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PENETRATION OF PLATES IN DENSE SNOW

by

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PREFACE

This report explores the implications of Kerr's viscous foundation theory by analyzing plate penetration data obtained in connection with the U. S. Army Cold Regions Research and Engineering Laboratory (USA CRREL) project, Mechanics of Deformation of Snow and Ice. The work was done jointly by Mr. René Ramseier, Research Division, Mr. J. A. Bender, Chief, and by Mr. Malcolm Mellor, Experimental Engineering Division, Mr. K. A. Linell, Chief.

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SUMMARY

As part of a program of snow studies in Greenland, instrumented struts were placed in rooms excavated below the surface of the ice cap. Closure rates were measured in the undersnow cavities, and strut loads developed by snow pressure on the circular end plates were recorded. The strut installations are described and the data are interpreted in terms of Kerr's viscous foundation theory. Emphasis is placed on the penetration of a circular rigid plate, the evaluation of the viscosity coefficients, comments on the values of the parameters for a given material, shear viscosity, and thickness in the Kerr model, and the effect of plate size on penetration rate. It is tentatively concluded that (1) until a suitable treatment for the viscoelastic continuum becomes available, the Kerr model offers a rational approach to problems of foundations in snow and to certain problems involving pressures on undersnow structures, (2) the viscosity coefficients of the model are not in themselves characteristic constants for a given snow, (3) as a consequence of (2), the viscosity coefficients may not be evaluated by comparing data for plates of widely differing sizes, and (4) if the thickness coefficient were found to be independent of plate size, as seems possible, it would become a valuable index for determining size effect.

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Introduction

As part of a program of snow studies in Greenland, instrumented struts were placed in rooms excavated below the surface of the ice cap. Closure rates were measured in the undersnow cavities, and strut loads developed by snow pressure on the circular end plates were recorded. It appears that some of the data obtained can be used to explore the applications and limitations of a viscoelastic foundation model proposed by Kerr (1962), perhaps shedding further light on the problems of snow foundations and stresses on undersnow structures.

Strut installations

At Camp Century, Greenland, 6 ft long vertical struts were placed in rooms excavated 12 m below the surface of the ice cap, their end plates bearing on roof and floor. Change of snow properties from roof to floor was slight. Strut loads were measured by Baldwin-Lima-Hamilton SP-4 load cells, Type C, indicating on an SR-4 transistorized strain gage indicator. Vertical closure of the cavities in the vicinity of the struts was measured by Starrett dial micrometers.

Data

Data from the Camp Century experiments are presented in Figures 1 and 2, where nominal plate pressures p_o and plate penetrations w_o are plotted against time. The curves of Figure 3 show the changes of snow density and overburden pressure with depth below the ice cap surface.

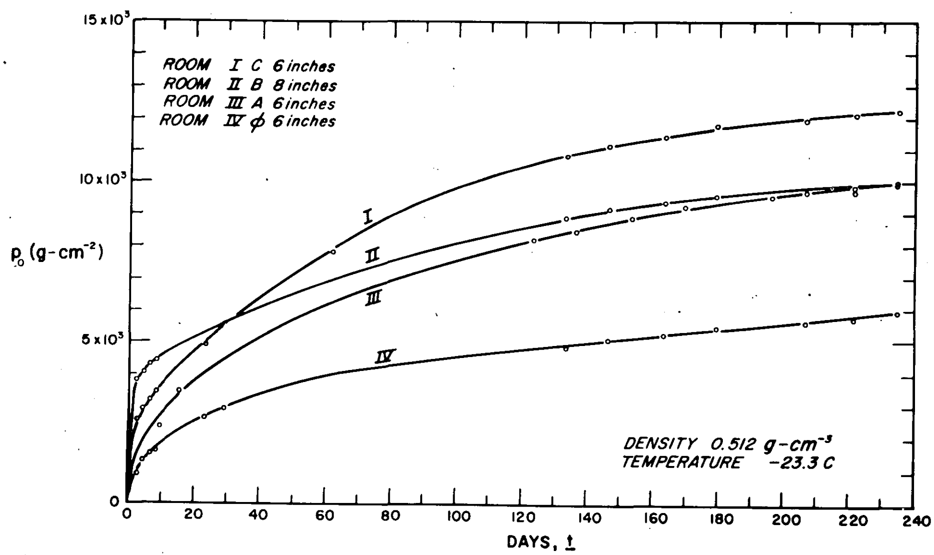

Figure 1. Plate pressure as a function of time.

