widths when the snowline elevation is a linear function of distance.

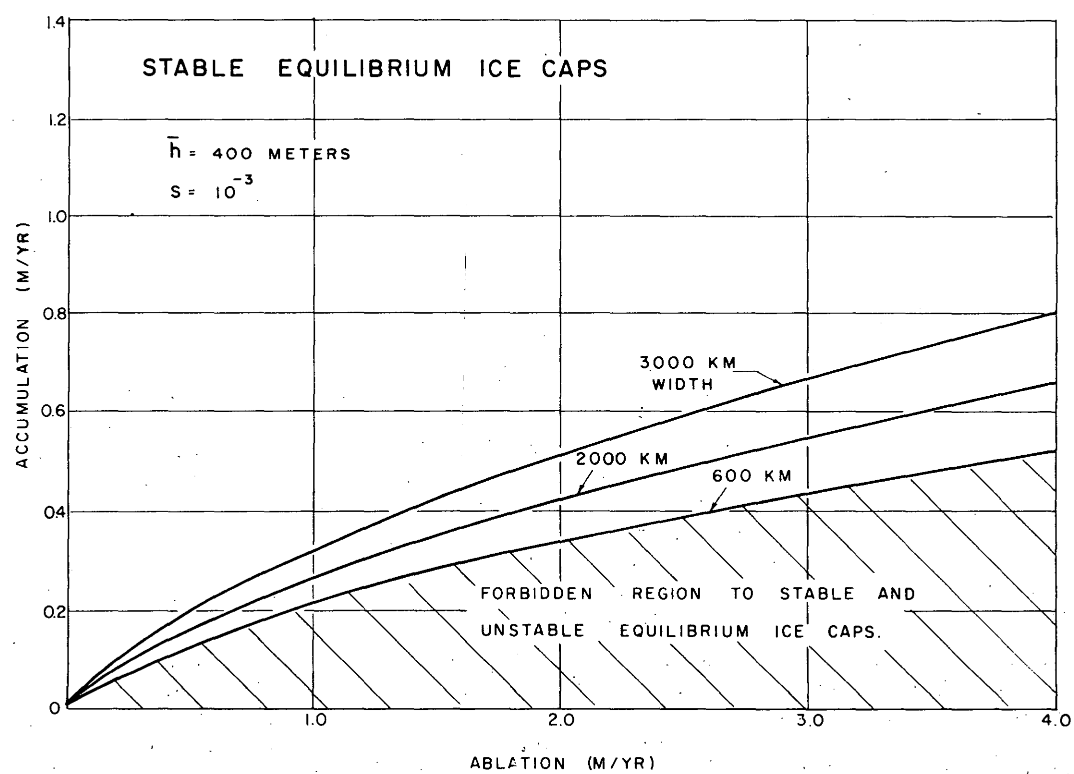

Figure 6. Curves of accumulation vs ablation for stable equilibrium ice caps of three different widths when the snowline elevation is a linear function of distance.

