ANALYSIS OF THE FREEZEBACK OF WATER IN A CYLINDRICAL BOREHOLE DRILLED IN AN ICE SHEET

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PREFACE

This report was prepared by Dr. Shunsuke Takagi, Research Physical Scientist, Physical Sciences Branch, Research Division, U.S. Army Cold Regions Research and Engineering Laboratory. The work was funded under the In-House Laboratory Independent Research program.

The numerical computation was performed by L. Stanley. Technical review of the report was performed by Dr. Yoshisuke Nakano.

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Introduction

The problem solved in this paper arose as the result of an exploratory drilling program in the Antarctic Ocean. When a borehole is drilled through an ice sheet the water filling it starts to freeze. However, the borehole must be kept open for a certain period of time so that the water below the ice sheet may be sampled and brought to the surface. This paper is part of an effort made at CRREL to solve this problem.

This paper presents the analytical solution of the freezeback of water in a borehole under the conditions that the temperature distribution does not depend on the vertical direction and the temperature of the water in the borehole is the freezing temperature everywhere in the region occupied by the water throughout the entire process. Carslaw and Jaeger* (p. 296) state that no exact solution of this problem was available when they wrote their book. To the author's knowledge, their statement is still true.

A new mathematical device is introduced in this paper to discover the analytical solution of this problem. The thickness h of the new ice layer formed inside the original cylindrical wall is used in place of time t. Conversion of t to h is performed by use of an expression of dh/dt as a power series of h. With this device the moving boundary between ice and water is mathematically transformed to a stationary boundary. The nonlinear problem of the freezing of water is thus broken down into a series of linear problems. In other words, temperature is determined successively by solving the linear ordinary differential equations arising from each term of the series. When the series expressing the temperature is determined, the series expressing dh/dt is given in terms of h. The problem can thus be solved.

The solution, however, cannot include more than the first few terms because of the insurmountable complexity involved in solving higher terms of the series. The solution is valid, therefore, only for a short initial time interval.

The method presented here is applicable to other problems of a similar nature.

The problem

Let the initial radius of the borehole be R_0. The radial distance R is nondimensionalized by using r defined by

$$R = R_0 r. \tag{1}$$

* Carslaw and Jaeger (1959) Conduction of heat in solids. Oxford.

Thus the ice/water interface is initially at

$$r = 1. \tag{2}$$

Let the nondimensional time τ be defined by

$$\tau = \frac{\alpha_i}{R_0^2} t \tag{3}$$

where α_i is the thermal diffusivity of ice and t is time. Then the differential equation to be solved is

$$\frac{\partial \theta_i}{\partial t} = \frac{\partial^2 \theta_i}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_i}{\partial r} \tag{4}$$

where θ_i is the temperature of ice. At $r = \infty$, the temperature of ice is:

$$\theta_i = \theta_\infty \tag{5}$$

and the initial temperature is:

$$\theta_i = \theta_\infty \qquad \text{for } 1 \leqq r < \infty. \tag{6}$$

Let h be the nondimensionalized thickness of the new ice layer formed inside the cylindrical wall. The boundary condition at $r = 1 - h$ is

$$\theta_i = \theta_f \tag{7}$$

where θ_f is the freezing temperature, which is assumed to be a fixed constant although, in reality, the freezing temperature of sea water drops with the progress of freezing due to the increase of salt content in the unfrozen sea water. In the following, the temperature of unfrozen water is assumed, for simple analysis, to be equal to θ_f everywhere in the unfrozen region and throughout the entire process of freezing.

The equation of energy balance at the ice/water interface is reduced, under the assumption introduced so far, to

$$-\left(\frac{\partial \theta_i}{\partial r}\right)_{1-h} = \frac{L}{c_i} \frac{dh}{dt} \tag{8}$$

where c_i is the specific heat per unit mass of ice, and L is the latent heat per unit mass of water.