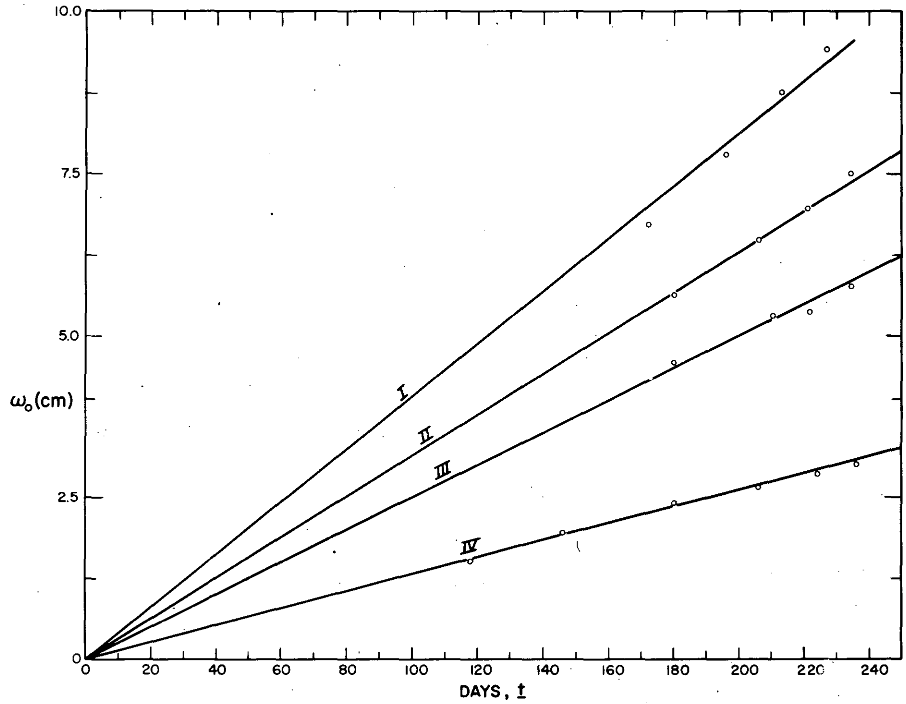

Figure 2. Plate penetration as a function of time.

Interpretation in terms of the Kerr model

The results of the closure and pressure measurements show that the plates are, in effect, slowly penetrating the snow at almost constant speed while the penetration resistance increases with time. Considering only the steady state condition after elastic straining and transient creep, it appears that penetration resistance is a function of time (and therefore of penetration) over the period of observation.

The situation is very similar to the settlement of foundations in snow, although ideally a foundation is usually subject to constant load. Kerr (1962) treated the problem of foundations on snow, representing the semi-infinite snow mass by a rheological model in the form of a finite layer having elastic and viscous elements (Fig. 4). Kerr's model is a viscoelastic modification of the Pasternak foundation model, and has the merit that shear interaction between snow under the loaded area and snow adjacent to the loaded area is simulated by viscous shear elements in the model. Elastic strain and initial transient creep are not considered, since interest centers on problems of long-duration loading.

Kerr assumed that a foundation under constant load settles at constant speed. For long-duration loading this is not quite so; the loaded snow densifies, its deformation resistance increases, and settlement speed decreases. For a similar reason, in the present strut experiments, deformation resistance increases with time. Nevertheless, it seems that initially we have to work over periods of limited duration, accepting the assumption that velocity and pressure at the loaded surface are invariant with time during the period considered. If this is not done, the parameters describing the shear and compression resistance of the snow become time-dependent variables.

Kerr suggests that the material parameters of his analysis may be evaluated by subjecting a circular plate to eccentric loading and he treats that case. It seems more convenient, however, to make the evaluation by solving the equations for different plate sizes, keeping loading symmetrical.

