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AN ANALYTICAL INVESTIGATION OF A MODIFIED STEFAN PROBLEM

by

Yin-Chao Yen

and

Chi Tien

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U.S. ARMY MATERIEL COMMAND
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PREFACE

This paper was prepared by Dr. Yen, Research Engineer and Dr. Tien, Expert, of the Physical Science Group, Research Division (J.A. Bender, Chief), U.S. Army Cold Regions Research and Engineering Laboratory (USA CRREL).

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SUMMARY

Approximate solutions of temperature distribution and melting rate of ice have been obtained for the case where the mode of heat transfer is natural convection due to the thermal instability caused by the heated lower surface. Extensive numerical solutions were obtained for the ice-water system corresponding to various thermal conditions in terms of parameters defined as $R_{\Delta T} = (T_s - T_m)/(T_m - T_0)$ and $\phi = L/C_p (T_m - T_0)$, where T_s is the temperature of the heat source, T_m is melting point of ice, T_0 is the initial temperature of ice, L is the latent heat of fusion, and C_p is heat capacity of ice.

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Introduction

The solution of heat conduction involving a change of phase has been studied extensively in the past because of its theoretical as well as practical importance. The classical Stefan problem describes the physical situation in which a semi-infinite solid, initially at a uniform temperature T_0 ($T_0 < T_m$, melting temperature) is subject to a high surface temperature T_s ($T_s > T_m$). The solution provides the expression of temperature distribution in both phases as well as the rate of melting. These are given as (Carslaw and Jaeger, 1959):

$$S(t) = 2\lambda(a_1 t)^{\frac{1}{2}}. \qquad (1)^*$$

λ is the root of the following expression:

$$\frac{\exp(-\lambda^2)}{\text{erf}\,\lambda} - \frac{k_2}{k_1}\sqrt{\frac{a_1}{a_2}}\left(\frac{T_m - T_0}{T_s - T_m}\right)\frac{\exp(-\lambda^2 \frac{a_1}{a_2})}{\text{erfc}\,[\lambda\sqrt{\frac{a_1}{a_2}}]} = \frac{\lambda L \sqrt{\pi}}{(C_p)_1(T_s - T_m)}. \qquad (2)$$

The temperature distributions in the liquid and solid phases are given as

$$\frac{T_1 - T_0}{T_s - T_0} = 1 - \left(\frac{T_s - T_m}{T_s - T_0}\right)\frac{\text{erf}\,[\frac{x}{2\sqrt{a_1 t}}]}{\text{erf}\,\lambda}, \quad 0 \leq x \leq S(t) \qquad (3)$$

$$\frac{T_2 - T_0}{T_m - T_0} = \frac{\text{erfc}\,[\frac{x}{2\sqrt{a_2 t}}]}{\text{erfc}\,[\lambda\sqrt{\frac{a_1}{a_2}}]}, \quad x \geq S(t). \qquad (4)$$

The subscripts 1 and 2 refer to the liquid phase and solid phase respectively, and the meanings of the symbols are given in the notation.

These equations have been applied to a large number of practical problems and sometimes have been used without proper recognition of the assumption that the only mode of heat transfer in both phases is by conduction. Although this is a reasonable assumption for the solid phase, it is not always true for the liquid phase. For example, if the problem is such that the higher temperature T_s is maintained at the lower side of the system, the liquid density decreases with the increase of temperature. The assumption of conduction in the liquid is valid only as long as the system remains stable, which is indicated by the magnitude of the Rayleigh number of the system. The onset of instability begins when the Rayleigh number exceeds its critical value, which is found (Chandrasekhar, 1961; Rayleigh, 1916) to be

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* See Appendix A for notation.

$$(N_{Ra})_c = \frac{g \beta \rho^2 C_p}{\mu k} \ell^3 (T_s - T_m) \tag{5}$$

where ℓ, the characteristic length, refers to the depth of the liquid. In the melting problem to be considered in this paper, ℓ becomes the same as S, the melting front or liquid-solid interface (this is true only if there is no change of density). From the definition of the Rayleigh number, it is obvious that with a given set of initial and boundary conditions the Rayleigh number for the liquid phase increases as the melting proceeds. The heat transfer mode changes from conduction to convection when the melting has reached the extent where the Rayleigh number exceeds the critical value. The classical Stefan solution (eq 1-4) would no longer be valid when the mode of heat transfer in the liquid phase changes to convection.

It is the purpose of the present investigation to consider this aspect of the problem when the liquid phase heat transfer is affected by convection. The problem was solved approximately using the heat balance integral techniques of Goodman (1958). This will be shown in the following sections.

Formulation of problem

For simplicity, the problem can be stated as follows: A semi-infinite solid extending $x \geqslant 0$ where x is taken opposite to the direction of gravitational force, initially at uniform temperature T_0 ($T_0 < T_m$, melting temperature), is subject to a change of surface temperature at $t \geqslant 0$, at $x = 0$. The surface temperature (T_s) is assumed to be constant and $T_s > T_m$. As a consequence, temperature change within the system as well as melting will take place. Figure 1 gives a diagrammatic description of the problem.

