AD

Research Note ETL-RN-74-9

PARALLEL OPTICAL PROCESSING TO CONVERT ELEVATION DATA TO SLOPE MAPS PHASE I, THEORETICAL ANALYSIS

October 1974

Approved for public release; distribution unlimited.

U.S. ARMY ENGINEER TOPOGRAPHIC LABORATORIES
FORT BELVOIR, VIRGINIA

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ETL-RN-74-9	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) PARALLEL OPTICAL PROCESSING TO CONVERT ELEVATION DATA TO SLOPE MAPS PHASE I, THEORETICAL ANALYSIS		5. TYPE OF REPORT & PERIOD COVERED Research Note July 72 – October 74
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Robert K. Brooke, Jr.		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS U. S. Army Engineer Topographic Laboratories Fort Belvoir, Virginia 22060		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 4A062112A854
11. CONTROLLING OFFICE NAME AND ADDRESS Technology Development Branch, Geographic Information Systems Division, U. S. Army Engineer Topographic Laboratories, Fort Belvoir, Virginia 22060		12. REPORT DATE October 1974
		13. NUMBER OF PAGES 23
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Coherent Optical Processing
Optical Data Processing
Optical Differentiation
Slope Mapping

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Given a photographic transparency with an amplitude transmittance function, $t(x_o,y_o)$, that is equivalent to a functional representation of elevation information, $h(x_o,y_o)$, for a particular geographic area, the slope information can be rapidly displayed using a coherent optical system and a one-dimensional differentiating filter. Implementation of this technique could provide an electro-optical system for the high speed conversion of elevation data to a

(continued)

Block 20 (continued):

slope map. It is concluded that slope map displays can be obtained from elevation function transparencies, though negative and positive slopes of the same magnitude are displayed identically. This is not expected to affect the utility of the output.

PREFACE

The analysis described in this report was authorized under Project 4A062112A854 and was conducted under the supervision of Mr. Bernard B. Scheps, Chief, Technology Development Branch and Dr. Kenneth R. Kothe, Director, Geographic Sciences Laboratory. The author wishes to acknowledge Mr. Scheps as the source of the concepts contained herein, and to thank Messrs. R. A. Hevenor and R. J. Orsinger for their assistance in validating the results. Mr. Hevenor recognized the mathematical relationship between slopes and elevations through their spectra, which triggered the initial concepts upon which this work is based.

This Research Note reports on Phase I of a 3-phase investigation of parallel optical processing to convert elevation data to slope maps.

CONTENTS

ILLUSTRATIONS

PARALLEL OPTICAL PROCESSING TO CONVERT ELEVATION DATA TO SLOPE MAPS PHASE I, THEORETICAL ANALYSIS

I. INTRODUCTION

The high-speed capabilities of coherent optical systems in processing two-dimensional data are well known,[1] however, the number of practical implementations of parallel optical processing is quite small. The following analysis is aimed at providing a basis for the possible operational use of coherent optical techniques to produce slope map displays from elevation data generated by the Universal Automatic Map Compilation Equipment (UNAMACE). Given a UNAMACE tape output re-encoded in the form of a photographic transparency with an amplitude transmittance, $t(x_o, y_o)$, that is equivalent to a functional representation of the elevation information, $h(x_o, y_o)$, for a particular geographic area, the slope information can be rapidly displayed using a coherent optical system and a one-dimensional differentiating filter. The required encoding of such an input from a UNAMACE tape transparency has already been accomplished in these Laboratories. The following analysis develops the theoretical aspects of optical slope extraction from UNAMACE output.

II. BACKGROUND

Based on Hevenor's recognition of the relationship between the Fourier transforms of elevations and slopes[2] and the Fourier transforming properties of coherent optical systems,[3] a theoretical analysis of an optical technique for extracting slope information from elevation data was undertaken. This background discussion describes the fundamental mathematical properties of the optical elements required for the optical slope extraction technique.

If an electromagnetic field described by $E(x_o, y_o)$ exists at an aperture in the (x_o, y_o) plane, as shown in Figure 1, the field $E(x_1, y_1)$ in the (x_1, y_1) plane is given, in scalar form, by the Raleigh-Sommerfield formula as follows:[4]

$$E(x_1, y_1) = \frac{1}{j\lambda} \iint_S \frac{E(x_o, y_o)}{r} e^{jkr} \cos(\underline{n}, \underline{r}) \, dx_o dy_o \quad (1)$$

[1] A. R. Shulman, *Optical Data Processing,* John Wiley & Sons, Inc., New York, 1970.