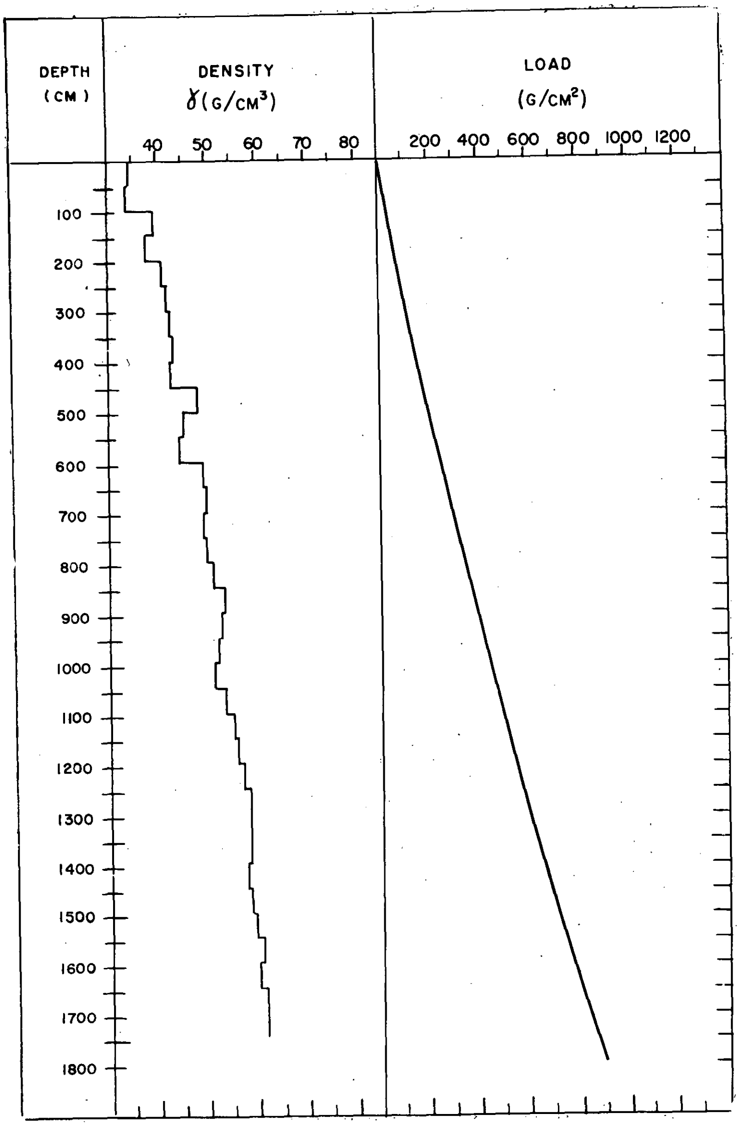

Figure 3. Snow density and overburden load plotted against depth below the ice cap surface (After Waterhouse and Steeves, 1960).

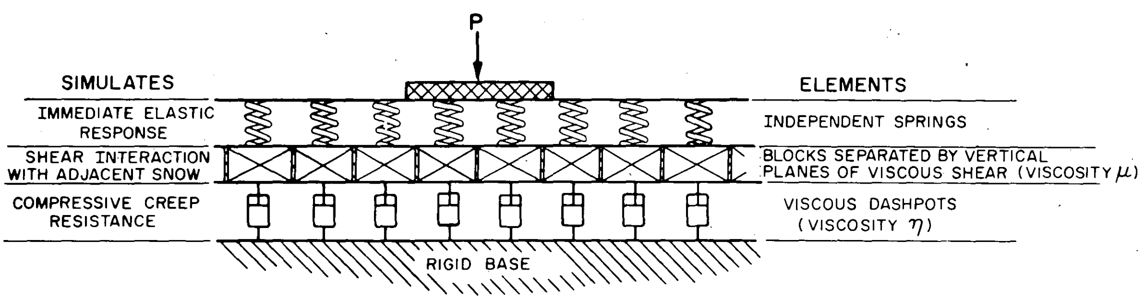

Figure 4. The Kerr viscoelastic foundation model.

Penetration of a circular rigid plate

A resumé of Kerr's derivation is given below for convenience, the particular case of a circular plate being considered from the start (see Fig. 5). Part of the argument has been restated to meet Nevel's criticism* of the original Kerr paper.

