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SPHERICAL WAVES
IN
VISCOELASTIC MEDIA

by

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PREFACE

This report was prepared under U. S. Army Cold Regions Research and Engineering Laboratory (USA CRREL) Project, Cold Regions Research - Wave Propagation Studies.

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SUMMARY

Closed form solutions to the propagation of spherical waves from an internal steady oscillating pressure source as well as to the transient response of a viscoelastic medium to a pressure impulse are presented. The assumption that the loss factors, quantities related to the dissipative property of a viscoelastic medium, are independent of frequency is used here for investigating the wave propagation phenomenon due to the action of an impulse.

Numerical examples for the displacement and stress functions have been used to illustrate the damping effect of a viscoelastic medium by comparing with its special case—the elastic medium. It is found that this effect shows its significance with increased traveling distance. Also, in the steady oscillation problems damping is of importance when the vibration frequency is high, while in the transient case it causes the decrease in the peak amplitude and also a time lag in the arrival of the wave front.

SPHERICAL WAVES IN VISCOELASTIC MEDIA

by

T. M. Lee

Introduction

The problem of stress wave propagations from a spherical source has received considerable attention from early investigators (Sharpe, 1942; Selberg, 1952; Blake, 1952; Mattice and Lieber, 1954). Their findings, however, are mainly confined to either an elastic medium or to information of interest only to seismic work. With the growth of engineering applications in this field, it seems worthwhile to explore this problem a bit further and to obtain results in a form suitable for these applications.

Within the framework of linear viscoelastic theory (Berry, 1958; Bland, 1960) it is possible to express the solution of a wave equation in terms of mechanical constants of the viscoelastic medium. The mathematical stress-strain relationship of a Voigt-solid enables one to obtain closed form solutions which clearly exhibit the damping effect of the medium. However, in choosing these basic constants, it appears to be more desirable to refer to the loss factors* rather than the viscous constants.* Thus, when studying the effects in the medium of the impact of an impulse one may use the assumption of constant loss factor, which has been verified in one-dimensional pulse propagation experiments (Kolsky, 1956).

The propagation of spherical waves from an internal steady oscillating pressure source and the transient response of the viscoelastic medium to a pressure impulse are examined in the present work. It is intended to express the displacement functions and the stress functions explicitly in terms of the input source and the basic mechanical constants of the medium. Therefore, when these are known,† one can readily calculate the displacements and stresses in the medium of interest.

Solutions of wave equations

The stress-strain tensor of a Voigt-solid is

$$\sigma_{ij} = \left(\lambda + \lambda' \frac{\partial}{\partial t} \right) \Delta \delta_{ij} + 2 \left(\mu + \mu' \frac{\partial}{\partial t} \right) \epsilon_{ij} \quad (1)$$

as compared to that of an isotropic elastic medium:

$$\sigma_{ij} = \lambda \Delta \delta_{ij} + 2\mu \epsilon_{ij} \quad i, j = 1, 2, 3, \quad (2)$$

where σ_{ij} and ϵ_{ij} are the components of the stress tensor and the strain tensor, respectively; δ_{ij} is the Kronecker delta; Δ is the dilatation, which equals $\epsilon_{11} + \epsilon_{22} + \epsilon_{33}$; λ and μ are Lamé's constants and λ' and μ' are viscous constants corresponding to λ and μ .

Thus, if we write $\left(\lambda + \lambda' \frac{\partial}{\partial t} \right)$ instead of λ and $\left(\mu + \mu' \frac{\partial}{\partial t} \right)$ instead of μ , the equation of motion in a Voigt-solid may be expressed as

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \left[(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t} \right] \frac{\partial \Delta}{\partial x_i} + \left(\mu + \mu' \frac{\partial}{\partial t} \right) \nabla^2 u_i \quad i = 1, 2, 3, \quad (3)$$

where u_i are the components of the displacement vector, ρ is the density of the solid, and ∇^2 denotes the Laplacian operator.

Introducing a scalar potential ϕ and a vector potential ψ such that

* The terms are defined in the next section.