[2] R. A. Hevenor, *A Relation Between the Spectrum of Surface Slopes and the Spectrum of the Surface Elevations and Its Usefulness in the Theory of Electromagnetic Wave Scattering from Rough Surfaces.* U.S. Army Engineer Topographic Labs., ETL-RN-70-2, July 1970.

[3] J. W. Goodman, *Introduction to Fourier Optics,* Chapter 3, McGraw-Hill Book Co., New York, 1968.

[4] J. W. Goodman, *Introduction to Fourier Optics,* Chapter 3, McGraw-Hill Book Co., New York, 1968.

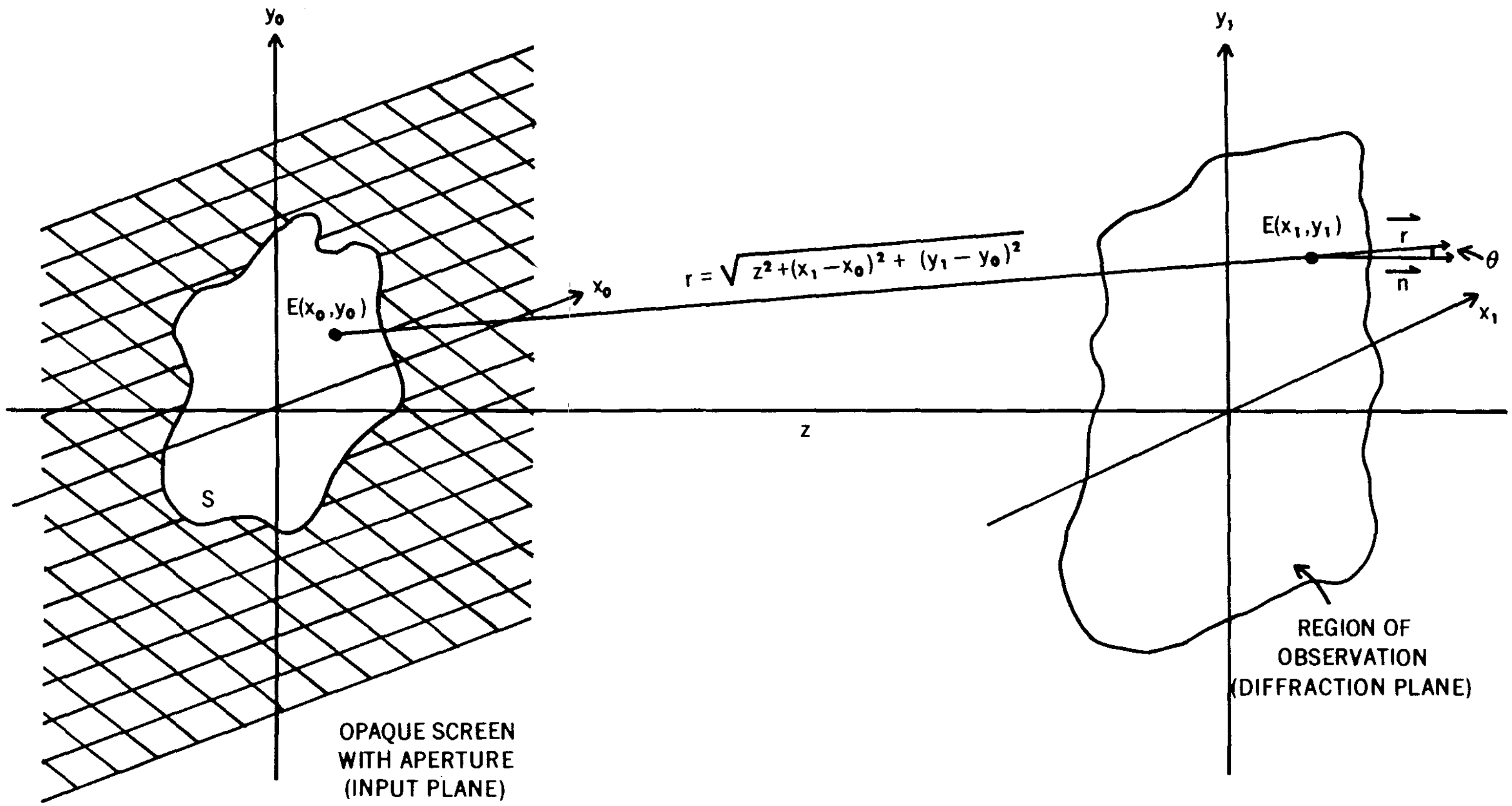

Figure 1. Diffraction at an Aperture.