Considering the equilibrium of an element in the shear layer for slow, non-accelerating flow gives, in polar coordinates:

$$p\left(r + \frac{\delta r}{2}\right)\delta r \delta\theta - q\left(r + \frac{\delta r}{2}\right)\delta r \delta\theta + \tau_r r \delta\theta - \left(\tau_r + \frac{\partial \tau_r}{\partial r}\delta r\right)\left(r + \delta r\right)\delta\theta = 0.$$

Neglecting second-order small quantities

$$p - q - \frac{1}{r}\tau_r - \frac{\partial \tau_r}{\partial r} = 0. \qquad (1)$$

In accordance with the model it is assumed that

$$\tau_r = -\mu \frac{\partial^2 w}{\partial r \, \partial t} \qquad (2)$$

$$q = \eta \frac{\partial w}{\partial t}. \qquad (3)$$

The viscosity coefficients μ and η are regarded as characteristic parameters for the particular snow type at the place and time considered.

Substituting for τ_r and q from eq 2 and 3 into eq 1:

$$\mu\left(\frac{\partial^3 w}{\partial r^2 \partial t} + \frac{1}{r}\frac{\partial^2 w}{\partial r \, \partial t}\right) - \eta \frac{\partial w}{\partial t} + p_{r,t} = 0. \qquad (4)$$

For a limited time period p may be regarded as invariant with time. Note that this assumption is consistent with the view that μ and η are invariant with time. Hence the p term of eq 4 can be removed by differentiation with respect to t:

$$\mu\left(\frac{\partial^4 w}{\partial r^2 \partial t^2} + \frac{1}{r}\frac{\partial^3 w}{\partial r \, \partial t^2}\right) - \eta \frac{\partial^2 w}{\partial t^2} = 0$$

or, setting $\eta/\mu = \lambda^2$

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \lambda^2\right) w = 0. \qquad (5)$$

A solution of eq 5 is obtained by separation of variables; it is assumed that

$$w = f(t)\,\phi(r)$$

and the general solution is obtained from the condition

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \lambda^2 \phi = 0. \qquad (6)$$

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*Nevel, D. E. (1965) A discussion of A. D. Kerr's "Settlement and tilting of footings on a viscous foundation," USA CRREL Technical Note.

This transforms to a modified Bessel equation of zero order when $z = \lambda r$, and the solution of eq 6 is in terms of modified Bessel functions of the first and second kind. The general solution of eq 5 for the external unloaded area, $r \geq R$ (where R is plate radius) thus becomes

$$w_{r,t} = f(t)\left[AI_0(\lambda r) + BK_0(\lambda r)\right]. \quad (7)$$

The required solution is determined from the boundary conditions of the problem as follows:

(1) Since the snow at large distances from the plate remains undeformed, $w_{r,t} = 0$ as $r \rightarrow \infty$. Also $K_0 \rightarrow 0$ and $I_0 \rightarrow \infty$ as $r \rightarrow \infty$, so that $A = 0$.

(2) In the area beneath the plate ($r \leq R$), deflection w_0 is constant for any time t, while in the area outside the plate ($r \geq R$) deflection w_r is according to eq 7. But at the plate perimeter ($r = R$), $w_{r,t} = w_{0,t}$ and hence

$$w_{0,t} = f(t)\, BK_0(\lambda R).$$

In the external area $r \geq R$ displacements are therefore given by

$$w_{r,t} = w_{0,t} \frac{K_0(\lambda r)}{K_0(\lambda R)}.$$

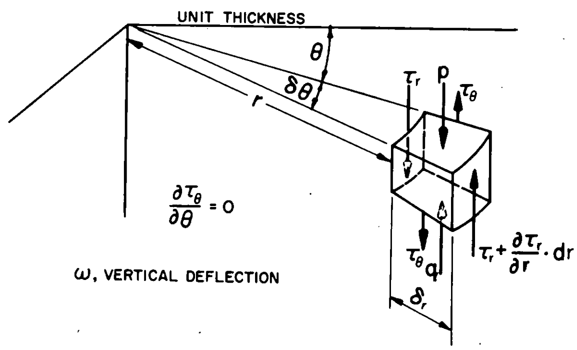

Figure 5. Stresses on an element in the shear layer.