For the classical Stefan problem, the following equations can be written:

$$\frac{\partial T_1}{\partial t} = \alpha_1 \frac{\partial^2 T_1}{\partial x^2}, \quad x < S(t) \tag{6}$$

$$\frac{\partial T_2}{\partial t} = \alpha_2 \frac{\partial^2 T_2}{\partial x^2}, \quad x > S(t) \tag{7}$$

$$T_2 = T_0, \quad x \geqslant 0, \quad t = 0 \tag{8}$$

$$T_1 = T_2 = T_m, \quad x = S(t) \tag{9}$$

$$-k_1 \frac{\partial T_1}{\partial x} = -k_2 \frac{\partial T_2}{\partial x} + \rho L \frac{dS}{dt}$$

$$T_2 \rightarrow T_0, \quad x \rightarrow \infty \tag{10}$$

$$T_1 = T_s, \quad x = 0 \tag{11}$$

and $S = 0, \quad t = 0.$

Complete solutions of eq 6-11 are given by eq 1-4 as stated before. Equation 6 presupposes that conduction prevails in the liquid phase. Since the liquid phase

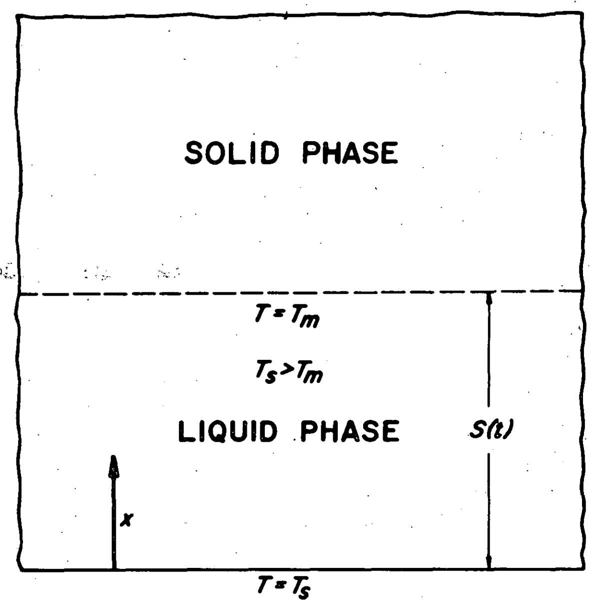

Figure 1. Schematic diagram of the melting problem.

is formed at the lower part of the system and is subject to a negative temperature gradient such as $T = T_s$ at $x = 0$, and $T = T_m$ at $x = S(t)$ and $T_s > T_m$, the system is inherently unstable for most liquids whose density dependence on temperature is negative. The liquid phase resembles the classical Rayleigh problem in which the hydrodynamic stability of a liquid confined between two horizontal plates and heated from below was studied. Numerous theoretical and experimental investigations have been reported in the literature concerning this subject and the criteria of stability for the case of two rigid surfaces is given as

$$(N_{Ra})_c = \frac{g \beta \rho^2 C_p}{\mu k} S^3 (T_s - T_m) = 1720.$$

Since the assumption of conduction in the liquid phase is valid only as long as the system remains stable, the classical Stefan's solution (eq 1-4) is applicable for $S(t) < S_c$ where S_c can be termed a transitional melting front which characterizes the change of heat transfer mode from conduction to convection in the liquid phase. S_c is given as

$$S_c = \left[1720 \frac{\mu_l k_l}{g \beta_l \rho_l^2 C_{p_l} (T_s - T_m)} \right]^{\frac{1}{3}}$$

$$= \left[1720 \frac{\mu_l^2}{g \beta \rho_l^2} \left(\frac{k_l}{C_{p_l} \mu_l}\right) \frac{1}{(T_s - T_m)} \right]^{\frac{1}{3}} \qquad (12)$$

or the corresponding value of t_c can be found by combining eq 12 and 1. This is given as

$$t_c = \frac{S_c^2}{4 \lambda^2 \alpha_l} . \qquad (13)$$

For water, $(g\,\beta\,\rho^2)/\mu$ is of the order of $10^6 \sim 10^7$ in ft^3/F and the Prandtl number is around 6~8. If $(T_s - T_m)$ is taken as 10F, $S_c(t)$ would be of the order of 10^{-2} to 10^{-1} ft. This seems to indicate that Stefan's solution is applicable only in the initial period of melting.

Once the melting front reaches the value of S_c, eq 6 will no longer be applicable to describe the changes of temperature of the liquid phase. In order to describe the problem adequately, a proper equation of change for the region $x < S(t)$ should be written taking into account the convective motion induced by thermal instability. Unfortunately, this cannot be done without difficulty since the actual mechanism of the convective motion is quite involved and not completely understood. This problem, however, can be greatly simplified by assuming that the rate at which the temperature distribution reaches its steady state in the liquid phase is much greater than the melting rate. Consequently, as far as the heat transfer in the liquid phase is concerned, one can ignore the change of geometric dimension due to melting. Under this assumption, the problem can be reduced to a simpler melting problem in which a prescribed heat flux is imposed on the solid-liquid interface. The heat flux imposed on the surface can be estimated from the available empirical correlation for natural convection heat transfer coefficient of liquid in confined space. Mathematically, the problem can be described as

$$\alpha_2 \frac{\partial^2 T_2}{\partial x^2} = \frac{\partial T_2}{\partial t}, \quad x > S(t), \ t > t_c \tag{14}$$

$$T_2 = T_m, \ x = S(t) \tag{15}$$

$$-k_2 \frac{\partial T_2}{\partial x} + \rho L \frac{dS}{dt} = H(t), \quad x = S(t) \tag{16}$$

$$T_2 \rightarrow T_0, \ x \rightarrow \infty \tag{17}$$

$$T_2 = f(x) \text{ for } x \geqslant S_c \text{ at } t = t_c. \tag{18}$$

$f(x)$ is given by eq 4 by letting $t = t_c$, which is given as

$$f(x) = T_0 + (T_m - T_0)\frac{\operatorname{erfc}\left[\sqrt{\frac{\alpha_1}{\alpha_2}}\ \frac{\lambda x}{S_c}\right]}{\operatorname{erfc}\left[\sqrt{\frac{\alpha_1}{\alpha_2}}\ \lambda\right]} \tag{19}$$