† For methods of laboratory evaluations of the mechanical constants, see, for instance, Lee (1963a; 1963b; 1964).

$$u_i = \frac{\partial \phi}{\partial x_i} + \left(\frac{\partial \psi_k}{\partial x_j} - \frac{\partial \psi_j}{\partial x_k} \right) \quad i, j, k = 1, 2, 3, \quad (4)$$

it can be shown that eq 3 will be satisfied if ϕ and ψ_i are solutions of

$$\left[(\lambda + 2\mu) + (\lambda' + 2\mu') \frac{\partial}{\partial t} \right] \nabla^2 \phi = \rho \frac{\partial^2 \phi}{\partial t^2} \quad (5)$$

and

$$\left(\mu + \mu' \frac{\partial}{\partial t} \right) \nabla^2 \psi_i = \rho \frac{\partial^2 \psi_i}{\partial t^2} \quad i = 1, 2, 3. \quad (6)$$

Equations 5 and 6 thus constitute the dilatational and distortional wave equations for a viscoelastic medium such as a Voigt-solid.

Let us now use eq 5 to consider the solution for spherical waves. When dealing with problems of spherical symmetry, however, it is more convenient to use the spherical coordinate and write

$$\left[(\lambda + 2\mu) + (\lambda' + 2\mu') \frac{\partial}{\partial t} \right] \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} \right) = \rho \frac{\partial^2 \phi}{\partial t^2} \quad (7)$$

since the Laplacian operator is now of the form

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}$$

where r denotes the radial distance to a point.

If we introduce

$$\phi(r, t) = \frac{\chi(r, t)}{r} \quad (8)$$

then eq 7 can be simplified to

$$\left[(\lambda + 2\mu) + (\lambda' + 2\mu') \frac{\partial}{\partial t} \right] \frac{\partial^2 \chi}{\partial r^2} = \rho \frac{\partial^2 \chi}{\partial t^2} \quad (9)$$

which obviously contains the general solution

$$\chi = \chi_1 \left(t - \frac{r}{\gamma} \right) + \chi_2 \left(t + \frac{r}{\gamma} \right) \quad (10)$$

where χ_1 and χ_2 are arbitrary functions and γ is the complex propagation constant depending on the properties of λ , λ' , μ , μ' and ρ .

For divergent waves, χ_2 may be excluded and, also, the particular solution of eq 10, which is useful in the present study, may be assumed of the form

$$\chi = A \exp \left[\omega \left(t - \frac{r}{\gamma} \right) \right] \quad (11)$$

where A is an arbitrary constant and

$$\gamma = \gamma_1 + i\gamma_2 = \left\{ \left[(\lambda + 2\mu) + i\omega(\lambda' + 2\mu') \right] / \rho \right\}^{\frac{1}{2}}.$$

Substituting eq 11 in eq 8, we obtain, after simplification,

$$\phi = \frac{A}{r} \exp \left[-a_1 r + i\omega \left(t - \frac{r}{C_1} \right) \right], \quad (12)$$

where a_1 = the attenuation coefficient

$$= \frac{\omega}{C_1} \tan \frac{\theta_1}{2} \quad (13a)$$

C_1 = the dilatational wave velocity

$$= \left(\frac{M_1}{\rho} \right)^{\frac{1}{2}} \sec \frac{\theta_1}{2} \quad (13b)$$

$$M_1 = (\lambda + 2\mu) \sec \theta_1 \quad (13c)$$

$$\text{and } \tan \theta_1 = \frac{\omega(\lambda' + 2\mu')}{\lambda + 2\mu}. \quad (13d)$$

In a similar manner, it can be shown that

$$\psi = \frac{B}{r} \exp \left[-a_2 r + i\omega \left(t - \frac{r}{C_2} \right) \right], \quad (14)$$

where B = constant

$$a_2 = \frac{\omega}{C_2} \tan \frac{\theta_2}{2} \quad (15a)$$

$$C_2 = \left(\frac{M_2}{\rho} \right)^{\frac{1}{2}} \sec \frac{\theta_2}{2} \quad (15b)$$

$$M_2 = \mu \sec \theta_2 \quad (15c)$$

$$\text{and } \tan \theta_2 = \frac{\omega\mu'}{\mu}. \quad (15d)$$

It is seen that the wave functions travel with phase velocities, which depend not only on the density and the modulus but also on the damping of the medium, unlike waves in elastic media. Their amplitudes, apart from the effect of expanding geometry, decay exponentially with distance according to a factor which again is related to the dissipative property of the medium.