(3) The condition for equilibrium in the axial direction is

$$P - \int_0^{2\pi}\int_0^{R} q_0 \; r\,dr\,d\theta - \int_0^{2\pi}\int_R^{\infty} q_e \; r\,dr\,d\theta = 0 \quad (8)$$

where P is the plate load and

$$q_0 = \eta\left(\frac{\partial w_0}{\partial t}\right) \qquad (r \leq R)$$

$$q_e = \eta\left(\frac{\partial w_0}{\partial t}\right)\frac{K_0(\lambda r)}{K_0(\lambda R)} \qquad (r \geq R).$$

Integration of eq 8 gives

$$P = \pi\eta R^2\left(\frac{\partial w_0}{dt}\right)\left[1 + \frac{2\,K_1(\lambda R)}{(\lambda R)\,K_0(\lambda R)}\right] \quad (9)$$

or

$$\frac{p_0}{\dot{w}_0} = \eta\left[1 + \frac{2}{\lambda R}\frac{K_1(\lambda R)}{K_0(\lambda R)}\right]. \quad (10)$$

Equation 10 provides a satisfactory basis for the present analysis. However, the solution may be completed by applying the initial condition $w = 0$ at $t = 0$ after integration of eq 10 with respect to time:

$$\frac{P}{\pi R^2}\cdot\frac{t}{w_{0,t}} = \frac{pt}{w_0} = \eta\left[1 + \frac{2}{\lambda R}\frac{K_1(\lambda R)}{K_0(\lambda R)}\right]. \quad (10a)$$

Table I. Plate penetration data for Camp Century tests.

Data source	Plate radius, R	Penetration rate, $\dot{w}_0$	$w_0 = 1.5$ cm		$w_0 = 2.5$ cm		$w_0 = 3.2$ cm	
			Plate pressure, p	$p/\dot{w}_0$	p	$p/\dot{w}_0$	p	$p/\dot{w}_0$
(Rm. no.)	(cm)	(cm sec^{-1})	(g cm^{-2})	(g sec cm^{-3})	(g cm^{-2})	(g sec cm^{-3})	(g cm^{-2})	(g sec cm^{-3})
I	7.62	4.70×10^{-7}	6.15×10^{3}	1.31×10^{10}	7.85×10^{3}	1.67×10^{10}	8.85×10^{3}	1.885×10^{10}
II	10.15	3.63×10^{-7}	6.45×10^{3}	1.78×10^{10}	7.50×10^{3}	2.065×10^{10}	8.15×10^{3}	2.245×10^{10}
III	7.62	2.88×10^{-7}	6.15×10^{3}	2.14×10^{10}	7.65×10^{3}	2.66×10^{10}	8.30×10^{3}	2.88×10^{10}
IV	7.62	1.51×10^{-7}	4.70×10^{3}	3.12×10^{10}	5.55×10^{3}	3.68×10^{10}	6.05×10^{3}	4.01×10^{10}

Inspection of eq 10 reveals that, assuming finite values for η and λ, the ratio of stress to strain rate becomes independent of plate size for large plates (the Bessel term tends to zero as R tends to infinity). The simplified form of eq 10 for R infinite

$$\frac{p_0}{\dot{w}_0} = \eta$$

is applicable to natural snow densification in the polar ice caps (in the Newtonian range).

Evaluation of the viscosity coefficients

To solve eq 10 for η and λ (and hence μ) we require simultaneous values of p_0 and $\dot{w}_0$ for at least two different plate sizes* without change of snow type (i.e. same density, temperature, grain structure). At Camp Century the bearing plates used in room II were slightly larger than those used in rooms I, III, and IV at the same level, so that a solution is possible. Unfortunately, strut data from other sites in Greenland and Antarctica cannot be utilized, since a uniform plate size was adopted. It is not to be expected that the Camp Century data will yield strong solutions; the ratio of plate radii is only 3:4, which is hardly sufficient to produce an effect exceeding experimental variability for a given plate size.

In Table I, data are listed for three different penetration values. Comparison for the same degree of total deformation† is probably a necessary condition for the solution of eq 10, since the viscosity coefficients may change as the snow is deformed. In effect, measurements on each strut are equivalent to two tests; all readings from the three struts in rooms I, III, and IV were averaged.

An equation in λ is obtained from eq 10 by eliminating η from the simultaneous equations resulting from substitution of values for $p/\dot{w}_0$ and R. Note that η has the dimensions (g cm^{-3}-sec), μ the dimensions (g cm^{-1}-sec), and consequently λ has the dimensions (cm^{-1}). λ is found by successive approximation**, η being given by re-substitution into one of the original equations. Values found in this way are given in Table II.

Table II. Parameters of the Kerr foundation theory evaluated from eq 10.