In order to estimate the surface heat flux $H(t)$ the correlation by O'Toole and Silveston (1961) of natural convection heat transfer for fluid confined between two parallel horizontal plates will be used. This correlation is given as follows:

$$N_{Nu} = \frac{hS}{k_1} = 0.00238\,(N_{Ra})^{0.816}, \quad 1720 \leqslant N_{Ra} \leqslant 3500 \tag{20a}$$

$$N_{Nu} = \frac{hS}{k_1} = 0.229\,(N_{Ra})^{0.252},\ 3500 \leqslant N_{Ra} \leqslant 10^5 \tag{20b}$$

$$N_{Nu} = \frac{hS}{k_1} = 0.104\,(N_{pr})^{0.084}\,(N_{Ra})^{0.305},\ 10^5 \leqslant N_{Ra} \leqslant 10^8. \tag{20c}$$

These three equations correspond to what is termed initial, laminar, and turbulent regime, respectively.

In terms of the extent of melting, the ranges of applicability for these three expressions are $1 \leqslant S/S_c \leqslant 1.26$, $1.26 \leqslant S/S_c \leqslant 3.9$, and $3.9 \leqslant S/S_c \leqslant 39.0$. Furthermore, the expression given by eq 20b differs from that of eq 20c only slightly for the case of water. For example, for $N_{Ra} = 10^4$, N_{Nu} is given as 2.34 from eq 20b as compared with 2.05 from eq 20c and for $N_{Ra} = 10^5$, N_{Nu} is found to be 4.17 from eq 20b and 4.16 from eq 20c. This seems to indicate that for practical purposes the heat transfer coefficient to be used for the estimation of heat flux can be obtained from eq 20c even for a Rayleigh number beyond its lower limit of applicability. This gives

$$h = (0.104)(N_{pr})^{0.084}\left[g\,\beta\,(T_s - T_m)\,\frac{\rho^2 C_p}{\mu k}\right]^{0.305} S^{-0.085} k_1 \tag{21}$$

and $H(t) = h\,(T_s - T_m)$

$$= (0.104)(N_{pr})^{0.084}\left[g\,\beta(T_s - T_m)\,\frac{\rho^2 C_p}{\mu k}\right]^{0.305} S^{-0.085} k_1\,(T_s - T_m). \tag{22}$$

Equations 14-22 can be written into the following dimensionless forms:

$$\frac{\partial^2 \theta}{\partial x^{+2}} = \frac{\partial \theta}{\partial t^+},\quad x^+ \geqslant S^+(t^+) \tag{23}$$

$$\theta = 1,\quad x^+ = S^+ \tag{24}$$

$$\theta \rightarrow 0,\quad x^+ \rightarrow \infty \tag{25}$$

$$-\frac{\partial \theta}{\partial x^+} + \phi\,\frac{dS^+}{dt^+} = H^+(t),\quad x^+ = S^+ \tag{26}$$

$$\theta = \frac{\operatorname{erfc}\left[\sqrt{\frac{\alpha_1}{\alpha_2}}\,\lambda x^+\right]}{\operatorname{erfc}\left[\sqrt{\frac{\alpha_1}{\alpha_2}}\,\lambda\right]},\quad x^+ \geqslant 1,\ t^+ = t_c^+ \tag{27}$$

$$S^+(t_c^+) = 1 \tag{28}$$

$$\theta = \frac{T_2 - T_0}{T_m - T_0} \tag{29}$$

$$x^+ = \frac{x}{S_c} \tag{30}$$

$$t^+ = \frac{\alpha_2 t}{S_c^2} \tag{31}$$

$$\phi = \frac{L}{(C_{p_2})(T_m - T_0)} \tag{32}$$

$$H^+(t) = \frac{S_c}{k_2(T_m - T_0)} H(t)$$

$$= (0.104)(N_{pr})^{0.084}\left[g\,\beta(T_s - T_m)\frac{\rho^2 C_p}{\mu k}\right]^{0.305} S_c\, S^{-0.085}\,\frac{T_s - T_m}{T_m - T_0}\left(\frac{k_1}{k_2}\right)$$

$$= (0.104)(N_{pr})^{0.084}\left[g\,\beta(T_s - T_m)\frac{\rho^2 C_p}{\mu k}\, S_c^{\,3}\right]^{0.305}\left(\frac{S_c}{S}\right)^{0.085}\frac{T_s - T_m}{T_m - T_0}\left(\frac{k_1}{k_2}\right)$$

$$= R_{\Delta T}\,(S^+)^{-0.085}\left(\frac{k_1}{k_2}\right) \tag{33}$$

where

$$R_{\Delta T} = \frac{T_s - T_m}{T_m - T_0}\,. \tag{34}$$

Equation 23 with its initial and boundary conditions was solved. Generally speaking, for a melting problem, closed form solution for the case with prescribed heat flux is not likely to be found and numerical techniques or approximate methods must be employed. For the present problem the "heat balance integral" method by Goodman (1958) will be used. The essence of this approach assumes that the effect of the heat flux at the surface is felt only within a finite distance from the surface and this distance f^+ is called the dimensionless thermal boundary layer thickness. The conduction equation (eq 23) will be satisfied on the average over this distance. Integrating eq 23 from $x^+ = S^+$ to $x^+ = f^+$, we have

$$\frac{\partial \theta}{\partial x^+}\bigg|_{f^+} - \frac{\partial \theta}{\partial x^+}\bigg|_{S^+} = \int_{S^+}^{f^+} \frac{\partial \theta}{\partial t^+}\, dx^+. \tag{35}$$