where $k = 2\pi/\lambda$ and $\cos(\underline{n}, \underline{r})$ is called the obliquity factor representing the cosine of the angle between the unit vector normal to the (x_1, y_1) plane and the unit vector $\underline{r}$. The letter S represents the area of the aperture in the (x_o, y_o) plane, and in accordance with the Kirchoff boundary conditions, the value of $E(x_o, y_o)$ is identically zero outside the aperture; r is the distance from the point (x_o, y_o) to point (x_1, y_1). Since it will be assumed that the distance z, separating the (x_o, y_o) and the (x_1, y_1) planes, is large compared to the wavelength λ of the propagating field, the Kirchoff conditions will yield satisfactory results. By assuming that the distance z, separating the (x_o, y_o) and (x_1, y_1) planes, is large compared to the dimensions of both the aperture in the (x_o, y_o) plane and the region of observation in the (x_1, y_1) plane, a number of simplifying approximations can be made for equation (1).

1. Since the angle $(\underline{n}, \underline{r})$ in the obliquity factor is given by

$$(\underline{n}, \underline{r}) = \tan^{-1} \frac{[(x_1 - x_o)^2 + (y_1 - y_o)^2]^{1/2}}{z}$$

$\cos(\underline{n}, \underline{r})$ is approximately equal to one for $(\underline{n}, \underline{r}) \geqslant 15°$ or

$$z > 4\,[(x_1 - x_o)^2 + (y_1 - y_o)^2]^{1/2}$$

2. The amplitude factor, $1/r$, can be approximated by $1/z$ without significant error. It should be noted that this approximation is not valid in the exponential term where a relatively small difference in r and z may produce a large phase variation.

3. An approximation for r in the exponent may be made in the following manner:

$$r = \sqrt{z^2 + (x_1 - x_o)^2 + (y_1 - y_o)^2}$$

This can be rewritten as:

$$r = z\sqrt{1 + \left[\frac{x_1 - x_o}{z}\right]^2 + \left[\frac{y_1 - y_o}{z}\right]^2}$$

From the binomial expansion

$$(1 + u)^{1/2} = 1 + \frac{u}{2} - \frac{u^2}{8} +, \ldots\ldots, u < 1$$

Using the first two terms of the expansion we write r as

$$r = z\left[1 + \frac{(x_1 - x_o)^2}{2z^2} + \frac{(y_1 - y_o)^2}{2z^2}\right]$$

Incorporating the above approximations, equation (1) now becomes

$$E(x_1, y_1) = \frac{1}{j\lambda z}\iint_S E(x_o, y_o)\exp\left\{jkz\left[1 + \frac{(x_1 - x_o)^2}{2z^2} + \frac{(y_1 - y_o)^2}{2z^2}\right]\right\} dx_o\, dy_o \quad (2)$$

And since $E(x_o, y_o)$ is identically zero outside the aperture S,we can integrate over the limits $-\infty$ to $+\infty$ without changing the value of the integral.

Thus equation (2) becomes

$$E(x_1, y_1) = \frac{1}{j\lambda z}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E(x_o, y_o)\exp\left\{jkz\left[1 + \frac{(x_1 - x_o)^2}{2z^2} + \frac{(y_1 - y_o)^2}{2z^2}\right]\right\} dx_o\, dy_o \quad (3)$$

Expanding the exponent in equation (3) yields,

$$E(x_1, y_1) = \frac{1}{j\lambda z}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E(x_o, y_o)\, e^{jkz} \cdot$$

$$\exp\left\{\frac{jk}{2z}\left[(x_1^2 + y_1^2) - 2(x_1 x_o + y_1 y_o) + (x_o^2 + y_o^2)\right]\right\} dx_o\, dy_o \quad (4)$$

Since the exponential terms not involving x_o and y_o are unaffected by the integration, they can be taken outside the integral to give

$$E(x_1, y_1) = \frac{e^{jkz}}{j\lambda z}\exp\left\{\frac{jk}{2z}(x_1^2 + y_1^2)\right\}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E(x_o, y_o) \cdot$$

$$\exp\left\{\frac{jk}{2z}(x_o^2 + y_o^2)\right\}\exp\left\{-\frac{jk}{z}(x_1 x_o + y_1 y_o)\right\} dx_o\, dy_o \quad (5)$$

Equation (5) represents the Fresnel or near-field formulation of the diffraction problem, and the resulting Fresnel integrals are, in general, quite difficult to solve.