Spherical waves from a steady oscillating source

We shall now use the results obtained in the previous section to examine wave propagation in a viscoelastic medium due to the action of a uniform, time-varying pressure source on a spherical cavity of radius r_0 inside the medium (Figure 1). Because of the symmetrical action of the pressure, the resultant particle motion will be radial and irrotational. Therefore, the potential function ϕ as given by eq 12 may be used alone for investigating the problem.

To study the steady-state oscillation problem, we assume that the pressure source $p(t)$ varies sinusoidally with time and its expression is given as:

Figure 1. Spherical cavity acted upon by a pressure source.

$$p(t) = p_0 \cos \omega t, \quad (16)$$

where p_0 is the amplitude of the pressure at a given frequency ω . Let ϕ be a displacement potential with its expression slightly modified to:

$$\phi = \frac{A}{r} e^{-a_1 (r - r_0)} \cos(\omega\tau - \epsilon_0), \quad (17)$$

where $\tau = t - \frac{r - r_0}{C_1}$ and ϵ_0 is a phase angle depending on the boundary conditions of the problem.

The boundary condition to be satisfied at the surface of the cavity is that

$$(\sigma_r)_{r=r_0} + p_0 \cos \omega t = 0, \quad (18)$$

where σ_r denotes the radial stress in the medium whose relationship with the radial displacement u_r is

$$\sigma_r = \left[(\lambda + 2\mu) + (\lambda' + 2\mu') \frac{\partial}{\partial t} \right] \frac{\partial u_r}{\partial r} + 2(\lambda + \lambda') \frac{\partial}{\partial t} \frac{u_r}{r}. \quad (19)$$

Thus, with the aid of eq 18 and 19, we find

$$\phi = -\frac{p_0 r_0^3}{4\mu D_0 r} e^{-a_1 (r - r_0)} \cos(\omega\tau - \epsilon_0), \quad (20)$$

where

$$D_0 = \left\{ (\xi + \eta \tan \theta_2)^2 + [\eta - \xi (\tan \theta_2 + \Gamma \xi)]^2 \right\}^{\frac{1}{2}} \quad (21a)$$

$$\epsilon_0 = \tan^{-1} \left[\frac{\xi + \eta \tan \theta_2}{\eta - \xi (\tan \theta_2 + \Gamma \xi)} \right] \quad (21b)$$

$$\xi = \frac{\omega r_0}{C_1}, \quad (22a)$$

$$\eta = 1 + a_1 r_0, \quad (22b)$$

$$\Gamma = \frac{\rho C_1^2}{4\mu}. \quad (22c)$$

It is seen that the potential function ϕ is now completely defined by the mechanical constants of the medium, the amplitude of the pressure source, and the size of the cavity. Therefore, when these are known, the potential function can be used to obtain information regarding the displacements and the stresses inside the medium due to the action of a pressure source.

To illustrate this point, let us derive further expressions for the radial displacement u_r and the radial stress σ_r . Making use of eq 19 and 20, it can be shown that

$$u_r = \frac{\partial \phi}{\partial r} = \frac{p_0 N_u}{4\mu D_0} \frac{r_0^3}{r^2} e^{-a_1 (r - r_0)} \cos(\omega\tau - \epsilon_0 + \epsilon_u) \quad (23)$$

and

$$\sigma_r = - \frac{p_0 N_\sigma}{D_0} \left(\frac{r_0}{r} \right)^3 e^{-a_1 (r - r_0)} \cos(\omega\tau - \epsilon_0 + \epsilon_\sigma), \quad (24)$$

where

$$N_u = \left(\xi_r^2 + \eta_r^2 \right)^{\frac{1}{2}}, \quad (25a)$$

$$\epsilon_u = \tan^{-1} \frac{\xi_r}{\eta_r}, \quad (25b)$$

$$N_\sigma = \left\{ (\xi_r + \eta_r \tan \theta_2)^2 + [\eta_r - \xi_r (\tan \theta_2 + \Gamma \xi_r)]^2 \right\}^{\frac{1}{2}} \quad (26a)$$