The right-hand side can be shown to be equal to

$$\int_{S^+}^{f^+} \frac{\partial \theta}{\partial t^+}\, dx^+ = \frac{d}{dt^+}\int_{S^+}^{f^+} \theta\, dx^+ - \theta\big|_{f^+}\frac{df^+}{dt^+} + \theta\big|_{S^+}\frac{dS^+}{dt^+} \tag{36}$$

or

$$\frac{\partial \theta}{\partial x^+}\bigg|_{f^+} - \frac{\partial \theta}{\partial x^+}\bigg|_{S^+} + \theta\big|_{f^+}\frac{df^+}{dt^+} - \theta\big|_{S^+}\frac{dS^+}{dt^+} = \frac{d}{dt^+}\int_{S^+}^{f^+} \theta\, dx^+. \tag{37}$$

A simple expression will be assumed such as

$$\theta = a + bx^+ + c(x^+)^2. \tag{38}$$

The coefficients a, b and c are to be determined by the compatibility requirement. These are

$$\theta = 1, \quad x^+ = S^+ \tag{39}$$

$$\theta = 0, \quad x^+ = f^+ \tag{40}$$

$$\frac{\partial \theta}{\partial x^+} = 0, \quad x^+ = f^+. \tag{41}$$

Combining eq 38-41, we have

$$\theta = \frac{1}{(f^+ - S^+)^2}\left[(f^+)^2 - 2f^+x^+ + (x^+)^2\right]. \tag{42}$$

Combining eq 42 and 37, we have

$$\frac{1}{3}\frac{d}{dt^+}\left[f^+ - S^+\right] = -\left[\frac{\partial \theta}{\partial x^+}\bigg|_{S^+} + \frac{dS^+}{dt^+}\right]. \tag{43}$$

Combining eq 43 with the boundary conditions given by eq 26 gives

$$\frac{1}{3}\frac{d}{dt^+}(f^+ - S^+) = -\left[\frac{\partial \theta}{\partial x^+}\bigg|_{S^+} + \frac{1}{\phi}\left(\frac{\partial \theta}{\partial x^+}\bigg|_{S^+} + H^+(t)\right)\right]. \tag{44}$$

From eq 42

$$\frac{\partial \theta}{\partial x^+}\bigg|_{S^+} = -\frac{2}{(f^+ - S^+)}. \tag{45}$$

Equation 44 becomes

$$\frac{dy^+}{dt^+} = \frac{3}{\phi}\left[\frac{2(1+\phi)}{y^+} - R_{\Delta T}(S^+)^{-0.085}\left(\frac{k_1}{k_2}\right)\right] \qquad (46)$$

where

$$y^+ = f^+ - S^+. \qquad (47)$$

Also, one can rewrite eq 26 as

$$\frac{dS^+}{dt^+} = \frac{1}{\phi}\left[R_{\Delta T}(S^+)^{-0.085}\left(\frac{k_1}{k_2}\right) - \frac{2}{y^+}\right]. \qquad (48)$$

The problem is now reduced to the solution of a pair of first order differential equations (eq 46 and 48). The initial conditions are given as

$$t^+ = t_c^+,\ S^+ = 1 \qquad (49)$$

$$y^+ = y_c^+ = f_c^+ - 1. \qquad (50)$$

f_c^+ is the initial thermal boundary layer thickness at $t^+ \doteq t_c^+$. This can be obtained by matching eq 27 with eq 42 at $t^+ = t_c^+$. It should be noted that eq 27 is an exact solution while eq 42 was obtained on the concept of "heat balance integral." It is, therefore, not possible to have an equivalence of these two equations. One can only attempt to have an approximate agreement between these two expressions. In the present investigation, the melting rate is of greater interest. Consequently, we will require that the temperature gradient at the interface be the same from both expressions. This gives

$$-\frac{2}{f_c^+ - 1} = -\frac{2}{y_c^+} = -\frac{\frac{2}{\sqrt{\pi}}\,\lambda\sqrt{\frac{a_1}{a_2}}\,\exp\left(-\frac{a_1}{a_2}\lambda^2\right)}{\mathrm{erfc}\left[\sqrt{\frac{a_1}{a_2}}\,\lambda\right]}$$

or

$$y_c^+ = \sqrt{\frac{a_2}{a_1}\pi}\left(\frac{1}{\lambda}\right)\mathrm{erfc}\left(\sqrt{\frac{a_1}{a_2}}\,\lambda\right)\exp\left(\frac{a_1}{a_2}\lambda^2\right). \qquad (51)$$

Solution and discussion

Numerical solutions of eq 46 and 48 for the system of ice-water were obtained because of the importance of this system in various applications. The physical properties of water and ice (in C. G. S. units) used in this computation are summarized as follows:

Water		Ice	
ρ_1	= 1.00	ρ_2	= 0.92
C_{p_1}	= 1.00	C_{p_2}	= 0.502
k_1	= 0.00144	k_2	= 0.0053
a_1	= 0.00144	a_2	= 0.0115
		L	= 80

The actual computation begins with the evaluation of λ from eq 2 for variations of ϕ and $R_{\Delta T}$. For the ice-water system, eq 2 becomes

$$\frac{\exp(-\lambda^2)}{\operatorname{erf}\lambda} - \frac{0.0053}{0.00144}\sqrt{\frac{0.00144}{0.0115}}\ \frac{1}{R_{\Delta T}}\ \frac{\exp\left(-\lambda^2\ \frac{0.00144}{0.0115}\right)}{\operatorname{erfc}\left[\lambda\sqrt{\frac{0.00144}{0.0115}}\right]}$$

$$= \sqrt{\pi}\ (\lambda)\left(\frac{0.502}{1.00}\right)\ \frac{\phi}{R_{\Delta T}}\ .$$

Because of the difficulty involved in obtaining values of λ in terms of $R_{\Delta T}$ and ϕ directly, instead, values of ϕ were calculated for given values of $R_{\Delta T}$ and λ. By interpolation, values of λ were obtained corresponding to the desired value of ϕ. A summary of the results is given in Table I.

