Research Report 255

UNSTEADY MOTION OF A SPHERE ALONG A CIRCULAR PATH IN A VISCOUS FLUID

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PREFACE

This paper was prepared by Dr. Fuat Odar, formerly of the Research Division, Cold Regions Research and Engineering Laboratory (CRREL), U.S. Army Terrestrial Sciences Center (USA TSC), presently at the Bettis Atomic Power Laboratory, Westinghouse, Inc., West Mifflin, Pennsylvania. This report was published under DA Task 1T061102B52A02, *Research in Earth Physics – Cold Regions and Related Environments.*

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Forces on a sphere moving unsteadily along a circular path in a viscous fluid are measured, and it is found that within the experimental range the formula valid for rectilinear motion has to be modified to account for the curvature of the path.

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by

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INTRODUCTION

The determination of the dynamic force exerted by a real fluid on a submerged object if the relative velocities between the two change with time presents a complex problem in fluid mechanics. The general situations in which both the fluid and the body move arbitrarily and unsteadily are very complicated. This paper is concerned with the unsteady motion of a sphere along a circular path in a real fluid. Since the present work is the outgrowth of the author's previous work on unsteady rectilinear motion of a sphere in a viscous fluid, a short summary of the previous findings will be presented.

PREVIOUS WORK ON UNSTEADY RECTILINEAR MOTION OF A SPHERE

The pioneering work on the unsteady motion of a sphere in a viscous fluid was done by Basset (1888) who, by neglecting the convective acceleration terms, solved the Navier-Stokes equations for the stream function and calculated the force on a sphere moving arbitrarily along a straight line in a viscous fluid. The expression for the force is

$$F = 6\pi r\mu V + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)\rho\frac{dV}{dt} + 6(\pi\mu\rho)^{1/2}r^2\int_0^t \frac{dV/dt'}{(t-t')^{1/2}}\,dt' \qquad (1)$$

where ρ and μ are the density and dynamic viscosity of the fluid respectively, r and V are the radius and the instantaneous velocity of the sphere in an otherwise motionless fluid respectively, and F denotes the dynamic force exerted by the fluid. The first term is the same as the Stokes formula valid for the steady motion of a sphere; the second term is the added mass term which can be obtained by considering the fluid irrotational and frictionless; and the last term is a history term which shows the effect of the previous accelerations upon the force at present time t. This formula is obviously not valid when the convective acceleration terms become important.

Basset's work was recently extended by Odar and Hamilton (1964) and the formula was modified to include the effect of the convective acceleration terms. The new formula is

$$F = \frac{1}{2}\rho C_D \pi r^2 V^2 + C_A\left(\frac{4}{3}\pi r^3\right)\rho\frac{dV}{dt} + C_H(\pi\mu\rho)^{1/2}r^2\int_0^t \frac{dV/dt'}{(t-t')^{1/2}}\,dt' \qquad (2)$$

in which C_D, C_A and C_H are the drag, added mass and history coefficients respectively. The first term is again the same as the steady state term calculated using the instantaneous velocity V, and C_D is the well known drag coefficient of the Reynolds number, $\mathrm{Re} = 2Vr/\nu$, where ν is the kinematic viscosity. The second term is again the added mass term, but the added mass coefficient, ½, valid for an irrotational frictionless fluid, is replaced by C_A which is a function of the acceleration number, $V^2/(2r|dV/dt|)$. Finally, the numerical value 6 in the history term is replaced by C_H which is also a function of the acceleration number. Both coefficients can be expressed as follows:

$$C_A = 1.05 - \frac{0.066}{Ac^2 + 0.12} \qquad (3)$$

$$C_H = 2.88 + \frac{3.12}{(Ac + 1)^3} \qquad (4)$$

where Ac is the acceleration number. These expressions were determined empirically from the results of experiments in which the motions of the sphere were simple harmonic with their frequency and amplitude varying between 17 and 159 rpm, and 1 and 4 in., respectively. The next step, the verification of the proposed equation (eq 2), was undertaken by Odar (1966) by comparing Moorman's (1955) free fall experimental data obtained by using different sizes of spheres and different types of fluids with the values computed from eq 2. The agreement between the calculated and experimental values clearly demonstrated that the proposed eq 2, which was valid for simple harmonic motion, was also valid for free fall. This finding suggests that eq 2 has a wide range of applications. In what follows the same type of equation will be applied to calculate the tangential force exerted by the fluid on a sphere moving along a circular path. Since the path is circular, some modification of this equation will be required.

EXPERIMENTS WITH A CIRCULAR PATH

A sketch of the apparatus used to impart an unsteady motion to a sphere is shown in Figure 1. This is the same apparatus with which the author (Odar, 1967) investigated the forces on a sphere moving steadily along a circular path. A detailed description of the apparatus and the force measurement system and some comments as well as some calculations on the accuracy of the measurements can be found in the above-mentioned reference. There are only two additions to the apparatus. The first one is a tachometer which is driven by friction at the edge of the flywheel to measure the instantaneous velocity of the sphere. The second is a series of wedges (approximately 120) attached on the circumference of the flywheel to produce electrical pulses with the motion of the flywheel. The pulses are recorded together with the velocity of the sphere which is proportional to the tachometer output and the tangential force on the sphere which is measured by the system explained in detail by Odar (1967). By referring to these pulses the location of the sphere during its unsteady motion can be determined.