Reduction to ordinary differential equations

The moving boundary at $r = 1 - h$ becomes a stationary boundary at $\xi = 0$ by the transformation

$$\xi = \frac{r - 1 + h}{h}. \tag{9}$$

The correspondence of r to ξ at three important points is:

r	$1-h$	1	∞
ξ	0	1	∞

Because h is monotonically increasing with time, h may be used in place of time. The temperature $\theta_i(r,\tau)$ of ice may then be expressed as

$$\theta_i(r,\tau) = \Theta_i(\xi, h). \tag{10}$$

Substituting eq 10 into eq 4 yields

$$(1 - h + \xi h)\left\{\frac{\partial \Theta_i}{\partial \xi}\frac{\dot{h}}{h}(\xi - 1) - \frac{\partial \Theta_i}{\partial h}\dot{h} + \frac{1}{h^2}\frac{\partial^2 \Theta_i}{\partial \xi^2}\right\} + \frac{1}{h}\frac{\partial \Theta_i}{\partial \xi} = 0 \tag{11}$$

where

$$\dot{h} = \frac{dh}{dt}. \tag{12}$$

We introduce another assumption that $\Theta_i(\xi, h)$ may be expressed as

$$\Theta_i(\xi, h) = \sum_{n=0}^{\infty} h^n \phi_n(\xi). \tag{13}$$

We assume that $\dot{h}$ can be expressed in terms of h:

$$\dot{h} = 2\beta^2 \sum_{n=-1}^{\infty} \nu_n h^n \tag{14}$$

where β is a constant to be determined and

$$\nu_{-1} = 1. \tag{15}$$

The series (eq 14) begins with $n = -1$ to yield $h(t)$ such that $\dot{h} \to \infty$ when $h \to 0$.

Substitute eq 13 and 14 into eq 11 and rearrange the result in the ascending order of h. The differential equations of ϕ_n ($n \geq 0$) are then found by equating the coefficients of h^n to zero. For systematic computation we transform the sum in the pair of braces in eq 11 to the following:

$$\frac{\partial \Theta_i}{\partial \xi}\frac{\dot{h}}{h}(\xi - 1) - \frac{\partial \Theta_i}{\partial h}\dot{h} + \frac{1}{h^2}\frac{\partial^2 \Theta_i}{\partial \xi^2} = \sum_{n=0}^{\infty} h^{n-2} F_n \tag{16}$$

where

$$F_n = \frac{d^2\phi_n}{d\xi^2} + 2\beta^2 \sum_{m=0}^{n} \nu_{n-m-1}\left[(\xi - 1)\frac{d\phi_m}{d\xi} - m\,\phi_m\right]. \tag{17}$$

The equation of ϕ_0 is given by

$$F_0 = 0 \tag{18}$$

and the equations of ϕ_n ($n \geqq 1$) are given by

$$F_n + (\xi - 1) F_{n-1} + \frac{d\phi_{n-1}}{d\xi} = 0. \tag{19}$$

The equations assume a familiar form when ξ is transformed to x, defined by

$$x = \beta(\xi - 1) \tag{20}$$

where, because $0 \leqq \xi \leqq \infty$,

$$-\beta \leqq x < \infty . \tag{21}$$

Thus we obtain

$$\frac{d^2\phi_0}{dx^2} + 2x \frac{d\phi_0}{dx} = 0 \tag{22}$$

and, for $n \geqq 1$,

$$\frac{d^2\phi_n}{dx^2} + 2x \frac{d\phi_n}{dx} - 2n\,\phi_n = f_n(x) \tag{23}$$

where $f_1(x)$ is given by

$$\beta f_1(x) = -(1 + 2\beta\nu_0 x) \frac{d\phi_0}{dx} \tag{24}$$

and $f_n(x)$ ($n \geqq 1$), by

$$\beta f_n(x) = -x \frac{d^2\phi_{n-1}}{dx^2} - \frac{d\phi_{n-1}}{dx} - 2 \sum_{m=0}^{n-1} \left(x \frac{d\phi_m}{dx} - m\,\phi_m\right)\left(\beta\nu_{n-m-1} + x\nu_{n-m-2}\right). \tag{25}$$