Total strain, w_0 (cm)	λ (cm^{-1})	η (g cm^{-3}-sec)	μ (g cm^{-1}-sec)
1.5	0.187	7.71×10^{9}	2.21×10^{11}
2.5	0.124	6.55×10^{9}	4.26×10^{11}
3.2	0.116	6.74×10^{9}	5.01×10^{11}

*We have no eccentric loadings which might permit use of the rotation method suggested by Kerr.

† Later arguments on the effective depth of the deforming layer suggest that perhaps comparison should be made for the same degree of relative deformation (deformation as a fraction of plate size).

** As an aid to solution, a graph of $\frac{K_1(x)}{K_0(x)}$ was drawn using values from the British Association Tables for $0<x<5$, with values computed from the asymptotic series for $K_0(x)$ and $K_1(x)$ for $x>5$.

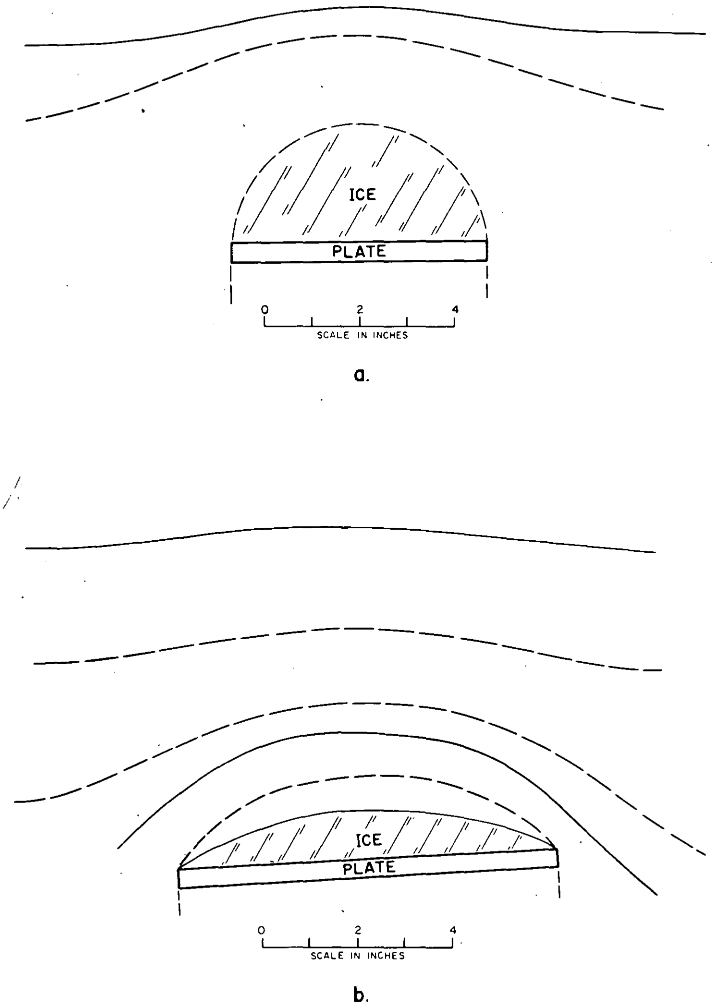

Figure 6. Deformation patterns in snow penetrated by circular rigid plates.

Comments on the values of λ, η, and μ

We first note that λ and μ change progressively as the plate penetrates, but no systematic variation of η with deformation is indicated. If a simple physical interpretation of the Kerr model is taken, this is rather surprising, as it is known that the deformation resistance of snow increases with compaction. However, the zone of influence may extend further into the mass as highly compressed snow builds up at the face of the plate.

The coefficient η. In the model, η is a true constant for a given material. The thickness of the deforming layer is undefined, and η has the dimensions (g cm^{-3} sec) rather than the usual dimensions of a viscosity coefficient (g cm^{-2} sec). In snow, however, it is known that a loaded plate influences the material only to a limited depth and, furthermore, that depth is related to the size of the plate.

At the conclusion of the Camp Century experiments the deformed snow was cut into thin section and examined by transmitted light. It was found that, at a perpendicular distance of one diameter from the plate, the axial displacement of the snow strata was almost an order of magnitude smaller than the penetration distance of the plate itself (Fig. 6). This is roughly in accordance with the findings of Wuori (1962) for deformation following short-duration loading, and is of the magnitude which might be inferred by analogy with the Boussinesq theory for stress distribution in an elastic continuum.