As stated by Odar (1967) the force measuring system recorded the gravity and buoyancy forces together with the hydrodynamic force exerted by the fluid, which was oil. Its viscosity and density were 1.65×10^{-2} slug/ft sec and 1.725 slug/ft^3 respectively. The unsteady motion could be imparted by varying the speed of the motor. To deduct the effects of gravity and buoyancy from the recordings, the sphere was first moved at a constant speed when the tank was full of oil, and the tangential force together with the electrical pulses coming from the wedges were recorded. The tangential force recording was a smooth harmonic at the frequency of the motion displaced by a constant amount corresponding to the constant hydrodynamic force as stated by Odar (1967). There was one to one correspondence between the smooth harmonic which was due to the gravity

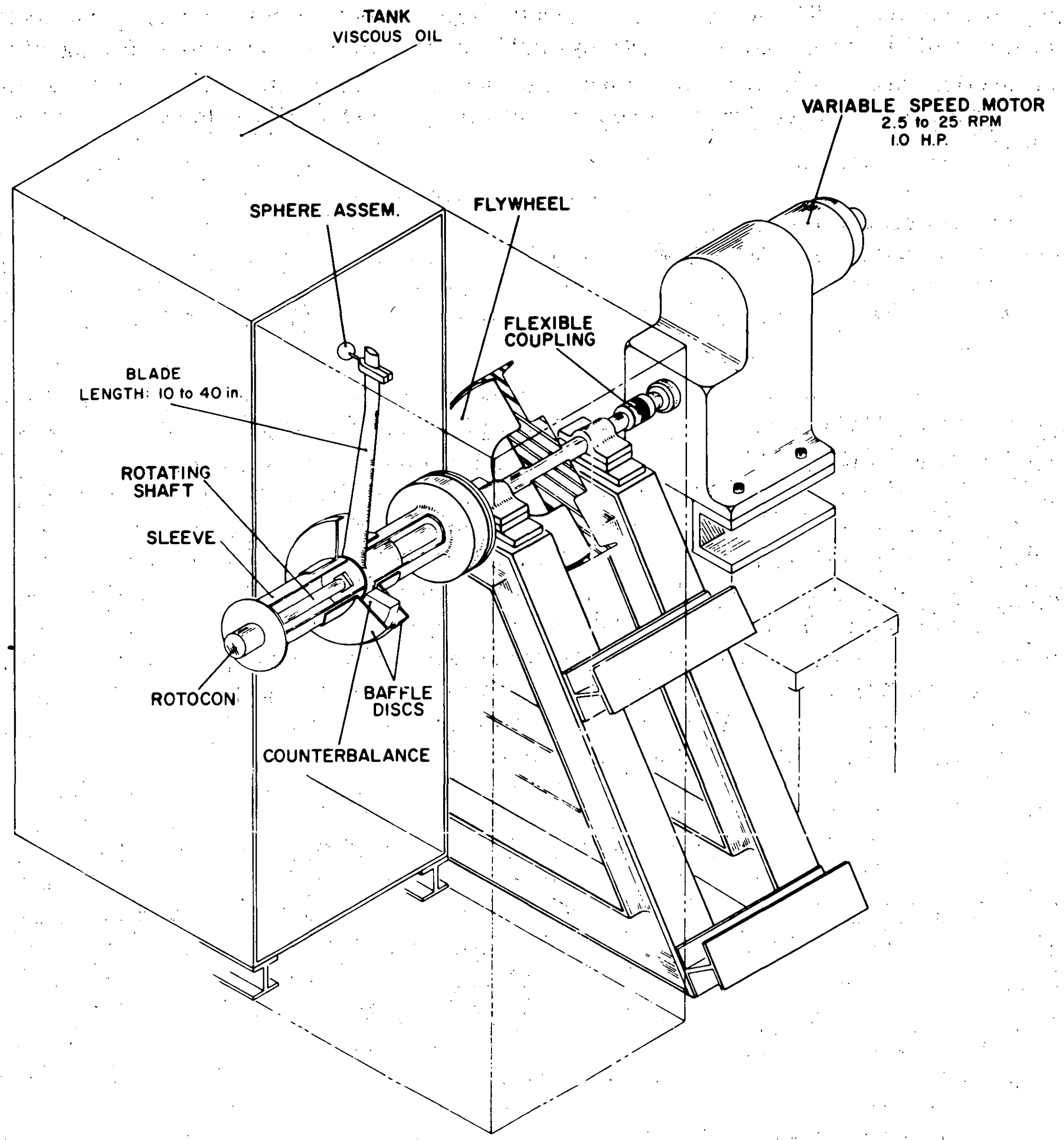


Figure 1. Apparatus.

and buoyancy and the pulses from the wedges. Thus, the tangential force recordings could be corrected for gravity and buoyancy by referring to the pulses in the recording of the steady motion. Obviously, the amount of this correction varied with the radius of the path.