The boundary condition 7 yields

$$\phi_0(x = -\beta) = \theta_F$$

and

$$\phi_n(x = -\beta) = 0 \qquad \text{for } n \geqq 1. \tag{26}$$

The boundary condition 5 yields

$$\phi_0(x = \infty) = \theta_\infty$$

and

$$\phi_n(x = \infty) = 0 \qquad \text{for } n \geq 1. \tag{27}$$

The initial condition 6 reduces to eq 27, because, as shown by eq 9, $h \to 0$ is equivalent to $\xi \to \infty$ for r such that $1 - h < r < \infty$.

General solution

One of the solutions of the homogeneous equation derived from eq 23,

$$\frac{d^2\phi_n}{dx^2} + 2x\frac{d\phi_n}{dx} - 2n\,\phi_n = 0 \tag{28}$$

is, as is well known*

$$\phi_n(x) = i^n \operatorname{erfc} x. \tag{29}$$

To obtain the second solution, note that eq 28 is found when ix is substituted for x in the differential equation satisfied by the nth degree Hermite's polynomial $H_n(x)$. Therefore the second solution of eq 28, expressed as a real function, is

$$E_n(x) = \frac{1}{i^n}H_n(ix). \tag{30}$$

Some of the functions $E_n(x)$ are shown below:

$$E_0(x) = 1$$

$$E_1(x) = 2x$$

$$E_2(x) = 4x^2 + 2$$

$$E_3(x) = 4x(2x^2 + 3).$$

The properties of $E_n(x)$ are found from the properties of $H_n(x)$. Among them, those that are useful to us are:

$$E_n(x) = e^{-x^2}\frac{d^n e^{x^2}}{dx^n} \tag{31}$$

$$E_{n+1}(x) - 2x\,E_n(x) - 2n\,E_{n-1}(x) = 0 \tag{32}$$

and

* National Bureau of Standards (1964) Handbook of mathematical functions with formulas, graphs, and mathematical tables. NBS Applied Mathematics Series 55, p. 484.

$$\frac{dE_n}{dx} = 2n\, E_{n-1}(x). \tag{33}$$

Use of $E_n(x)$ enables us to rewrite $i^n \operatorname{erfc}(-x)$* to a real form:

$$i^n \operatorname{erfc}(-x) = -(-1)^n\, i^n \operatorname{erfc} x + \frac{2^{-n+1}}{n!} E_n(x). \tag{34}$$

The solution of the inhomogeneous equation 23 satisfying the boundary conditions given in eq 27 takes the following form:

$$\phi_n(x) = A_n\, i^n \operatorname{erfc} x + G_n(x)\, E_n(x) \tag{35}$$

where A_n is a constant and $G_n(x)$ is to be determined. Substituting eq 35 into eq 23 yields

$$\frac{dG_n}{dx} = \frac{1}{E_n^2(x)\, e^{x^2}} \int f_n(x)\, E_n(x)\, e^{x^2}\, dx \tag{36}$$

where the indefinite integral is used on the right-hand side.

To reduce eq 8 to the boundary conditions of ϕ_n, use eq 10, 13, 12 and 14 in eq 8. Comparing the coefficients of h^n, we find

$$\left(\frac{d\phi_0}{dx}\right)_{x=-\beta} = -\frac{2L}{c_i}\beta \tag{37}$$

and

$$\left(\frac{d\phi_{n+1}}{dx}\right)_{x=-\beta} = -\frac{2L}{c_i}\beta\nu_n. \tag{38}$$

The first solution

The solution $\phi_0(x)$ of eq 22 satisfying the boundary conditions specified in eq 26 and 27 is:

$$\phi_0(x) = \theta_\infty + A_0 \operatorname{erfc} x \tag{39}$$

where

$$A_0 = \frac{\theta_F - \theta_\infty}{\operatorname{erfc}(-\beta)}.$$

Note that $\operatorname{erfc}(-\beta)$ is given by letting $n = 0$ in eq 34 as follows:

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* Op cit., Formula 7.2.11.

$$\mathrm{erfc}(-\beta) = 2 - \mathrm{erfc}\beta. \tag{40}$$

Substituting eq 39 into eq 37, we find the equation for determining β:

$$\beta\, e^{\beta^2}(2 - \mathrm{erfc}\beta) = \frac{c_i}{\sqrt{\pi}L}(\theta_F - \theta_\infty). \tag{41}$$

The second solution

Substituting the first solution $\phi_0(x)$ into eq 24, we find:

$$\beta f_1(x) = \frac{2A_0}{\sqrt{\pi}}(1 + 2\beta\nu_0 x)e^{-x^2}. \tag{42}$$

Then use of eq 36 yields $G_1(x)$:

$$G_1(x) = B_1 - \frac{A_0}{\beta\sqrt{\pi}} \int_x^\infty e^{-x^2}\left(\frac{1}{2} + \frac{2\beta\nu_0}{3}x\right)dx \tag{43}$$

which, on integration, becomes

$$G_1(x) = B_1 - \frac{A_0}{\beta\sqrt{\pi}}\left(\frac{1}{2}\mathrm{erfc}x + \frac{2\beta\nu_0}{3\sqrt{\pi}}e^{-x^2}\right). \tag{44}$$

Thus we have the general solution of $\phi_1(x)$:

$$\phi_1(x) = A_1\,\mathrm{erfc}x + B_1 x - \frac{A_0}{\beta\sqrt{\pi}}\left(\frac{1}{2}\mathrm{erfc}x + \frac{2\beta\nu_0}{3\sqrt{\pi}}e^{-x^2}\right). \tag{45}$$

The boundary condition $\phi_1(x = \infty) = 0$ is satisfied by letting

$$B_1 = 0.$$

The boundary condition $\phi_1(x = -\beta) = 0$ yields one of the equations for determining the two remaining unknowns A_1 and ν_0:

$$A_1\,\mathrm{i\,erfc}(-\beta] - \nu_0\frac{2A_0}{3\pi}e^{-\beta^2} = \frac{A_0}{2\beta\sqrt{\pi}}\mathrm{erfc}(-\beta). \tag{46}$$

The second equation for determining A_1 and ν_0 is:

$$A_1\,\mathrm{erfc}(-\beta) + \nu_0\left(\frac{4A_0\beta}{3\pi}e^{-\beta^2} - \frac{2L}{c_i}\beta\right) = \frac{A_0}{\beta\pi}e^{-\beta^2} \tag{47}$$

which is obtained by substituting eq 45 into an equation found by letting $n = 1$ in eq 38.

The third solution

Use of eq 25 yields

$$f_2(x) = g_2(x) + \nu_1 \frac{4A_0}{\sqrt{\pi}} x\, e^{-x^2} \tag{48}$$

where

$$\beta g_2(x) = A_1\left(\operatorname{erfc} x + \frac{2\beta\nu_0}{\sqrt{\pi}} e^{-x^2}\right) - \frac{A_0\nu_0}{\sqrt{\pi}} \operatorname{erfc} x -$$

$$- \frac{2A_0}{\sqrt{\pi}} e^{-x^2}\left[\frac{3+4\beta^2\nu_0^2}{6\beta\sqrt{\pi}} + \left(\frac{1}{\beta} + \frac{5\nu_0}{3\sqrt{\pi}}\right)x + \frac{4\beta\nu_0^2}{3\sqrt{\pi}} x^2\right]. \tag{49}$$