A simpler formulation of the diffraction problem arises in the Fraunhofer or far-field case. This situation occurs if we make the further assumption that

$z \gg k(x_o{}^2 + y_o{}^2)/2$ (Fraunhofer Condition) for (x_o, y_o) on S. For the Fraunhofer case, equation (5) becomes

$$E(x_1 y_1) = \frac{e^{jkZ}}{j\lambda z} \exp\left\{\frac{jk}{2z}(x_1^2 + y_1^2)\right\} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E(x_o, y_o) \exp\left\{-\frac{2\pi j}{\lambda z}(x_1 x_o + y_1 y_o)\right\} dx_o\, dy_o \quad (6)$$

Examining equation (6) we see that the field at a point $(x_1 y_1)$ in the (x_1, y_1) plane is equal to the Fourier Transform of the field at the aperture, S, multiplied by an amplitude and phase factor independent of the field at S. We can interpret the quantities $2\pi x_1/\lambda z$ and $2\pi y_1/\lambda z$ as angular spatial frequencies ω_x and ω_y respectively. (The spatial frequencies themselves are $f_x = x_1/\lambda z$ and $f_y = y_1/\lambda z_1$.)

The assumption, $z \gg k(x_o{}^2 + y_o{}^2)/2$ at optical wavelengths, requires that z be very large. For $\lambda = 500$ nanometers and a maximum aperture radius of 1 cm, the required value of z is $z \gg \pi / 5 \times 10^{-7} (10^{-4} + 10^{-4}) \simeq 1.2 \times 10^3$ meters. Distances of this magnitude are impractical for most purposes. However, there is an alternative practical approach to obtaining Fraunhofer diffraction. This is achieved by passing the field, $E(x_o, y_o)$, through an equiconvex thin lens. The effect of an equiconvex thin lens in transmitting a plane wave is to convert the plane wave to a converging spherical wave, i.e., the lens can be considered as a quadratic phase adjuster. We can write the transmission function of an equiconvex lens as a phase adjustment of the form,[6] (see Appendix A, Phase Effect on a Thin Lens),

$$T(x_\ell, y_\ell) = e^{jknt} \exp\left\{-\frac{jk}{\lambda f}(x_\ell^2 + y_\ell^2)\right\} \quad (7)$$

where x_ℓ and y_ℓ are the coordinates of a point on the lens, t is the thickness of the lens, and n is its index of refraction. It is noted that this expression is valid only in that region of the lens where the paraxial approximation holds, or for small x_ℓ and y_ℓ. If we now write the diffraction formula for an afocal system employing an equiconvex thin lens having focal length f, we have

$$E(x_1, y_1) = \frac{1}{2j\lambda f} e^{jk(nt + 2f)} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E(x_o, y_o) \exp\left\{-\frac{2\pi j}{\lambda f}(x_1 x_o + y_1 y_o)\right\} dx_o\, dy_o \quad (8)$$

where (x_1, y_1) is a point in the back focal plane of the lens and (x_o, y_o) a point in the front focal plane (see Appendix B, Effect of a Thin Lens on the Diffraction Formula). Thus, from equation (8) we see that the field at the back focal plane of an equiconvex thin lens is equal to a constant amplitude and phase factor times the Fourier Transform of the field at the front focal plane.

---

[5] J. W. Goodman, *Introducton To Fourier Optics*, Chapter 3, McGraw-Hill Book Co., New York, 1968.

III. SLOPE EXTRACTION ANALYSIS

Based on the fact that an equiconvex thin lens in an afocal arrangement displays the Fourier Transform of an input field at its back focal plane, we now develop an optical technique for extracting the slope information from an input transparency $t(x_o, y_o)$ representing elevation as a function of transmission, i.e. $t(x_o, y_o) = h(x_o, y_o)$.