Numerous experiments were made by changing the speed of the motor which could be varied between 2.5 and 25 rpm. The Reynolds numbers varied between 0 and about 200. Obviously the tangential force exerted on the sphere would be composed of the same type of effects as in the rectilinear motion. These effects were represented by the steady state drag, added mass and history terms; however, since the path of the motion was curved, some effect of the curvature was conceivable. In order to include the effect of the curvature, eq 2 was modified to

$$F = \frac{1}{2} \rho C_D \pi r^2 V^2 + \lambda C_A \left(\frac{4}{3} \pi r^3\right) \rho \frac{dV}{dt} + \chi C_H (\pi \rho \mu)^{1/2} r^2 \int_0^t \frac{dV/dt'}{(t-t')^{1/2}} dt' \qquad (5)$$

where λ and χ are functions of $a = R/r$ where R is the radius of curvature of the path. The first term is the same as the steady state drag term. It had been found previously (Odar, 1967) that when the motion of the sphere was steady, the curvature had no effect on the tangential force; namely the force was the same as that in rectilinear motion. This is the reason why the first term in eq 5, which is analogous to eq 2, remained unmodified. Experimental results showed that the other two terms were indeed affected by the curvature, or a, a dimensionless quantity which could be derived by dimensional reasoning and also which, in essence, represented the geometry of the system.

The values of a varied from 7.0 to 27.0 during the experiments. The experimental results are illustrated in the Appendix. The measured forces are shown by solid curves. The points indicate the values calculated from eq 5 with properly chosen values of λ and χ. These values were mostly chosen by trial and error. However, the values of χ could easily be verified since for each value of a a special run was made where after an arbitrary change of the velocity, the velocity was set at a constant value. Thus, at this range of the velocity the added mass term became zero and the force was only dependent on the steady state drag term, the calculation of which could easily be made, and the history term.

The forces were calculated by using the method of finite differences. The velocities were scaled at intervals of 0.1 sec from the recordings and the accelerations were calculated. Since it was impossible to scale the velocities with extreme accuracy due to the thickness of the trace – 0.01-0.02 in. – the accelerations could not be determined to desired accuracy with the numerical differentiation. These inaccuracies caused slight scattering of the calculated points, especially at the beginning of the motion where the measurement of small velocities was extremely difficult. An effort to measure accelerations with an accelerometer did not produce satisfactory results since the accelerometer measured acceleration due to gravity also. The highest acceleration was about 9.0 ft/sec^2 calculated at $t = 0.7$ sec in Run 14. However, the values of acceleration during the remaining part of the run were lower; for example, the acceleration at $t = 6.0$ sec was about 1.5 ft/sec^2. Because of the thickness of the trace the accuracy could not be better than 1%, and therefore the measurement of small accelerations such as 1.5 ft/sec^2 was not reliable. Since the scattering of the calculated points was not really bad, as can be judged from the runs shown in the Appendix, slight inaccuracies due to numerical differentiation were thought to be acceptable.

The history integral was calculated according to the following formula:

$$\int_0^{n\Delta t} \frac{dV/dt'}{(t-t')^{1/2}} dt' = 2(\Delta t)^{1/2} \sum \left(\frac{\overline{dV}}{dt}\right)_i [(n-i+1)^{1/2} - (n-i)^{1/2}] \qquad (6)$$

in which n is the number of time intervals and

$$\left(\frac{\overline{dV}}{dt}\right)_i = \frac{1}{2}\left[\left(\frac{dV}{dt}\right)_i + \left(\frac{dV}{dt}\right)_{i-1}\right] \qquad (7)$$

is the average acceleration between i'th and $(i-1)$'th interval. Equation 6 is obtained from term-by-term integration of the integral at the left-hand side.

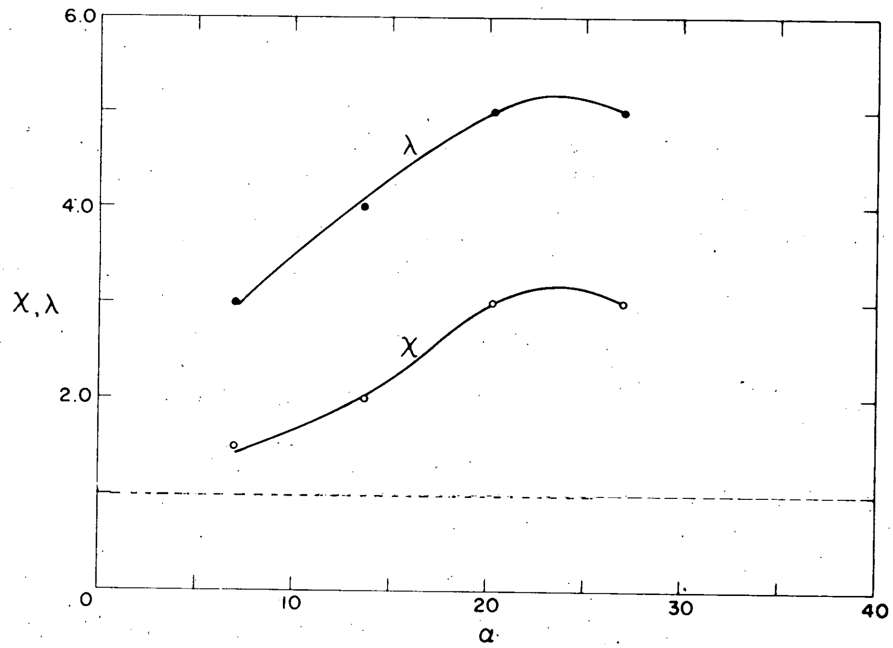


Figure 2. Variations of λ and χ.