$$\epsilon_\sigma = \tan^{-1} \left[\frac{\xi_r + \eta_r \tan \theta_2}{\eta_r - \xi_r (\tan \theta_2 + \Gamma \xi_r)} \right], \quad (26b)$$

$$\xi_r = \frac{\omega r}{C_1} = \frac{r}{r_0} \xi, \quad (27a)$$

$$\eta_r = 1 + a_1 r. \quad (27b)$$

These results indicate that the displacement function u_r and the stress function σ_r are similar in form to the potential function ϕ . Thus, when the mechanical constants mentioned before are known, these functions can be calculated from eq 23 and 24.

Figure 2. Decaying of radial stress with scaled distance r/r_0 from a steady oscillating pressure source.

As an example, we choose the special case of $\lambda = \mu$ and $\tan \theta_1 = \tan \theta_2$ for computing the amplitude of σ_r at various scaled distances r/r_0 (Fig. 2). For comparison, the elastic case is shown by a solid line while the viscoelastic case ($\tan \theta_1 = 0.1$) by a dotted line. It is seen that the stress wave diminishes rather quickly with distance. The attenuation effect resulting from traveling through a viscoelastic medium does not seem to be too severe at low frequency, $\frac{\omega r_0}{C_1} = 0.5$. But when the frequency is high, for instance

$\frac{\omega r_0}{C_1} = 5$, the amplitude of a viscoelastic wave decays much more rapidly with distance

than that of an elastic wave. This is mainly because the attenuation coefficient α_1 , which is lacking in the elastic case, is directly proportional to the vibration frequency.*

The radial displacement decays in a manner similar to the radial stress in traveling through the medium. But when the input frequency reaches the resonant frequency of the medium the amplitude of the displacement increases substantially. This phenomenon is useful in obtaining information regarding the mechanical properties of the medium since they are closely related (see Lee, 1963a; 1963b; 1964). As another example, we compute the radial displacement at the surface of the cavity for an input pressure source of varying frequency. The "resonance" phenomenon is clearly indicated in Figure 3.

Figure 3. Motion at the surface of the cavity from a steady oscillating pressure source of varying frequency.

Response to transient loadings

To investigate the response of the medium to internal loadings of finite duration, we assume that the pressure function $p(t)$, acting on the spherical cavity, is expressible in the form of a Fourier integral:

* Generally, C_1 and $\tan \theta_1$ of viscoelastic material change rather slowly with frequency. Thus, if we insert the condition of constant loss factor in a Voigt-solid, it leads to α_1 varying directly with frequency. This assumption will be discussed in the next section.

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\zeta) e^{i\omega(t-\zeta)} d\zeta d\omega. \quad (28)$$

If ϕ as given by eq 12 is a particular solution of the wave equation, the sum of such solutions with different coefficients $A(\omega)$ is also a solution. Thus,

$$\phi(r, t) = \frac{1}{2\pi r} \operatorname{Re} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\omega) e^{-\alpha_1(r-r_0)} p(\zeta) e^{i\omega(t-\frac{r-r_0}{C_1}-\zeta)} d\zeta d\omega, \quad (29)$$

where Re denotes "the real part of" and the spectrum of the pulse

$$\overline{p(\omega)} = \int_{-\infty}^{\infty} p(\zeta) e^{-i\omega\zeta} d\zeta \quad (30)$$

has been introduced as part of the coefficient $A(\omega)$ for the convenience of manipulation.

Applying the boundary condition at the surface of the cavity, it can be shown that

$$A(\omega) = - \frac{r_0^3}{4\mu [\eta - \xi (\tan \theta_2 + \Gamma \xi) + i(\xi + \eta \tan \theta_2)]}. \quad (31)$$

From the substitution of eq 31 in eq 29, we obtain

$$\phi(r, t) = - \frac{r_0^3}{8\pi r \mu} \operatorname{Re} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-\alpha_1(r-r_0)} p(\zeta) e^{i\omega(t-\zeta)} d\zeta d\omega}{[\eta - \xi (\tan \theta_2 + \Gamma \xi) + i(\xi + \eta \tan \theta_2)]}. \quad (32)$$

To illustrate the effect in the medium from an impulse, we let

$$p(\zeta) = p_0 H(\zeta) e^{-\beta\zeta}, \quad (33)$$

where $H(\zeta)$ is the Heaviside's step function and is defined as

$$H(\zeta) = 0, \quad \zeta < 0; \quad H(\zeta) = 1, \quad \zeta \geq 0;$$

p_0 is the initial amplitude of the impulse, β is the decay factor and they are both constants.