Mathematically, it can be shown[6] that the spectrum of the surface slopes in the x-direction, $T_x(\omega_x, \omega_y)$, is related to the spectrum of the surface elevations as follows:

$$T_x(\omega_x, \omega_y) = j\omega_x T(\omega_x, \omega_y)$$

Where $T(\omega_x, \omega_y)$ is the Fourier Transform of the surface elevation, $t(x_o, y_o)$, the two dimensional Fourier Transform being defined as

$$F(\omega_x, \omega_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{-j(\omega_x x + \omega_y y)}\, dxdy$$

Similarly,

$$T_y(\omega_x, \omega_y) = j\omega_y T(\omega_x, \omega_y)$$

where $T_y(\omega_x, \omega_y)$ is the spectrum of the surface slopes in the y-direction. The inverse Fourier Transform defined as

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\omega_x, \omega_y)\, e^{j(\omega_x x + \omega_y y)}\, d\omega_x d\omega_y$$

of $T_x(\omega_x, \omega_y)$ and $T_y(\omega_x, \omega_y)$ will then give the surface slopes $t'_x(x_o, y_o)$ and $t'_y(x_o, y_o)$, in the x and y directions respectively, where

$$t'_x(x_o, y_o) = \frac{\partial t(x_o, y_o)}{\partial x_o} \quad \text{and} \quad t'_y(x_o, y_o) = \frac{\partial t(x_o, y_o)}{\partial y_o}$$

The above mathematical operations for obtaining slope information from elevation data can be implemented optically in the following manner employing the optical system shown in Figure 2, where lenses L_1 and L_2 are equiconvex

---

[6] R.A. Hevenor, *A Relation Between the Spectrum of the Surface Slopes and the Spectrum of the Surface Elevations and its Usefulness in the Theory of Electromagnetic Wave Scattering from Rough Surfaces,* U.S. Army Engineer Topographic Labs., ETL-RN-70-2, July 1970.

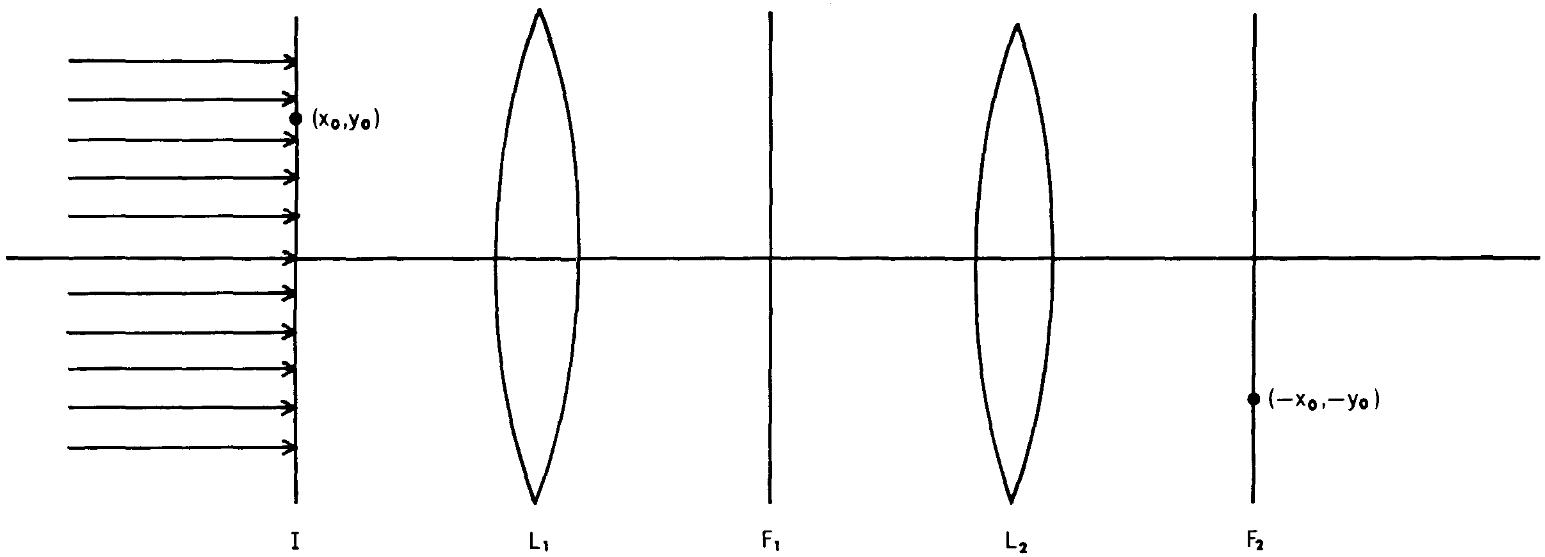

Figure 2. Coherent Optical Processing System.