To simplify the calculations of the first term, it is changed to $C\mu\pi rV$ in which C is the new drag coefficient. The relationship between C and C_D is

$$C = \frac{1}{4} C_D \,\text{Re}. \tag{8}$$

Using the numerical values of Lapple (1951) this coefficient can be formulated as follows:

$$C = 0.75\,\text{Re}^{0.72} + 6.0 \text{ for Re} \leqslant 1000 \tag{9}$$

which proved to be very convenient for calculations with a digital computer.

Despite some inaccuracies in the calculation of accelerations the agreement between the calculated points and the experimental values is excellent as can be seen from comparisons in the Appendix. In many places there is practically no difference between the calculated values which are plotted at 0.1-sec intervals and the experimental values.

Experiments indicated that λ and χ are some functions of $a = R/r$ as illustrated in Figure 2. The experimental range for a was between 7 and 27. For larger values of a, the path approaches a straight line, and therefore, λ and χ should approach unity. For smaller values of a the variations of λ and χ are not known; but the experimental values of λ and χ for $7 \leqslant a \leqslant 27$ suggest that both λ and χ approach unity as a approaches zero. More experiments with various values of a are needed to determine the variations of λ and χ outside of the range $7 \leqslant a \leqslant 27$. It will also be useful to make more experiments within the range in order to determine the derivatives of λ and χ with respect to a.

CONCLUSION

A new formula expressing the unsteady tangential hydrodynamic force on a sphere moving along a circular path is found. Since the tangential acceleration is completely arbitrary and the agreement between the calculated and experimental values is excellent for each run, it is logical to assume that eq 5 is valid for all types of accelerations. When the path is curved, the effects of the added mass and the history of the motion increase whereas the contribution from the steady state drag remains the same as that in a rectilinear motion.

LITERATURE CITED

Basset, A.B. (1888) *A treatise on hydrodynamics*. Cambridge, England: Deighton, Bell and Co., vol. 2, chap. 21. Also Dover Publications, 1961.

Lapple, C.E. (1951) *Fluid and particle mechanics*. University of Delaware, chap. 13.

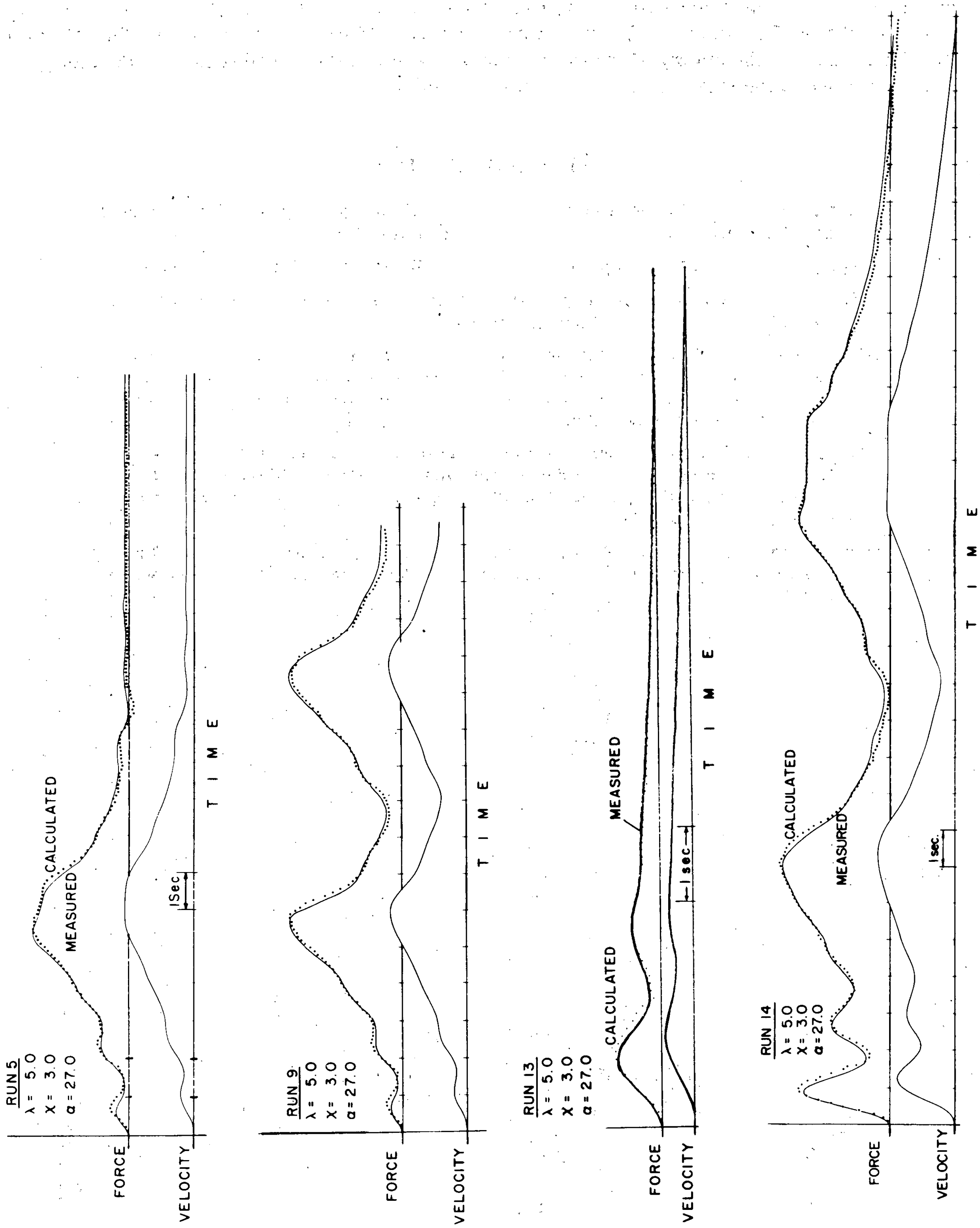
Moorman, R.B. (1955) Motion of a spherical particle in the accelerated portion of free fall. Ph.D. Dissertation, State University of Iowa.