Inserting eq 33 in eq 32 and performing the integration for ζ , we find

$$\phi = - \frac{p_0 r_0^3}{8\pi r \mu} \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{-\alpha_1(r-r_0) + i\omega\tau} d\omega}{(\beta + i\omega) [\eta - \xi (\tan \theta_2 + \Gamma \xi) + i(\xi + \eta \tan \theta_2)]}. \quad (34)$$

Before attempting to integrate above, let us first examine some of the properties of viscoelastic materials over the very wide frequency range. As is indicated from the experimental results of Lethersich (1950), the loss factor $\tan \theta_2$ for the viscoelastic materials changes very slightly with frequency. Based on the assumption of constant loss factor,* Kolsky (1956) calculated the pulse propagation along a bar and found the results agree very well with the measured values. Since the present work also deals with pulse propagation, except that it is a three-dimensional case, it seems reasonable

*In the sense that these factors are independent of frequency.

to adopt the same assumption. This leads automatically to $\tan \frac{\theta_1}{2}$, $\tan \theta_2$ and Γ being independent of frequency, and the quantity η can be expressed in terms of ξ :

$$\eta = 1 + \xi \tan \frac{\theta_1}{2}. \quad (35)$$

Changing from variable ω to ξ , eq 34 can be rewritten in a more suitable form for integration:

$$\phi = \frac{p_0 r_0^3}{\rho C_1^2 r} \operatorname{Re} \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-s|\xi| + i\tau'\xi} d\xi}{(\xi - i\beta_0) [\xi - (a_1 + ib_1)] [\xi - (a_2 + ib_2)]} \right\}, \quad (36)^\dagger$$

$$\text{where } a_1 = \frac{(\tan \frac{\theta_1}{2} - \tan \theta_2) + (4\Gamma - 1)^{\frac{1}{2}}}{2\Gamma}, \quad (37a)$$

$$b_1 = \frac{(4\Gamma - 1)^{\frac{1}{2}} + [(\tan \frac{\theta_1}{2} - \tan \theta_2) + 2\Gamma \tan \theta_2]}{2\Gamma(4\Gamma - 1)^{\frac{1}{2}}}, \quad (37b)$$

$$a_2 = \frac{(\tan \frac{\theta_1}{2} - \tan \theta_2) - (4\Gamma - 1)^{\frac{1}{2}}}{2\Gamma}, \quad (37c)$$

$$b_2 = \frac{(4\Gamma - 1)^{\frac{1}{2}} - [(\tan \frac{\theta_1}{2} - \tan \theta_2) + 2\Gamma \tan \theta_2]}{2\Gamma(4\Gamma - 1)^{\frac{1}{2}}}, \quad (37d)$$

$$s = \left(\frac{r}{r_0} - 1 \right) \tan \frac{\theta_1}{2}, \quad (38a)$$

$$\beta_0 = \frac{r_0}{C_1} \beta, \quad (38b)$$

$$\tau' = \frac{C_1}{r_0} \tau. \quad (38c)$$

It is clear that the integrand has three simple poles in the upper half complex plane. Choose a closed contour with the integration path taken along the real axis and a semicircle of radius R in the upper half plane. As $R \rightarrow \infty$ the integral along the semicircle vanishes and the value of eq 36 is obtained by summing up the residues. After some manipulation, we find

$$\begin{aligned} \phi = \frac{p_0 r_0^3}{\rho C_1^2 r} & \left\{ e^{-\beta_0 \tau'} (f_0 \cos \beta_0 s - g_0 \sin \beta_0 s) \right. \\ & + e^{-(a_1 s + b_1 \tau')} [f_1 \cos (a_1 \tau' - b_1 s) + g_1 \sin (a_1 \tau' - b_1 s)] \\ & \left. - e^{(a_2 s - b_2 \tau')} [f_2 \cos (a_2 \tau' - b_2 s) + g_2 \sin (a_2 \tau' - b_2 s)] \right\}, \end{aligned} \quad (39)$$