thin lenses with focal length f. Planes I, L_1, F, L_2, and F_2 are each separated by a distance f, i.e., the system is afocal. The first operation requires taking the Fourier Transform of the surface elevations, $t(x_o, y_o)$. This is accomplished by placing a transparency with transmission function $t(x_o, y_o)$ at plane I of Figure 2 with a coherent plane wave incident from the left. Then, from Equation (8), the field at the F_1 plane is given by;

$$T(x_1,y_1) = \frac{1}{2j\lambda f} e^{jk(nt+2f)} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} t(x_o,y_o)\, e^{-\frac{2\pi j}{\lambda f}(x_1 x_o + y_1 y_o)}\, dx_o dy_o \qquad (9)$$

or

$$T(x_1,y_1) = \frac{1}{2j\lambda f} e^{jk(nt+2f)} T(\omega_x,\omega_y)$$

where $T(\omega_x,\omega_y)$ is the Fourier Transform of $t(x_o,y_o)$

The next step is to multiply $T(\omega_x,\omega_y)$ by either $j\omega_x$ or $j\omega_y$, depending on the slope direction desired. For slopes in the x-direction, we want $j\omega_x T(\omega_x, \omega_y)$. This quantity is obtained by inserting a filter at the F_1 plane with a transmission function of $j\omega_x$ or $j2\pi x_1/\lambda f$. Such a filter can be physically approximated[7] by a transmission function of the form

$$F(\omega_x) = |\omega_x|\, j \,\mathrm{sgn}\, \omega_x$$

with a thin opaque strip in the ω_y direction at $\omega_x = 0$ where

$$\mathrm{sgn}\,\omega_x = \begin{cases} -1, & \omega_x < 0 \\ 0, & \omega_x = 0 \\ +1, & \omega_x > 0 \end{cases}$$

This filter is a combination of an amplitude transmission of $|\omega_x|$, and a half-wave phase plate that is coated over the $-\omega_x$ region of the F_1 plane. It is positioned to produce a phase shift of $+\pi/2$ in the $+\omega_x$ region and $-\pi/2$ in the $-\omega_x$ region. After transmission by the filter, the field immediately to the right of F_1 is given by

$$j\omega_x T(x_1,y_1) = \frac{1}{2j\lambda f} e^{jk(nt+2f)}\, j\omega_x T(\omega_x,\omega_y)$$

---

[7] R. L. Cody, *A Comparison of Various Coherent Optical Filtering Operations*, Arnold Engineering Development Center AEDC-TR-71-137, June 1971, AD 726 091.

It is now necessary to take the inverse Fourier Transform to obtain the desired slope information. This is achieved by again using an equiconvex thin lens L_2 and observing (or recording) the field at F_2 given by

$$F_2 = \frac{1}{2j\lambda f} e^{jk\,(nt + 2f)} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} j\omega_x T(x_1, y_1)\, e^{-\frac{2\pi j}{\lambda f}(x_1 x_2 + y_1 y_2)}\, dx_1 dy_1$$

which can be rewritten as

$$F_2 = \frac{1}{(2j\lambda f)^2} e^{2jk\,(nt + 2f)} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} j\omega_x T(\omega_x, \omega_y)\, e^{-\frac{2\pi j}{\lambda f}(x_1 x_2 + y_1 y_2)}\, dx_1 dy_1$$

or

$$F_2 = \frac{1}{(2j\lambda f)^2} e^{2jk\,(nt + 2f)} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} j\omega_x \left(\frac{\lambda f}{2\pi}\right)^2 T(\omega_x, \omega_y)\, e^{-\frac{2\pi j}{\lambda f}(x_1 x_2 + y_1 y_2)}\, d\omega_x d\omega_y$$

where the substitutions, $\omega_x = 2\pi x_1/\lambda f$ and $\omega_y = 2\pi y_1/\lambda f$, have been introduced. F_2 can now be written as

$$F_2 = \frac{1}{4} e^{2jk\,(nt + 2f)} \left[\frac{1}{4\pi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} j\omega_x T(\omega_x, \omega_y)\, e^{-j(\omega_x x_2 + \omega_y y_2)}\, d\omega_x d\omega_y\right]$$

and from our definition of the inverse Fourier Transform

$$F_2 = -\frac{1}{4} e^{2jk\,(nt + 2f)}\, t'_x(-x_2, -y_2). \qquad (10)$$

For the system shown in Figure 2, the point $(-x_2, -y_2)$ is equal to the point $(-x_o, -y_o)$. Thus, the field at F_2 represents the slopes in the x-direction of the elevation function, $t(x_o, y_o)$ at $(-x_o, -y_o)$, in the F_2 plane, multiplied by a constant amplitude and phase factor.