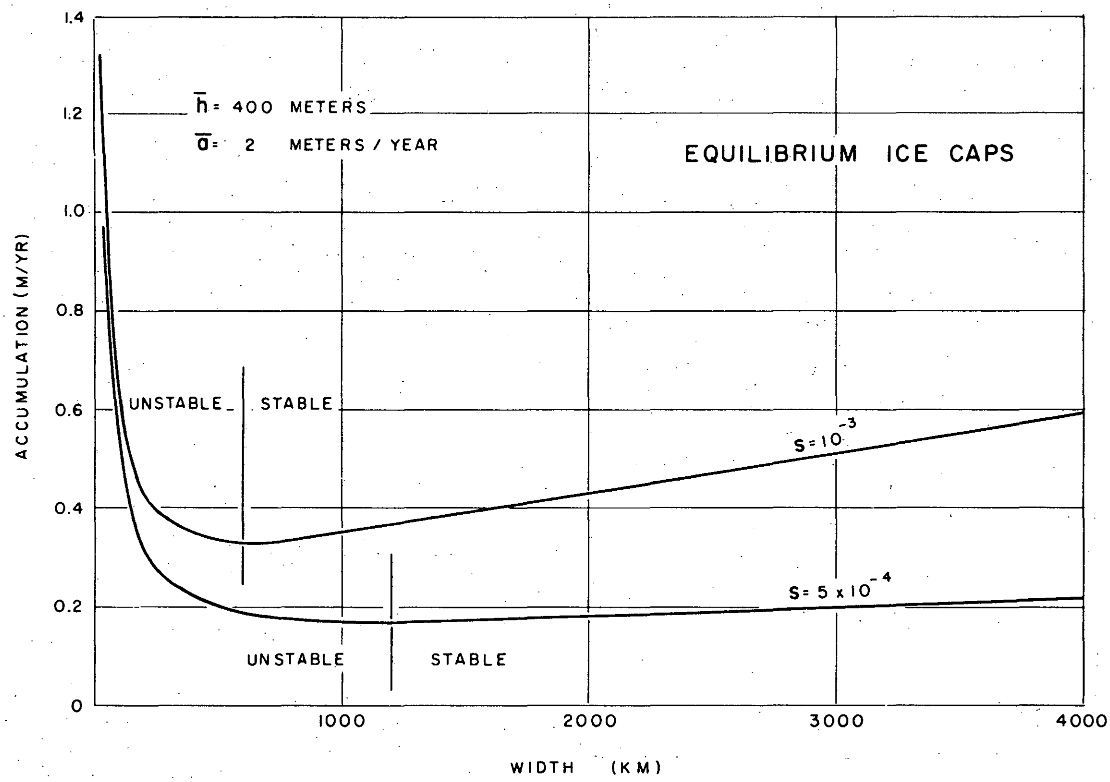

Figure 7. Curves of accumulation vs equilibrium width of ice caps when the snowline is a linear function of distance and the ablation rate is 2 m/yr.

smaller slope in the stable region. Hence, changing the rate of accumulation results in a larger effect on the width of the equilibrium ice cap for the curve with the smaller value of s. This situation will be true in general. The smaller s is, the more sensitive the width of a stable equilibrium ice cap is to changes in accumulation (and also ablation). The larger s is, the more insensitive the ice cap is to changes in these variables. In an extreme case, s might be infinite (and $\bar{h}$ equal to minus infinity), which represents the case of accumulation out to a fixed distance from the center of an ice cap and ablation at greater distance. The width of a stable equilibrium ice cap now is twice the product of this fixed distance and $(1 + a/\bar{a})$; when $\bar{a}$ is greater than a, the width is essentially constant. An example of this situation is the Antarctic (and, to a lesser extent, Greenland). Here the boundary between land and ocean fixes the position of the border between the accumulation and ablation areas (the ocean causes $\bar{a} = \infty$) and permits the existence of a stable equilibrium ice cap whose width is insensitive to changes in rates of accumulation.

In contrast to the Antarctic ice sheet, ice-age ice sheets would be expected to have a snowline with a small value of s. It seems reasonable to believe that, as an ice-age ice sheet advances, the snowline will be lowered below present-day values. When an ice-age ice cap reaches a large size, the equation for h_s possibly could be approximated by a linear relationship similar to the one we have used, but with a smaller value of s than exists today.* Hence, the stable equilibrium width of an ice-age ice sheet should depend very strongly on the rates of accumulation and ablation. Even if the rate of change of

* The value of $\bar{h}$ would have to be increased in order to insure that h_s be a function that increases continuously with distance, but with a decreasing slope.

h_s is of the order of present-day values (those used in calculating the curves of Fig. 7), the width of stable equilibrium is quite sensitive to accumulation and ablation conditions. For example, in the case of the curve in Figure 7 with $s = 5 \times 10^{-4}$, a change in the accumulation rate from about 22 cm ice/yr to 17 cm ice/yr would decrease the width from 4000 km to 1200 km. Any accumulation rate lower than 17 cm/yr would lead to the complete disappearance of the ice cap.

Application to glaciers

The analysis of the stability of an ice sheet resting on a flat base also can be applied to a glacier descending a mountain or to a small ice cap which has formed in an elevated region and whose edge must descend into a lower elevation as it expands.