[†] The higher orders of $\tan \frac{\theta_1}{2}$ and $\tan \theta_2$ have been neglected, since these quantities themselves are small.

where

$$f_0 = \frac{a_1 a_2 - d_1 d_2}{(a_1^2 + d_1^2)(a_2^2 + d_2^2)}, \quad (40a)$$

$$g_0 = \frac{a_1 d_2 + a_2 d_1}{(a_1^2 + d_1^2)(a_2^2 + d_2^2)}, \quad (40b)$$

$$f_1 = \frac{a_1(a_1 - a_2) - d_1(b_1 - b_2)}{(a_1^2 + d_1^2)[(a_1 - a_2)^2 + (b_1 - b_2)^2]}, \quad (40c)$$

$$g_1 = \frac{a_1(b_1 - b_2) + d_1(a_1 - a_2)}{(a_1^2 + d_1^2)[(a_1 - a_2)^2 + (b_1 - b_2)^2]}, \quad (40d)$$

$$f_2 = \frac{a_2(a_1 - a_2) - d_2(b_1 - b_2)}{(a_2^2 + d_2^2)[(a_1 - a_2)^2 + (b_1 - b_2)^2]}, \quad (40e)$$

$$g_2 = \frac{a_2(b_1 - b_2) + d_2(a_1 - a_2)}{(a_2^2 + d_2^2)[(a_1 - a_2)^2 + (b_1 - b_2)^2]}, \quad (40f)$$

$$d_1 = b_1 - \beta_0, \quad (41a)$$

$$d_2 = b_2 - \beta_0. \quad (41b)$$

The potential function ϕ consists of three major components. The first component is a non-oscillating term with its damping factor inherited from the input, while the other two terms oscillate behind the wave front with their amplitudes damped not only with time but also with distance.

From the standpoint of practical applications, it is of interest to know the displacements and the stresses in the medium induced by such an impulse. Following a procedure similar to that outlined in the last section, we find

$$\begin{aligned} u_r = \frac{p_0 r_0}{4\mu\Gamma(r/r_0)^2} & \left\{ e^{-\beta_0 \tau'} [(F_0 \frac{r}{r_0} - f_0) \cos \beta_0 s - (G_0 \frac{r}{r_0} - g_0) \sin \beta_0 s] \right. \\ & + e^{-(a_1 s + b_1 \tau')} [(F_1 \frac{r}{r_0} - f_1) \cos(a_1 \tau' - b_1 s) + (G_1 \frac{r}{r_0} - g_1) \sin(a_1 \tau' - b_1 s)] \\ & \left. - e^{(a_2 s - b_2 \tau')} [(F_2 \frac{r}{r_0} - f_2) \cos(a_2 \tau' - b_2 s) + (G_2 \frac{r}{r_0} - g_2) \sin(a_2 \tau' - b_2 s)] \right\} \quad (42) \end{aligned}$$

and

$$\begin{aligned}
\sigma_r = & -\frac{p_0}{\left(\frac{r}{r_0}\right)} \left(e^{-\beta_0 \tau'} \left\{ \left[\frac{1}{\Gamma\left(\frac{r}{r_0}\right)^2} \left(F_0 \frac{r}{r_0} - f_0 \right) - \beta_0^2 f_0 \right] \cos \beta_0 s - \left[\frac{1}{\Gamma\left(\frac{r}{r_0}\right)^2} \left(G_0 \frac{r}{r_0} - g_0 \right) - \beta_0^2 g_0 \right] \sin \beta_0 s \right\} \right. \\
& + e^{-(a_1 s + b_1 \tau')} \left\{ \left[\frac{1}{\Gamma\left(\frac{r}{r_0}\right)^2} \left(F_1 \frac{r}{r_0} - f_1 \right) + K_1 \right] \cos(a_1 \tau' - b_1 s) + \left[\frac{1}{\Gamma\left(\frac{r}{r_0}\right)^2} \left(G_1 \frac{r}{r_0} - g_1 \right) + L_1 \right] \sin(a_1 \tau' - g_1 s) \right\} \\
& \left. - e^{(a_2 s - b_2 \tau')} \left\{ \left[\frac{1}{\Gamma\left(\frac{r}{r_0}\right)^2} \left(F_2 \frac{r}{r_0} - f_2 \right) + K_2 \right] \cos(a_2 \tau' - b_2 s) + \left[\frac{1}{\Gamma\left(\frac{r}{r_0}\right)^2} \left(G_2 \frac{r}{r_0} - g_2 \right) + L_2 \right] \sin(a_2 \tau' - g_2 s) \right\} \right) \quad (43)
\end{aligned}$$