The inversion of coordinates is a result of taking forward Fourier Transforms with lenses L_1 and L_2, instead of a forward and inverse transform as is mathematically required to give $t'_x(x_o, y_o)$. It is a simple matter to invert the function $t'_x(-x_o, -y_o)$ to produce the desired result. The function $t'_y(-x_o, -y_o)$ is displayed by rotating the filter 90°. In fact, the slope in any direction can be obtained by proper orientation of the filter. It should be noted that the filter used in this procedure is an approximation of the "$j\omega$" differential filter and the fact that it, as well as the optical components, have finite bandwidth, has been ignored.

One additional factor is also notable. Since the output is viewed or recorded as an intensity by a square law detector, what is actually observed is $1/16\,(|t'_x\,(x_o, y_o)|^2)$ and $1/16\,(|t'_y\,(x_o, y_o)|^2)$. Thus, negative and positive slopes of the same magnitude are displayed identically. Slope maps generally do not show the sign of the slope direction. Thus, no loss of utility results from this fact.

IV. CONCLUSIONS

Based on the results presented here, the following conclusions are presented:

1. Slope map displays can be obtained from elevation function transparencies using a coherent optical system.

2. Negative and positive slopes of the same magnitude are displayed identically. This is not expected to impose any burden on utility of the output.

APPENDIX A. PHASE EFFECT OF A THIN LENS

In calculating the phase effects introduced by passing a plane wave through a lens, the following assumptions are made:

1. The paraxial approximation is valid, i.e., $R^2 \gg x_\ell^2 + y_\ell^2$ where R is the radius of curvature of the lens surface, and x_ℓ, y_ℓ are coordinates in the plane of the lens.

2. The thickness of the lens produces negligible lateral displacement of the transmitted rays.

3. The standard sign conventions[8] of geometric optics are to be followed.

To determine the required phase adjustment introduced by the lens, we must calculate the optical path from a point (x_ℓ, y_ℓ) of plane A of Figure A1 through the lens to point (x_ℓ, y_ℓ) of plane D. This will be done in two steps.

First we calculated the optical path AB from A to B at (x_ℓ, y_ℓ)

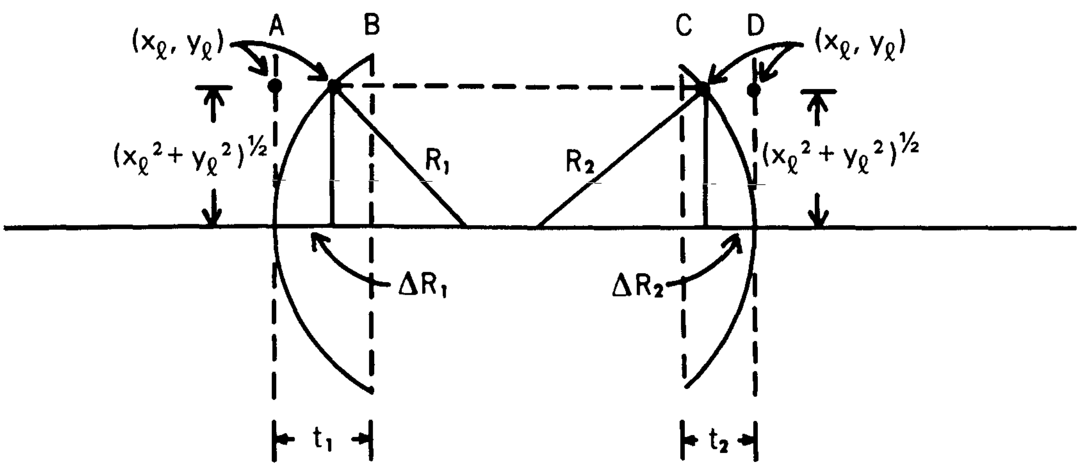

Figure A1. Thin Lens Representation.

[8] F. A. Jenkins, and H. E. White, *Fundamentals of Optics*, Third Edition, p. 33, McGraw-Hill Book Co., New York, 1957.

$$AB = n(t_1 - \Delta R_1) + \Delta R_1 \qquad \text{(A1)}$$

where n is the index of refraction of the lens, t_1 is the lens thickness at (x_ℓ, y_ℓ), and the index of refraction of air is equal to one. We can write ΔR_1 as

$$\Delta R_1 = R_1 - \sqrt{R_1^2 - (x_\ell^2 + y_\ell^2)}$$

Substituting this expression into the equation (A1), we get

$$AB = n\left[t_1 - R_1 + \sqrt{R_1^2 - (x_\ell^2 + y_\ell^2)}\right] + \left[R_1 - \sqrt{R_1^2 - (x_\ell^2 + y_\ell^2)}\right]$$

or

$$AB = nt_1 - (n-1)\left[R_1 - \sqrt{R_1^2 - (x_\ell^2 + y_\ell^2)}\right] \qquad \text{(A2)}$$