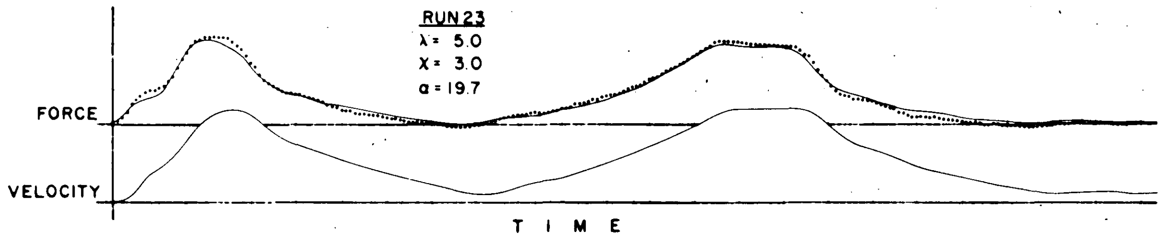
Odar, F. (1966) Verification of the proposed equation for calculation of the forces on a sphere accelerating in a viscous fluid. *Journal of Fluid Mechanics*, vol. 25, p. 591, part 3. Also U.S. Army Cold Regions Research and Engineering Laboratory (USA CRREL) Research Report 190.

________ (1967) Forces on a sphere moving steadily along a circular path in a viscous fluid. USA CRREL Research Report 229. Also *Transactions of the American Society of Mechanical Engineers, Journal of Applied Mechanics*, vol. 35, Series E, no. 2, p. 238, June 1967.

Odar, F. and Hamilton, W.S. (1964) Forces on a sphere accelerating in a viscous fluid. *Journal of Fluid Mechanics*, vol. 18, part 2, p. 302. Also USA CRREL Research Report 128.