Table I. Numerical values of λ.

$R_{\Delta T} = \frac{T_s - T_m}{T_m - T_0}$	$\phi = \frac{L}{(C_{p_2})(T_m - T_0)}$	λ
0.1250	4.00	0.06955
0.1667	5.33	0.08413
0.2500	4.00	0.12164
0.2500	8.00	0.10450
0.3333	5.33	0.14175
0.3333	10.66	0.11770
0.5000	4.00	0.20147
0.5000	8.00	0.16755
0.5000	16.00	0.13341
0.6667	5.33	0.22665
0.6667	10.66	0.18315
1.0000	16.00	0.20080
1.0000	32.00	0.15213
1.3333	10.66	0.27411
2.0000	8.00	0.37640
2.0000	16.00	0.29304
2.0000	32.00	0.22083

Once these values of λ are available, the initial condition y_c^+ can be obtained readily from eq 51, and eq 46 and 48 are solved numerically using the Runge-Kutta method. The results give values of S^+ and y^+ as functions of t^+. The temperature profile in the solid phase can be estimated from these values and eq 42. Numerical results were obtained from S^+ up to approximately 50, although the values beyond $S^+ = 39$ are considered unreliable. A more detailed discussion on this point will be given later. Results are graphically represented in Figures 2-7.

One of the important factors in carrying out numerical solutions of this type of problem is the selection of the increment size of the independent variable (t^+) since this directly determines the accuracy of the results. In this work, various increment sizes of t^+ up to 0.4 were used and the respective results were compared in Table II.

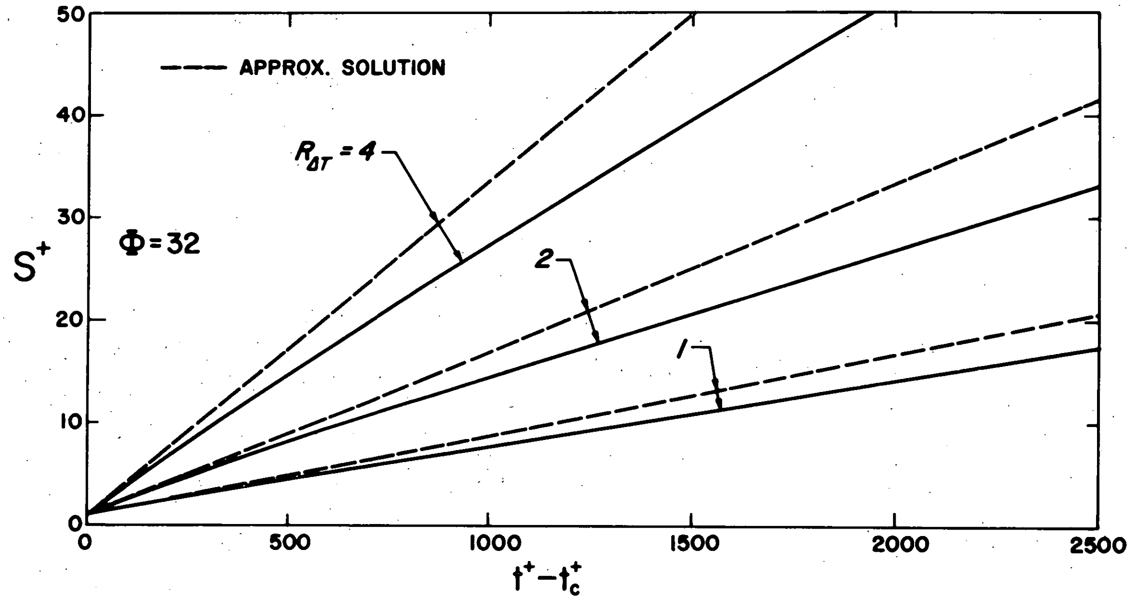

Figure 2. Relationship between S^+ and $t^+ - t_c{}^+$ from exact and approximate solutions for $\phi = 32$, $R_{\Delta T} = 1$, 2, and 4.