Note that $g_2(x)$ does not contain the unknown constant ν_1. Substituting eq 48 into eq 36 yields

$$\frac{dG_2}{dx} = \frac{1}{\beta}\left(A_1 - \frac{A_0\nu_0}{\sqrt{\pi}}\right)\frac{2x}{E_2^2(x)} \operatorname{erfc} x +$$

$$+ \frac{1}{(2x^2+1)^2} e^{-x^2} \sum_{n=1}^{5} p_n x^n + \frac{\nu_1 A_0}{\sqrt{\pi}} \frac{x^4 + x^2}{(2x^2+1)^2} e^{-x^2} \tag{50}$$

where

$$p_1 = \frac{\nu_0}{\sqrt{\pi}} A_1 - \frac{3+4\beta^2\nu_0^2}{6\beta^2\pi} A_0$$

$$p_2 = \frac{1}{2\beta\sqrt{\pi}}\left[-A_0\left(\frac{1}{\beta} + \frac{8\nu_0}{3\sqrt{\pi}}\right) + A_1\right]$$

$$p_3 = \frac{1}{3}\left(\frac{2\nu_0}{\sqrt{\pi}} A_1 - \frac{3+8\beta^2\nu_0^2}{3\beta^2\pi} A_0\right)$$

$$p_4 = -\frac{A_0}{2\beta\sqrt{\pi}}\left(\frac{1}{\beta} + \frac{5\nu_0}{3\sqrt{\pi}}\right)$$

$$p_5 = -\frac{8\nu_0^2}{15\pi} A_0 .$$

Because the right-hand side of eq 50 cannot be integrated to a simple function, the third solution $\phi_2(x)$ is very complicated. To integrate eq 50 we define functions $H_n(x)$ and $K_n(x)$:

$$H_n(x) = \int_x^\infty \frac{2\lambda}{(2\lambda^2 + 1)^n} e^{-\lambda^2} d\lambda \tag{51}$$

and

$$K_n(x) = \int_x^\infty \frac{1}{(2\lambda^2 + 1)^n} e^{-\lambda^2} d\lambda. \tag{52}$$

Then eq 50 is integrated to:

$$G_2(x) = I_2(x) + \frac{\nu_1 A_0}{\sqrt{\pi}} J_2(x) \tag{53}$$

where $J_2(x)$ and $I_2(x)$ are given by

$$J_2(x) = -\frac{\sqrt{\pi}}{8} \operatorname{erfc} x + \frac{1}{4} K_2(x)$$

and

$$I_2(x) = -\frac{1}{4\beta}\left(A_1 - \frac{A_0\nu_0}{\sqrt{\pi}}\right)\left[\frac{1}{E_2(x)} \operatorname{erfc} x - \frac{1}{\sqrt{\pi}} K_1(x)\right]$$
$$- q_0 K_2(x) - q_1 H_2(x) - q_2 K_1(x) - q_3 H_1(x) - q_4 \frac{\sqrt{\pi}}{2} \operatorname{erfc} x - q_5 e^{-x^2}$$

in which q_n are:

$$q_0 = \frac{A_0}{8\beta\sqrt{\pi}}\left(\frac{1}{\beta} + \frac{11\nu_0}{2\sqrt{\pi}}\right) - \frac{A_1}{4\beta\sqrt{\pi}}$$

$$q_1 = \frac{\nu_0 A_1}{3\sqrt{\pi}} - \frac{A_0}{6\beta^2\pi}\left(1 + \frac{8}{5}\beta^2\nu_0^2\right)$$

$$q_2 = \frac{1}{4\beta\sqrt{\pi}}\left(A_1 - \frac{A_0\nu_0}{\sqrt{\pi}}\right)$$

$$q_3 = \frac{\nu_0 A_1}{6\sqrt{\pi}} - \frac{A_0}{36\pi\beta^2}\left(3 + \frac{16}{5}\beta^2\nu_0^2\right)$$

$$q_4 = -\frac{A_0}{8\beta\sqrt{\pi}}\left(\frac{1}{\beta} + \frac{5\nu_0}{3\sqrt{\pi}}\right)$$

$$q_5 = -\frac{A_0\nu_0^2}{15\pi}.$$

Thus the general solution of $\phi_2(x)$ is found as follows:

$$\phi_2(x) = A_2\, i^2\mathrm{erfc}x + (G_2(x) + B_2)E_2(x). \tag{54}$$

The boundary condition $\phi_2(x = -\beta) = 0$ yields

$$B_2 = 0.$$

The boundary condition $\phi_2(x = \infty) = 0$ yields one of the two equations for determining two unknown constants A_2 and ν_1:

$$A_2\, i^2\mathrm{erfc}(-\beta) + \nu_1 \frac{A_0}{\sqrt{\pi}} J_2(-\beta)E_2(-\beta) = -I_2(-\beta)E_2(-\beta). \tag{55}$$

The second equation for determining A_2 and ν_1 is obtained by substituting eq 54 into an equation found by putting $n = 2$ in eq 38; thus we get

$$A_2\, \mathrm{ierfc}(-\beta) - \nu_1\left(Q_2 + \frac{2L\beta}{c_i}\right) = P_2 \tag{56}$$

where

$$Q_2 = \frac{2A_0}{\sqrt{\pi}} \frac{\beta^2 + \beta^4}{2\beta^2 + 1} e^{-\beta^2} - \frac{2A_0}{\sqrt{\pi}} \beta J_2(-\beta)$$

$$P_2 = \frac{2}{2\beta^2 + 1}\left[e^{-\beta^2}(-p_1\beta + p_1\beta^2 - p_3\beta^3 + p_4\beta^4 - p_5\beta^5)\right.$$

$$\left. - \frac{1}{2}\left(A_1 - \frac{A_0\nu_0}{\sqrt{\pi}}\right)\mathrm{erfc}(-\beta)\right].$$

No solution of higher order than three has been computed.

Determination of $h(t)$

Transform eq 14 to

$$h\,dh = 2\beta^2(1 + \nu_0 h + \nu_1 h^2)d\tau. \tag{57}$$

The first approximation of h is found by taking only the first term in the right-hand side to be

$$h = 2\beta\tau^{1/2}. \tag{58}$$

Substituting eq 58 into the second term of eq 57 and neglecting the third term, the second approximation is found to be

$$h = 2\beta\tau^{1/2} + \frac{4}{3}\nu_0\beta^2\tau. \tag{59}$$

Substituting eq 59 into the right-hand side of eq 57, the third approximation is found to be

$$h = 2\beta\tau^{1/2} + \frac{4}{3}\nu_0\beta^2\tau + 2\left(\nu_1 - \frac{1}{3}\nu_0^2\right)\beta^3\tau^{3/2}. \tag{60}$$

Numerical computation

The following values are chosen as the input parameters: $\alpha = 0.0114$ cm²/sec, $C = 0.5$ cal/g, $L = 80$ cal/g, $\theta_\infty = -15°$C and $\theta_f = -1.8°$C. It is believed that the freezing point of the sea water in the Antarctic Ocean is adequately represented by $\theta_f = -1.8°$C.

The following relations simplify the program:

$$p_1 = 2(3q_3 + q_5)$$

$$p_2 = 2(q_2 + 2q_4)$$

$$p_3 = 4(q_3 + 25q)$$

$$p_4 = 4q_4$$

$$p_5 = 8q_5$$

$$q_0 = -(q_2 + q_4)$$

$$q_1 = \frac{1}{3}(6q_3 + 4q_5).$$

Thus only q_2 through q_5 are independent.

Temperature was computed at $r = 1, 2, 3, 5, 7$ as functions of τ in consideration of the convenience of actual measurement. The result is shown in Figure 1. With given values eq 60 becomes

$$h = 0.08850\tau^{1/2} + 0.01529\tau - 0.01249\tau^{3/2} \tag{61}$$

which is plotted in Figure 2. In eq 61, h reaches maximum in the neighborhood of $\tau = 4$. Therefore the solution given in this paper by taking the first three terms of the series expressing temperature is not valid for $\tau > 4$. Letting $\tau = 4$ in eq 3 we get the uppet limit of the time of applicability in terms of R_0:

$$R_0^2(\text{cm}) > 0.00285t\ (\text{sec}) = 0.0171t\ (\text{min}) = 10.26t\ (\text{hr})$$

where the units in parentheses show the units of corresponding quantities.