APPENDIX A: COMPARISONS OF THE MEASURED AND CALCULATED FORCES

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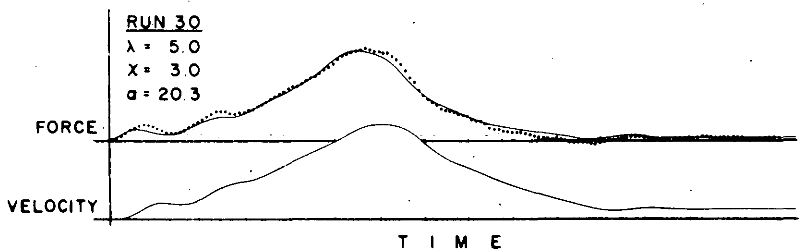
a = 19.7
FORCE
VELOCITY
T I M E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a = 20.3
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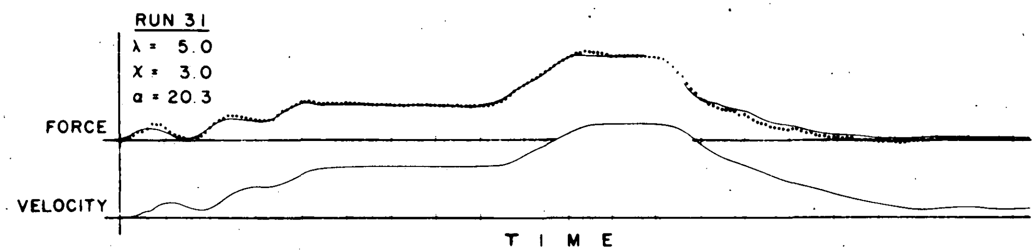
VELOCITY
T I M E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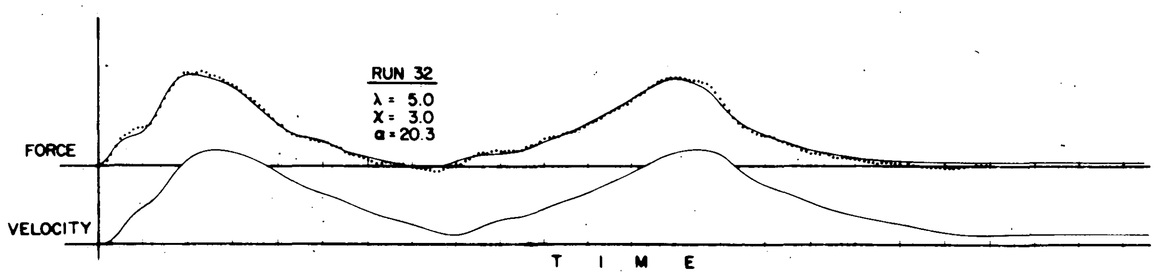
VELOCITY
T I M E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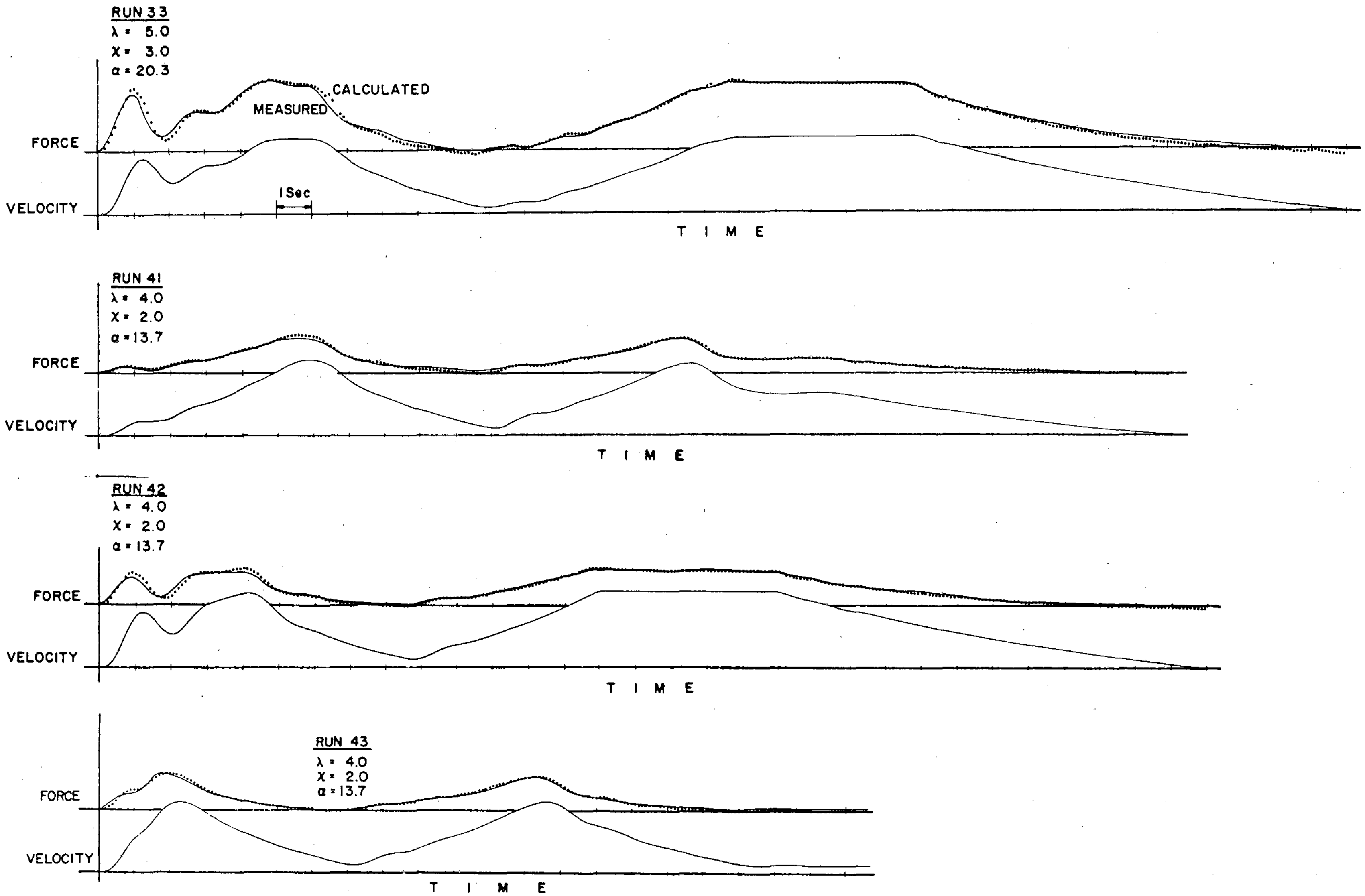
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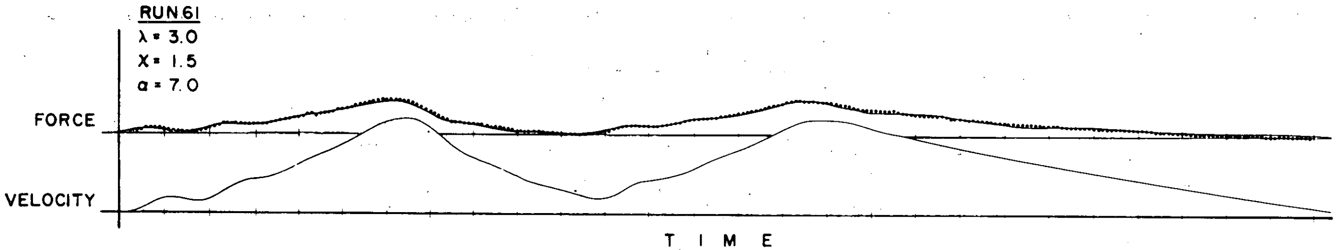
λ = 5.0
χ = 3.0
α = 20.3
CALCULATED
MEASURED
FORCE