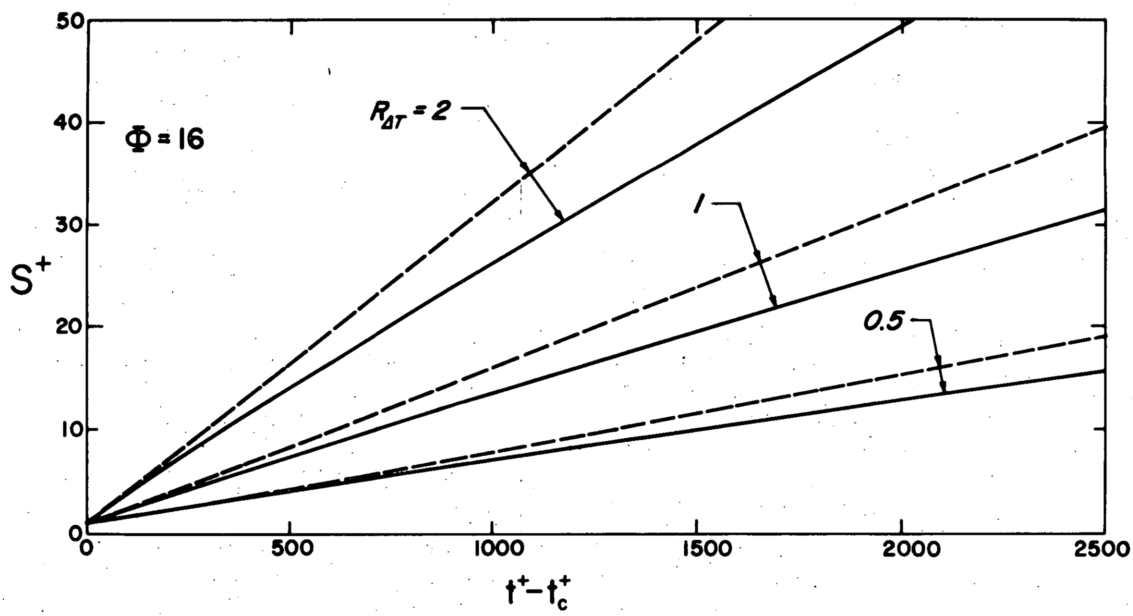

Figure 3. Relationship between S^+ and $t^+ - t_c{}^+$ from exact and approximate solutions for $\phi = 16$, $R_{\Delta T} = 0.5$, 1 and 2.

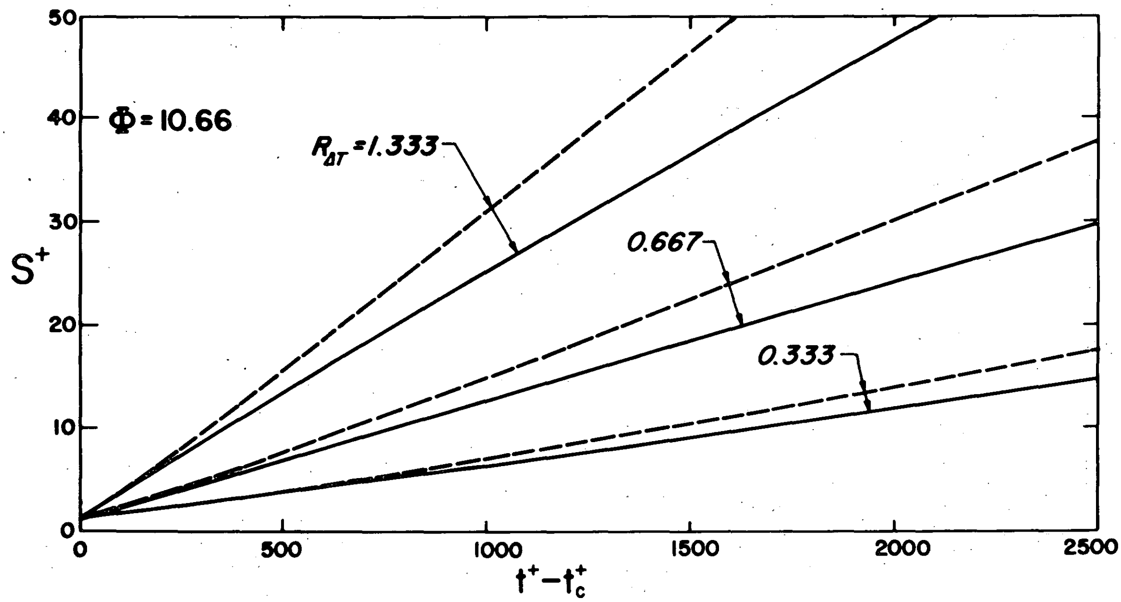

Figure 4. Relationship between S^+ and $t^+ - t_c{}^+$ from exact and approximate solutions for $\phi = 10.66$, $R_{\Delta T} = 0.333$, 0.667 and 1.333.

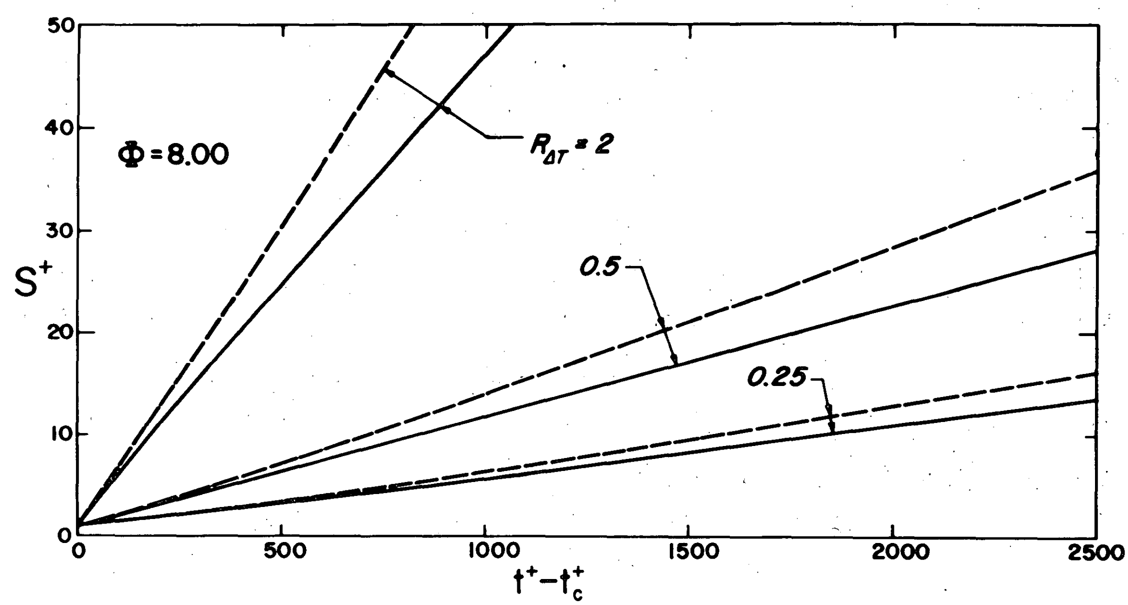

Figure 5. Relationship between S^+ and $t^+ - t_c{}^+$ from exact and approximate solutions for $\phi = 8$, $R_{\Delta T} = 0.25$, 0.5 and 2.