where

$$\left. \begin{aligned}
F_0 &= \beta_0 (f_0 - g_0 \tan \frac{\theta_1}{2}), \quad G_0 = \beta_0 (f_0 \tan \frac{\theta_1}{2} + g_0) \\
F_1 &= f_1 (b_1 - a_1 \tan \frac{\theta_1}{2}) - g_1 (a_1 + b_1 \tan \frac{\theta_1}{2}) \\
G_1 &= f_1 (a_1 + b_1 \tan \frac{\theta_1}{2}) + g_1 (b_1 - a_1 \tan \frac{\theta_1}{2}) \\
F_2 &= f_2 (b_2 + a_2 \tan \frac{\theta_1}{2}) - g_2 (a_2 + b_2 \tan \frac{\theta_1}{2}) \\
G_2 &= f_2 (a_2 + b_2 \tan \frac{\theta_1}{2}) + g_2 (b_2 - a_2 \tan \frac{\theta_1}{2})
\end{aligned} \right\} \quad (44)$$

$$\left. \begin{aligned}
K_1 &= a_1 (a_1 f_1 + b_1 g_1) + b_1 (a_1 g_1 - b_1 f_1) \\
L_1 &= a_1 (a_1 g_1 - b_1 f_1) - b_1 (a_1 f_1 + b_1 g_1) \\
K_2 &= a_2 (a_2 f_2 + b_2 g_2) + b_2 (a_2 g_2 - b_2 f_2) \\
L_2 &= a_2 (a_2 g_2 - b_2 f_2) - b_2 (a_2 f_2 + b_2 g_2)
\end{aligned} \right\} \quad (45)$$

Note that the displacement function u_r and the stress function σ_r basically have the same structure as the potential function ϕ . However, in order to illustrate their properties, it is best to use the numerical examples. For convenience of computation, we shall use the same set of constants where $\lambda = \mu$ and $\tan \theta_1 = \tan \theta_2$.

As can be observed from Figure 4, the motion in viscoelastic media induced by an external impulse lags behind that in elastic ones. This phenomenon becomes more prominent as the distance from the source increases, especially when the input is not a step function. The manner of stress decrease with distance is similar in both the elastic and viscoelastic cases, except that the latter can no longer be governed by the rule that dynamic stresses from a spherical source decrease inversely proportional to the radial distance (Fig. 5). Generally speaking, the damping effect in a viscoelastic medium becomes significant when the traveling distance is large, especially when the input is of an exponential decaying type.

The fact that only the inertial term is of significance in σ_r at a large distance may be used to simplify eq 43. Thus, when r/r_0 is large, we write

 a. Decaying factor $\beta_0 = 0$.

 b. Decaying factor $\beta_0 = 0.5$.

 c. Decaying factor $\beta_0 = 1.0$.

Figure 4. Radial displacements produced by an impulse.

a. Decaying factor $\beta_0 = 0$. b. Decaying factor $\beta_0 = 0.5$. c. Decaying factor $\beta_0 = 1.0$.

Figure 5. Radial stresses produced by an impulse.

$$\sigma_r = - \frac{P_0}{\left(\frac{r}{r_0}\right)} \left\{ \beta_0^2 e^{-\beta_0 \tau'} (g_0 \sin \beta_0 s - f_0 \cos \beta_0 s) \right. \\ \left. + e^{-(a_1 s + b_1 \tau')} [K_1 \cos (a_1 \tau' - b_1 s) + L_1 \sin (a_1 \tau' - g_1 s)] \right. \\ \left. - e^{(a_2 s - b_2 \tau')} [K_2 \cos (a_2 \tau' - b_2 s) + L_2 \sin (a_2 \tau' - g_2 s)] \right\}. \quad (46)$$

This equation can approximate the result from eq 43 within 10% when $\frac{r}{r_0} = 10$. For larger $\frac{r}{r_0}$, the accuracy increases.