A glacier normally rests on a sloping bed. If the slope of the bed is β, the equation which describes the profile of the glacier is

$$B\,(\rho g)^2\, h^3\, (\beta + dh/dx)^2 = \int_0^x A\, dx \tag{28}$$

rather than eq 7, which applies to an ice cap on a flat base. In eq 28, x is measured parallel to the bed. A profile similar to that of an ice cap will be obtained from this equation.

Since the coordinate system for the glacier is tilted at a slope β compared to a system whose horizontal distance is parallel to the earth's surface, the snowline elevation will be given by the equation

$$h_s = \bar{h} - \beta x \tag{29}$$

where s has been set equal to $-\beta$ (β is less than 0). (We assume in eq 29 that the elevation of the snowline in the mountains remains at a constant elevation above sea level.) The value of $\bar{h}$ in eq 29 depends on the location of the origin of x. Typical values of β range around 10^{-2}, an order of magnitude larger than the estimates we have used for s. From eq 22, which should be qualitatively correct when applied to glaciers, it can be seen that the length of a stable equilibrium profile depends inversely on s to the 5/2 power, if $sR > \bar{h}$. Stable equilibrium profiles for glaciers should occur at lengths which are 100 to 1000 times smaller than those of continental ice caps. This situation results from the fact that β is at least 10 times larger than the values of s appropriate to large ice caps. Since continental ice sheets had dimensions of the order of 2000 km, stable equilibrium glaciers could be expected to, and do, exist for lengths of 1 to 10 km.

Discussion

The analysis presented in this paper suggests that a small ice cap can become unstable and grow to a large size if the ice cap exceeds a critical width. The critical nucleation size can be of the order of the dimensions of existing ice caps in the Arctic ($\sim$ 30 km) when the snowline elevation is 100 to 400 m above sea level and the product $(\bar{a})^{2/3}/a$ is less than, or equal to, 1 to 10 $(m/yr)^{-1/3}$ (for example, an ablation rate of 1 m/yr and an accumulation rate of 1 m/yr). If an ice cap is less than this critical size, it should disappear completely. The fact that existing ice caps persist can be explained by the reason mentioned in the previous section or by local weather pecularities.

Theories of the ice ages* usually assume that an ice age starts as a result of increased accumulation or decreased ablation, just those conditions which would be required to make a small ice cap "go critical." Ewing and Donn (1956), for example, propose that an ice age begins when the Arctic Ocean becomes ice-free leading to increased snowfall on the surrounding lands. It may not be necessary, however, to invoke some special event to make a small ice cap go critical. Normal weather fluctuations may produce a century of greater than normal accumulation and less than normal ablation. These conditions then may induce a small ice cap to start growing to a large size.

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* Flint (1947) has reviewed the more important work in this field.

If a small ice cap did grow into a large ice-age ice sheet, is it possible for the sheet ever to shrink again without a significant change in the world's weather conditions (other than that change produced by the ice sheet itself)? The accumulation on the ice sheet itself would be expected to decrease as it becomes bigger, both because the accumulation area will be at a high elevation and because the cooling of the earth by the presence of the sheet may lead to reduced precipitation. The Antarctic ice sheet has low rates of accumulation ($\sim$ 10 to 20 cm of ice/yr), as does the smaller Greenland Ice Cap ($\sim$ 10 to 40 cm of ice/yr), and a large ice-age ice sheet also might be expected to have such low values. An inspection of Figures 3 to 7 will show that when accumulation rates fall to these values, an ice cap easily could become unstable and shrink to nothing. If low accumulation rates once set in, and if they persist, there would be no serious problem in explaining the shrinkage of ice-age ice sheets. However, one might expect that as a large ice cap began shrinking, the accumulation rate would increase again (and the ablation rate decrease as the edge shifts northward). Ewing and Donn circumvent this difficulty by their proposal that the Arctic Ocean, which was ice-free at the beginning of the ice age, freezes over again and the ice cover persists until after the complete disappearance of the continental ice caps. According to their theory, the snow precipitation rates are controlled by the Arctic Ocean and low rates occur so long as the Arctic Ocean is covered with ice. If their theory is correct, there is no difficulty in understanding why an ice-age ice cap suddenly becomes unstable and shrinks to nothing. As the ice cap grows and the Arctic Ocean freezes over, the accumulation rates decrease until the instability discussed in this paper overtakes the ice sheet, which then shrinks until it disappears.