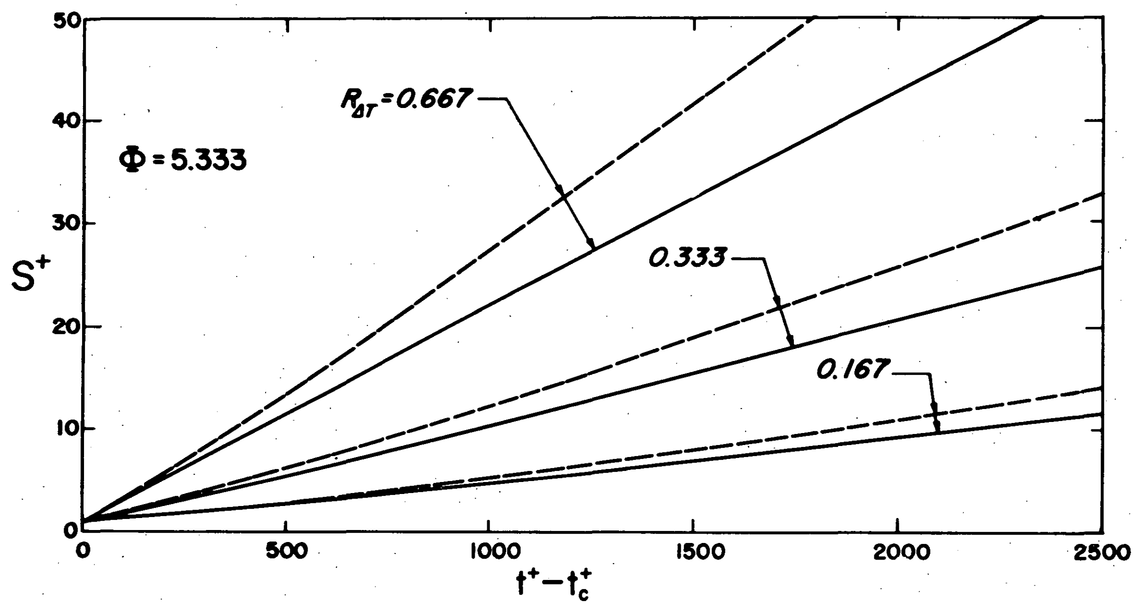

Figure 6. Relationship between S^+ and $t^+ - t_c^+$ from exact and approximate solutions for $\phi = 5.333$, $R_{\Delta T} = 0.167$, 0.333 and 0.667.

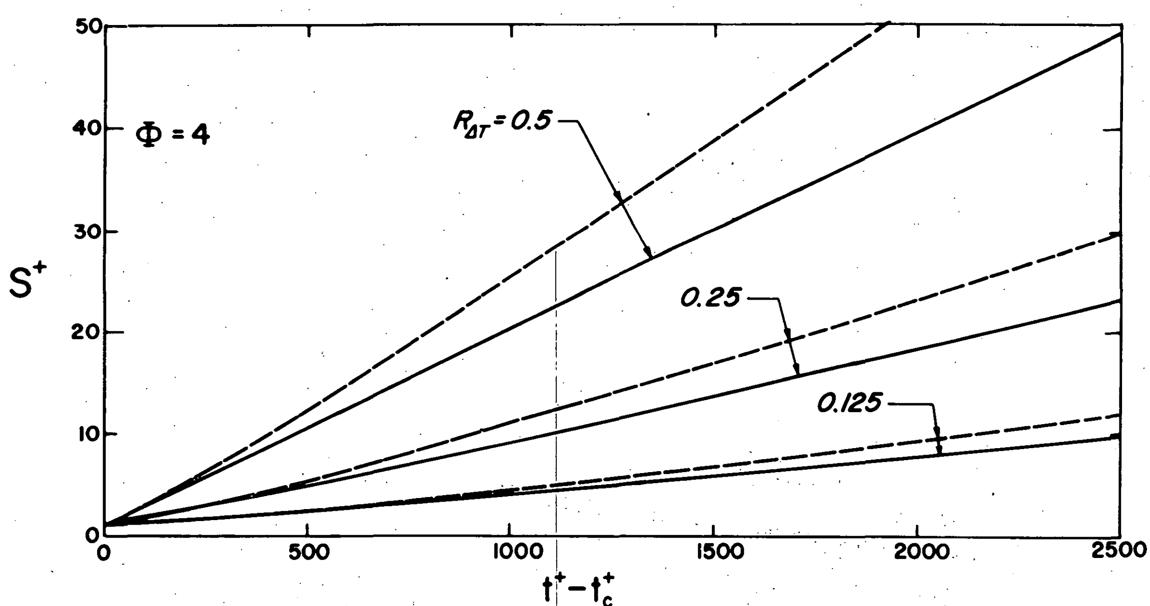

Figure 7. Relationship between S^+ and $t^+ - t_c^+$ from exact and approximate solutions for $\phi = 4$, $R_{\Delta T} = 0.125$, 0.25 and 0.5.