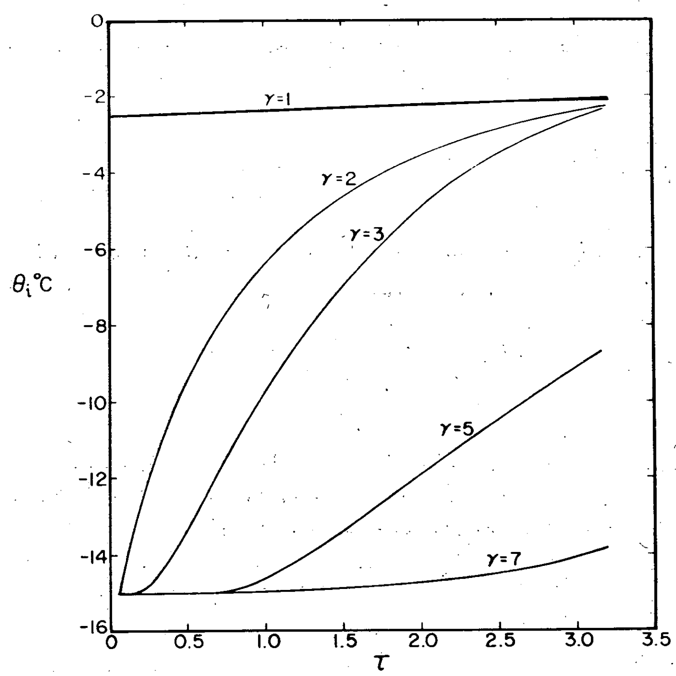

Figure 1. Temperature of ice ($\theta_2^\circ C$) versus nondimensional time (τ).

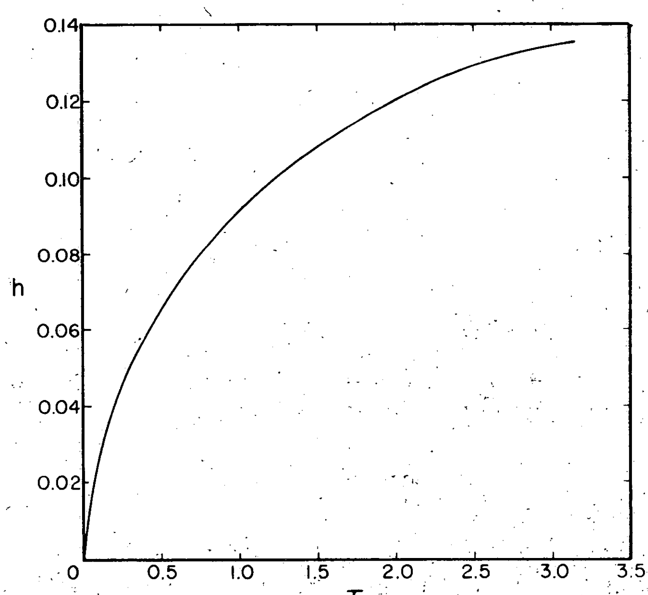

Figure 2. Nondimensional thickness (h) of new ice versus nondimensional time (τ).

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13. ABSTRACT

Herewith presented is the rigorous solution of the freezeback of water in a cylindrical borehole drilled in an ice sheet floating on water, based on the assumption that the temperature distribution does not depend on the vertical direction and the temperature of the water in the borehole is the freezing temperature. The solution is found by using the thickness of the newborn ice in place of time. Because of the complexity of the analysis, the solution can be found only for the first few terms of the series solution. Numerical computation of the solution thus found by use of the first few terms of the series solution yields the growth curve of the newborn ice that reaches maximum at a certain time. The solution ceases to be valid before the time of maximum is reached.

14. KEY WORDS

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