Conclusions

The amplitude of the viscoelastic waves from a spherical source is seen to decay exponentially, aside from the effect of geometrical expansion, with an attenuation factor related to the dissipative property of the medium (eq 13a and 15a). This factor is responsible for causing the amplitude of a viscoelastic wave to depart from that of an elastic wave when the vibration frequency is high, and also when the distance from the source is large (Fig. 2 and 5).

The "resonant frequency" for a viscoelastic medium can be observed from the motion at the surface of the cavity when under the forced harmonic vibration of a spherical pressure source. The damping effect of the medium has not only damped the amplitude of the vibration but also lowered the magnitude of the "resonant frequency" (Fig. 3).

The motion in a viscoelastic medium caused by an impulse lags behind that in an elastic one. This becomes more prominent as distance from the source increases (Fig. 4). The front of elastic stress waves decreases inversely proportional to the radial distance, while that of viscoelastic waves can no longer be governed by this rule on account of damping.

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<p>AD</p> <p>Accession No.</p> <p>U. S. Army Cold Regions Research and Engineering Laboratory, Army Materiel Command, Hanover, N.H. SPHERICAL WAVES IN VISCOELASTIC MEDIA—T. M. Lee</p> <p>Research Report 158, Apr 1965, 14p - illus. DA Task IV025001A13001 Unclassified Report</p> <p>Closed form solutions to the propagation of spherical waves from an internal steady oscillating pressure source and the transient response of the viscoelastic medium to a pressure impulse are presented. The displacement functions and the stress functions are expressed in terms of the input source and the basic mechanical constants of the medium. In the investigation of impulse-caused propagation phenomenon it is assumed that the loss factors are independent of frequency. Numerical examples for the displacement and stress functions illustrate the damping effect of a viscoelastic medium by comparison with its special case—the elastic medium. The amplitude of the viscoelastic waves from a spherical source decays exponentially; this factor causes the amplitude to depart from that of an elastic wave when the vibration frequency is high and when the distance from the source is large. The "resonant frequency" for a viscoelastic medium (over)</p>	<p>UNCLASSIFIED</p> <ol style="list-style-type: none"> 1. Solids--Mechanical properties 2. Solids--Wave transmission 3. Materials--Elasticity <ol style="list-style-type: none"> I. Lee, Tung-Ming II. U. S. Army Cold Regions Research and Engineering Laboratory 	<p>AD</p> <p>Accession No.</p> <p>U. S. Army Cold Regions Research and Engineering Laboratory, Army Materiel Command, Hanover, N.H. SPHERICAL WAVES IN VISCOELASTIC MEDIA—T. M. Lee</p> <p>Research Report 158, Apr 1965, 14p - illus. DA Task IV025001A13001 Unclassified Report</p> <p>Closed form solutions to the propagation of spherical waves from an internal steady oscillating pressure source and the transient response of the viscoelastic medium to a pressure impulse are presented. The displacement functions and the stress functions are expressed in terms of the input source and the basic mechanical constants of the medium. In the investigation of impulse-caused propagation phenomenon it is assumed that the loss factors are independent of frequency. Numerical examples for the displacement and stress functions illustrate the damping effect of a viscoelastic medium by comparison with its special case—the elastic medium. The amplitude of the viscoelastic waves from a spherical source decays exponentially; this factor causes the amplitude to depart from that of an elastic wave when the vibration frequency is high and when the distance from the source is large. The "resonant frequency" for a viscoelastic medium (over)</p>	<p>UNCLASSIFIED</p> <ol style="list-style-type: none"> 1. Solids--Mechanical properties 2. Solids--Wave transmission 3. Materials--Elasticity <ol style="list-style-type: none"> I. Lee, Tung-Ming II. U. S. Army Cold Regions Research and Engineering Laboratory
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