There is another way in which an instability in an ice-age ice sheet might be brought about. A small ice cap starting to grow in the far North is likely to be frozen to its bed. If such is the case, the ice cap will not be able to slide over its bed and the effective value of $\underline{B}$ in eq 5 will be reduced and the value of $\underline{c}$ in eq 11 increased. As the ice sheet grows, it still may remain frozen to its bed until it reaches a large size. Small thickness and large accumulation rates favor a cold ice cap being frozen to its bottom (Robin, 1955; Weertman, 1961b) and large thickness and small accumulation rates increase the probability that the bottom will be at the melting point. Now suppose that, when the ice sheet reaches a large size, the temperature at the bottom of the cap rises until the ice there is at the melting point and thus the ice cap can slide over its bed. The value of $\underline{c}$ will be decreased. From studies made on temperate glaciers of the fraction of motion which is due to sliding and the fraction which is due to differential motion within the glaciers, one can estimate that $\underline{c}$ will be decreased by a factor of about one-half. An inspection of the equations developed in this paper will show that a decrease in $\underline{c}$ by one-half is essentially equivalent to an increase in the ablation rate by a factor of two. Such a change could lead to the instability of an ice-age ice sheet and its subsequent shrinkage. The decreased value of $\underline{c}$ would persist as the ice sheet decreases in size for the following reason. The bottom of a cold ice cap can be at the melting point because of the geothermal heat flowing up through the earth. In addition, if an ice cap is sliding on its bottom, the heat of sliding helps keep the bottom at the melting point. The heat of sliding is of the same order of magnitude as the geothermal heat (Weertman, 1957, 1961b). Because of this extra heat from the sliding process itself (Weertman, 1961b), once an ice cap originally frozen to its bed starts to slide, it could continue to slide even though brought back to the conditions where formerly it had been frozen.

Another factor which may be responsible for initiating instability in a continental ice cap is the isostatic sinking of the ice-cap bed as the ice sheet becomes large. The ice-cap profile can be found quite easily when this sinking occurs (Weertman, 1961a). In such a situation, eq 7 is replaced

$$B(\rho g)^2 \gamma^3 h^3 (dh/dx)^2 = \int_0^x A\,dx \qquad (30)$$

where $\underline{h}$ now is the present height of the upper ice cap surface above the original position of the bed before sinking occurred and $\gamma = (1 - \rho/\rho_r)^{-1}$, where ρ_r is the average density of rock below the ice cap ($\rho_r \approx 3\rho$). Eq 30 can be considered to be the same as eq 7 but with a $\underline{B}$ which is $\gamma^3 \approx 3.4$ larger. As before, this apparent increase in $\underline{B}$ can be considered to be equivalent to an increase in the ablation rate by a factor of 1.8 to 1.9.

Such an increase possibly could set off an instability in an ice cap. If this mechanism is to be effective, however, the time for isostatic sinking to occur must be longer than the time required to build up a large ice sheet. That is, a large ice sheet must build up before any sinking occurs. Then the sinking, when it does take place, becomes equivalent to an increase in ablation rate, which allows the sheet to decrease in size. The decrease must take place slowly so that an isostatic rise can keep up with the decreased volume of ice.

The difficulty in maintaining a low accumulation rate or high ablation rate while a large ice cap shrinks actually may not be so troublesome as it appears. If the accumulation rate depends solely on the average height of an ice cap, there will be only a small variation in the rate once the cap becomes large. From eq 21 or 26, it can be seen that the height at the center of a stable equilibrium ice cap actually can be independent of the width of the ice cap. Hence, there may be little change in the accumulation rate until the ice cap has decreased its width by a factor of two or three.

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