It therefore seems likely that there is an implicit relation between η and $\underline{R}$, since the effective depth of the deforming layer depends on $\underline{R}$. If this is so, the true compactive viscosity is obtained by multiplying η by the effective depth of the deforming layer $\underline{H}$. (The "effective depth" would represent the thickness of a fictitious layer subject to uniform stress distribution.) This possibility can be explored by comparing the present values of η with compactive viscosity coefficients for the same snow found by creep tests and by analysis of natural snow densification in the ice cap.

As a preliminary to such a comparison, it is necessary to convert the calculated coefficient η to an "equivalent Newtonian" coefficient η_n, because snow exhibits visco-plastic behavior rather than Newtonian viscosity at high stresses, such as were developed in the plate tests.

In the model, η is defined by

$$\eta = \frac{q_0}{\dot{w}_0} .$$

If we acknowledge a hyperbolic sine dependence of strain rate on stress, the equivalent Newtonian coefficient is given by

$$\eta_n = \frac{\sigma_0 \sinh\left(\frac{q_0}{\sigma_0}\right)}{\dot{w}_0} \qquad (11)$$

where σ_0 is a characteristic parameter for a given snow type which has the dimensions of a stress. For Camp Century a value of $\sigma_0 = 700 \text{ g cm}^{-2}$ may be assumed (Bader, 1962). Taking average values of η and $\dot{w}_0$ from Tables I and II, we find

$$\eta_n = 2.93 \times 10^{10} \text{ g cm}^{-3} \text{ sec.}$$

No suitable confined creep data are available for Camp Century, but since unconfined creep tests by Ramseier and Pavlak show strain rates for Camp Century to be almost identical to those for Byrd Station (Antarctica) snow at the same pressure and temperature, we may draw upon confined creep data for Byrd. Taking a value of compactive viscosity η_{c1} for snow density 0.51 g cm^{-3} from Mellor and Hendrickson ($\eta_{c1} = 4.6 \times 10^{11} \text{ g cm}^{-2} \text{ sec}$), a value for the effective depth of deformation (H_1) in the Camp Century plate tests can be calculated:

$$H_1 = \frac{\eta_{c1}}{\eta_n} = \frac{4.6 \times 10^{11}}{2.93 \times 10^{10}} = \underline{15.7 \text{ cm}} ,$$

Using an alternative value, η_{c2}, caculated from Bader's data for snow densification at Byrd Station and adjusted to a temperature of -23C, a second estimate of effective depth, H_2, is obtained:

$$H_2 = \frac{\eta_{c2}}{\eta_n} = \frac{6.1 \times 10^{11}}{2.93 \times 10^{10}} = \underline{20.8 \text{ cm}} .$$

The lower value H_1 is about equal to the diameter of the smaller plate, while the higher value H_2 is about equal to the diameter of the larger plate. There is thus reasonable agreement between the effective depths of deformation calculated in this way and the effective depths observed directly in the snow cross-sections.

The coefficient μ. Turning to consideration of the shear viscosity μ, we may enquire whether change of effective depth for the deforming layer will affect the value of this parameter. If a physical similarity to the model's shear layer is admitted, then it does appear that μ will increase as the effective depth H increases, since μ defines a shear strength per unit width for a layer of undefined thickness. We have, as yet, no means of testing this.

In Table II, μ increases as the strain w_0 increases. It is probably unprofitable to attempt an interpretation of this trend in terms of the model, since the physical situation changes appreciably from that represented by the model when penetrations are large. Examination of the snow cross-sections revealed that, as penetration increases, a cupola of incompressible ice develops on the plate, and gradually tends to become a solid hemisphere. Thus the geometry of the penetrating surface is effectively changed, and it seems that eventually the plate and its ice bulb will simply shear through the snow like a wedge, displacing snow by compression in directions radial to the ice bulb.

The parameter λ. In reality, the values of η and μ must depend on the thickness of the deforming layer H (an effect bypassed in the theory by the assumptions of eq 2 and 3), and are therefore likely to be dependent on plate size R. It is possible, however, that λ ($= \sqrt{\eta/\mu}$) could be independent of R. For example, if both η and μ are linearly dependent on R, then λ should be unaffected when R changes. If this should prove to be the case, λ would be a valuable parameter for determining the size effect for plates bearing on various types of snow.