VELOCITY
1 Sec
T I M E
RUN 41
λ = 4.0
χ = 2.0
α = 13.7
FORCE
VELOCITY
T I M E
RUN 42
λ = 4.0
χ = 2.0
α = 13.7
FORCE
VELOCITY
T I M E
RUN 43
λ = 4.0
χ = 2.0
α = 13.7
FORCE
VELOCITY
T I M E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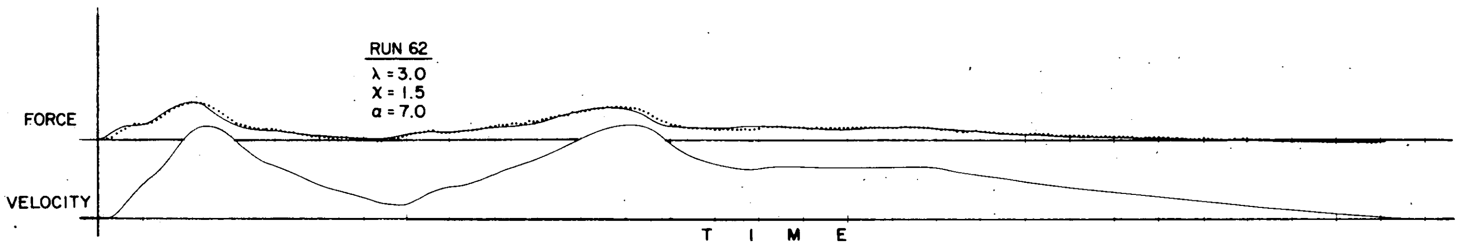
χ = 1.5
α = 7.0
FORCE
VELOCITY
T I M E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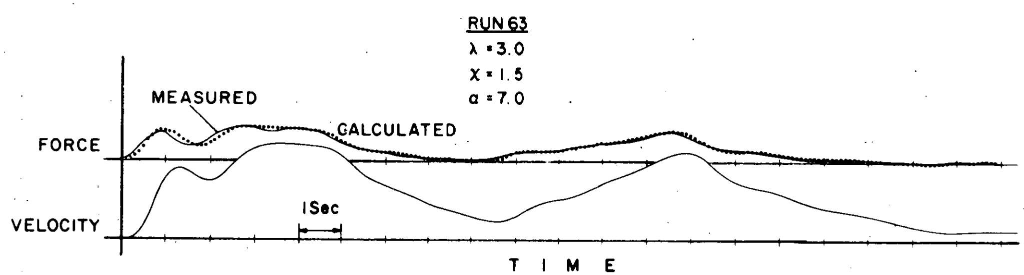
VELOCITY
T I M E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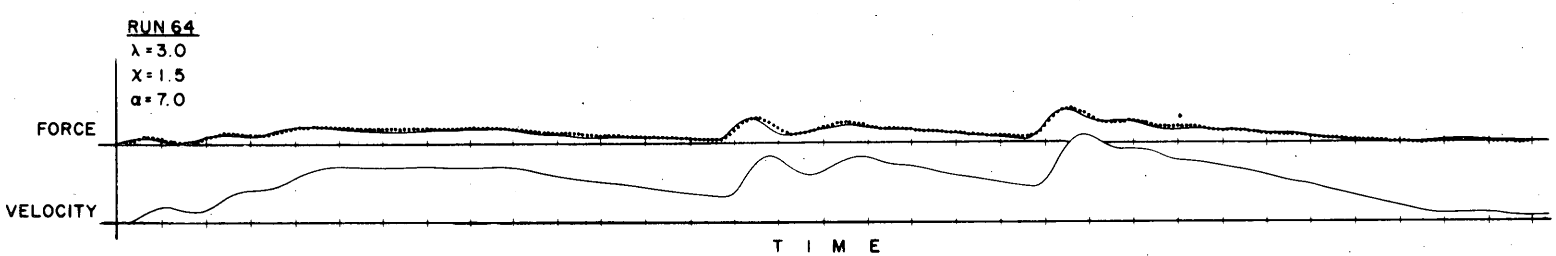
MEASURED
CALCULATED
FORCE
VELOCITY
1Sec
T I M E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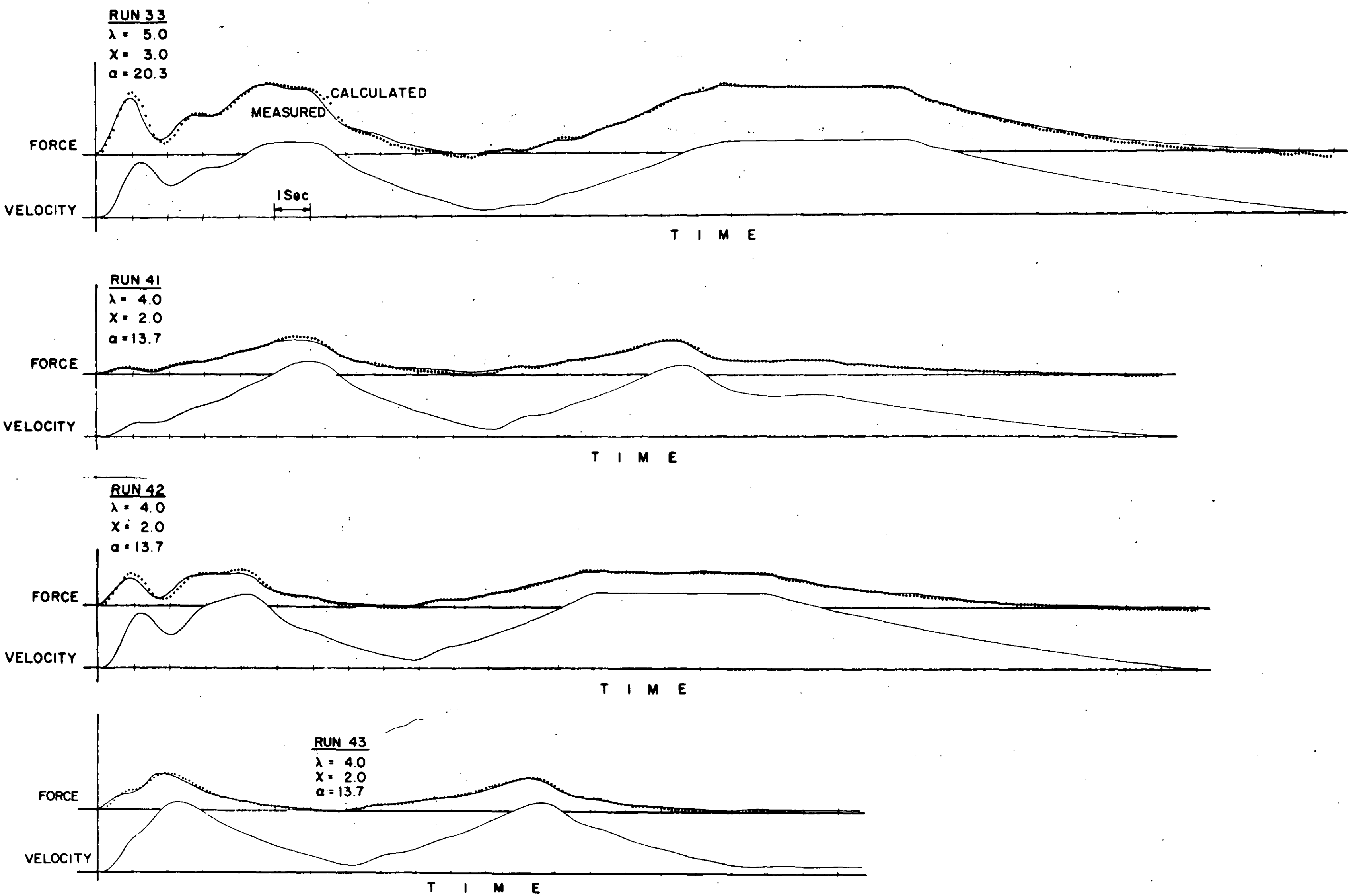
T I M E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α = 20.3
CALCULATED
MEASURED
FORCE
VELOCITY
1 Sec
T I M E
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X = 2.0
α = 13.7
FORCE
VELOCITY
T I M E
RUN 42
λ = 4.0
X = 2.0
α = 13.7
FORCE
VELOCITY
T I M E
RUN 43
λ = 4.0
X = 2.0
α = 13.7
FORCE
VELOCITY
T I M E

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13. ABSTRACT

Forces on a sphere moving unsteadily along a circular path in a viscous fluid are measured, and it is found that within the experimental range the formula valid for rectilinear motion has to be modified to account for the curvature of the path.

14. Key Words

Fluid mechanics
Sphere
Viscous fluid
Hydrodynamic force

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