Table II. Comparison of numerical solutions S^+ using different dimensionless time increments for the case

$R_{\Delta T} = 0.5$, $\phi = 8$.

$t^+ - t_c^+$	$\Delta t^+ = 0.1$	$\Delta t^+ = 0.4$
0.0000	1.0000	1.0000
65.1000	2.0009	2.0026
121.1000	3.0008	3.0027
329.9000	7.0006	7.0027
533.2000	11.0016	11.0039
684.8000	14.0003	14.0012
836.7000	17.0009	17.0043
1192.6000	24.0018	24.0006
1448.4000	29.0003	29.0040
1705.8000	34.0000	34.0007
2016.8000	40.0012	40.0070
2329.9000	46.0009	46.0021
2539.8000	50.0006	50.0045

Although eq 46 and 48 are nonlinear, the exponent of the nonlinear term is rather small. For reasonably small S^+, therefore, one can assume $(S^+)^{-0.085}$ to be equal to unity. Equations 46 and 48 become

$$\frac{dy^+}{dt^+} = \frac{3}{\phi}\left[\frac{2(1+\phi)}{y^+} - R_{\Delta T}\left(\frac{k_1}{k_2}\right)\right] \tag{52}$$

$$\frac{dS^+}{dt^+} = \frac{1}{\phi}\left[R_{\Delta T}\left(\frac{k_1}{k_2}\right) - \frac{2}{y^+}\right]. \tag{53}$$

By direct integration y^+ is found to be

$$\frac{k_2}{k_1}\left[\frac{2(1+\phi)}{R_{\Delta T}\left(\frac{k_1}{k_2}\right)} \ln \frac{2(1+\phi) - R_{\Delta T}\left(\frac{k_1}{k_2}\right) y_c^+}{2(1+\phi) - R_{\Delta T}\left(\frac{k_1}{k_2}\right) y^+} - (y^+ - y_c^+)\right] = \frac{3R_{\Delta T}}{\phi}(t^+ - t_c^+). \tag{54}$$

and S^+ is found to be

$$S^+ - 1 = \frac{R_{\Delta T}}{\phi}\left(\frac{k_1}{k_2}\right)(t^+ - t_c^+) - \frac{2}{\phi}\int_{t_c^+}^{t^+} \frac{dt^+}{y^+}. \tag{55}$$

Numerical solutions for these approximate expressions were also obtained. Comparisons between the exact and approximate results are shown in Figures 2-7.

It is also interesting to consider the situation $S^+ > 39$. In such an event, the correlation given by Silveston and O'Toole is no longer applicable. Consequently, expression 22 is no longer valid. A different heat flux expression should be used instead. The two surfaces become so far apart that one can no longer talk about a confined space, but must consider these two surfaces separately. The heat flux to the solid phase can be estimated from the following expression

$$H(t) = h_1 (T_s - T_b) = h_2 (T_b - T_m) \tag{56}$$

where h_1 is the heat transfer coefficient for a heated plate facing upward and h_2 that for a cold plate facing downward. T_b is the bulk temperatur of the liquid. From existing correlations (McAdams, 1954; O'Toole and Silveston, 1959), h_1 and h_2 can be estimated, provided T_b is known. The actual procedure is a trial and error one. A value of T_b is assumed. h_1 and h_2 can, therefore, be estimated and the equality of eq 56 can be checked. Since T_s and T_m are constant for a given situation, H(t) in this case also becomes constant. The melting problem reduces to one with constant heat input and continual removal of the melting liquids which can be solved by the method established previously (Goodman, 1958).

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APPENDIX A: NOTATION

C_p = heat capacity
$f(x)$ = temperature distribution given by eq 19
g = gravitational acceleration
$H(t)$ = heat flux
$H^+(t)$ = dimensionless heat flux given by eq 33
k = thermal conductivity
h = heat transfer coefficient
ℓ = characteristic length
L = latent heat of fusion
N_{pr} = Prandtl number
$R_{\Delta T}$ = dimensionless parameter given by eq 34
N_{Ra} = Rayleigh number
$(N_{Ra})_c$ = critical Rayleigh number
$S(t)$ = solid-liquid interface position
$S^+(t)$ = dimensionless solid-liquid interface position defined as S/S_c
S_c = transitional melting front
t = time
t^+ = dimensionless time, defined by eq 31
T_1, T_2 = temperature in the liquid and solid phase respectively
T_b = bulk liquid temperature
T_m = melting temperature
T_0 = initial ice temperature
T_s = temperature at the hot surface
x = distance
x^+ = dimensionless distance defined by eq 30
y^+ = defined as $f^+ - S^+$
$\alpha_{1,2}$ = thermal diffusivity of liquid and solid respectively
β = thermal expansion coefficient
f^+ = dimensionless thermal boundary layer thickness
ϕ = dimensionless parameter defined by eq 32
ρ = density
λ = constant defined by eq 2
μ = viscosity
θ = dimensionless temperature difference ratio defined by eq 27

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13. ABSTRACT

Approximate solutions of temperature distribution and melting rate of ice have been obtained for the case where the mode of heat transfer is natural convection due to the thermal instability caused by the heated lower surface. Extensive numerical solutions were obtained for the ice-water system corresponding to various thermal conditions in terms of parameters defined as $R_{\Delta T} = (T_s - T_m)/(T_m - T_0)$ and $\phi = L/C_p(T_m - T_0)$, where T_s is the temperature of the heat source, T_m is the melting point of ice, T_0 is the initial temperature of ice, L is the latent heat of fusion, and C_p is the heat capacity of ice.

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