Size effect for plate penetration

Making the assumption that λ is invariant with R, the effect of plate size on penetration rate or penetration resistance can be investigated. Figure 7 shows the ratio $p/\dot{w}_0$ as a function of R, the curves being obtained by substituting the results of Table II into eq 10. There is an increasingly powerful size effect as plate radius drops below 25 cm, while size becomes relatively unimportant for plates of radius greater than 25 cm (the $p/\dot{w}_0$ ratio for a plate of infinite radius would be roughly two-thirds of the value for a 25 cm diameter plate).

If the curves of Figure 7 proved valid for snow of the type employed and for the general magnitudes of stress and strain rate of the experiments, one implication would be that size is of little practical significance in the design of heavily loaded foundations on medium-density snow; size effect would only be important in the design of pile foundations, and in interpreting data from small test foundations. Since the present results were obtained from ill-conditioned equations, further data will be needed to determine λ firmly and to investigate possible variations of η with snow type and stress and strain rate magnitudes.

Conclusions

The data analyzed here are too slight to provide more than a tantalizing glimpse of the mechanics of plate penetration. Nevertheless, the following tentative conclusions have been reached.

1. Until a suitable treatment for the visco-elastic continuum becomes available, the Kerr model offers a rational approach to problems of foundations in snow and to certain problems involving pressures on undersnow structures. The validity of the model is restricted to small penetrations and moderate stresses (or strain rates). For large penetrations (relative to plate size) and high stresses (relative to snow type) deformation probably becomes a flow phenomenon, giving the effect of plastic yielding.

2. The viscosity coefficients of the model, η and μ, are not in themselves characteristic constants for a given snow. They are related to the thickness of the deforming snow layer, which in turn depends on the plate size. It is also expected that they will vary with the magnitude of the stress, since the true viscosity coefficients to which they are related are functions of stress.

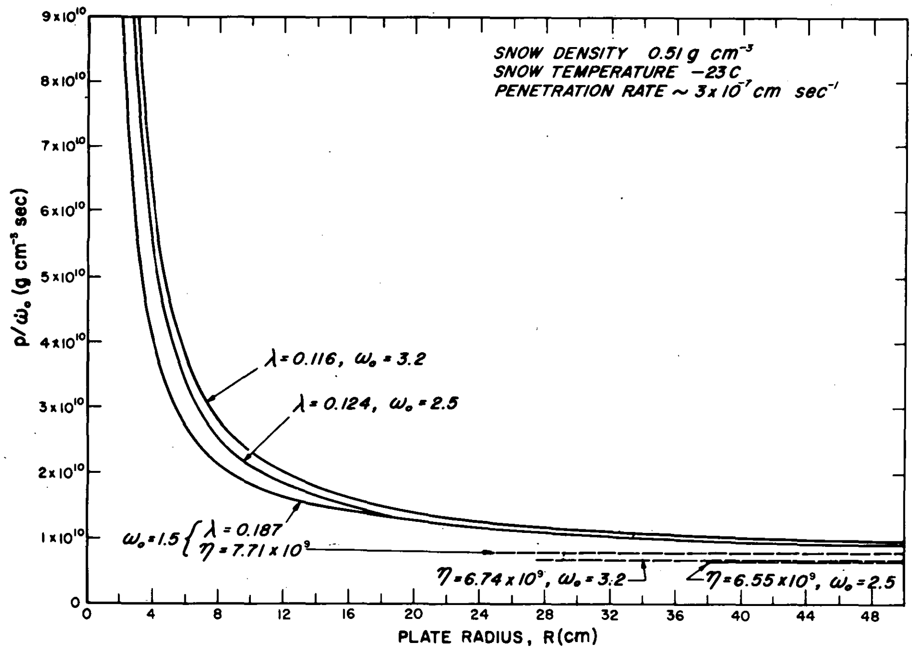

Figure 7. Effect of plate size on penetration rate and penetration resistance.

3. One consequence of 2. is that η and μ may not be evaluated by comparing data for plates of widely differing sizes.

4. If λ were found to be independent of plate size, as seems possible, it would become a valuable index for determining size effect. Since η can be found independently from creep tests or, for low stresses, from snow densification data, λ could thus be evaluated from experiments with a single plate size.

Plate penetration experiments seem worth pursuing. Further work should clear up some practical questions and advance the knowledge of snow mechanics.

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