Similarly,

$$CD = n\left[t_2 + R_2 + \sqrt{R_2^2 - (x_\ell^2 + y_\ell^2)}\right] + \left[-R_2 - \sqrt{R_2^2 - (x_\ell^2 + y_\ell^2)}\right]$$

where $(-R_2)$ is used to give a positive path length, since R_2 itself is negative by convention. CD can be rewritten as

$$CD = nt_2 - (n-1)\left[-R_2 - \sqrt{R_2^2 - (x_\ell^2 + y_\ell^2)}\right] \qquad \text{(A3)}$$

Summing equations (A2) and (A3), we get

$$AD = nt - (n-1)\left[\left\{-R_2 - \sqrt{R_2^2 - (x_\ell^2 + y_\ell^2)}\right\} + \left\{R_1 - \sqrt{R_1^2 - (x_\ell^2 + y_\ell^2)}\right\}\right]$$

where $t = t_1 + t_2$

Since we have assumed $R^2 \gg x_\ell^2 + y_\ell^2$, we can use the binomial theorem to expand the bracketed expression. Taking the first two terms of the expansion gives

$$AD = nt - (n-1)\left[-R_2 + R_2 - \frac{x_\ell^2 + y_\ell^2}{2R_2} + R_1 - R_1 + \frac{x_\ell^2 + y_\ell^2}{2R_1}\right] \qquad \text{(A4)}$$

where the positive number $(-R_2)$ has been factored out of the square root prior to expansion. We can rewrite equation (A4) to give the optical path

$$AD = nt - (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]\frac{(x_\ell^2 + y_\ell^2)}{2}$$

But for a paraxial system we define the focal length, f, of a lens as

$$\frac{1}{f} \stackrel{\Delta}{=} (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

Thus,

$$AD = nt - \frac{x_\ell^2 + y_\ell^2}{2f} \tag{A5}$$

and the phase delay due to the lens is given by

$$e^{jknt}\, e^{-jk/2f\left[x_\ell^2 + y_\ell^2\right]} \tag{A6}$$

APPENDIX B. EFFECT OF A THIN LENS ON THE DIFFRACTION FORMULA

Referring to Figures B1 and B2, we will now show that the effect of a thin lens on an input E (x_o, y_o) is to produce the Fourier Transform of E (x_o, y_o) at the back focal plane of the lens when the input is positioned in the front focal plane. We assume that the paraxial approximation is valid and that the displacement of the rays passing through the lens is negligible, i.e., figure B1 may be replaced by figure B2.

Since point F in the (x_1, y_1) plane is in the focal plane of the lens, all incoming rays passing through this point are parallel to the left of the lens as illustrated in Figure B2. Ray AF is the principal or undeviated ray, and ray CF passes through the front focal point. From Figure B2 we can write the optical path, BF, as

$$BF = r + \left[nt - \frac{1}{2f}(x_\ell^2 + y_\ell^2) \right] + Q \qquad \text{(B1)}$$

where B is any point in the (x_o, y_o) plane located a distance, d, in front of the lens and F is a point (x_1, y_1) at the back focal plane. Rewriting equation (B1) in terms of coordinates in the $(x_o y_o)$ and (x_1, y_1) plane, we get

$$BF = r + \left\{ nt - \frac{1}{2f}\left[(x_o - \underline{x})^2 + (y_o - \underline{y})^2 \right] \right\} + \left[f^2 + (x_1 - x_o + \underline{x})^2 + (y_1 - y_o + \underline{y})^2 \right]^{1/2} \qquad \text{(B2)}$$

where r is the optical path to the left of the lens,

$$nt - \frac{1}{2f}\left[(x_o - \underline{x})^2 + (y_o - \underline{y})^2 \right]$$

is the optical path due to the lens (from Appendix A),

$$\left[f^2 + (x_1 - x_o + \underline{x})^2 + (y_1 - y_o + \underline{y})^2 \right]^{1/2}$$

is the optical path to the right of the lens, and $(\underline{x}, \underline{y})$ are the coordinates of the intersection of the principal ray with the (x_o, y_o) plane.

By similar triangles

$$\frac{r}{d} = \frac{s}{f} \text{ or } r = d\frac{s}{f}$$

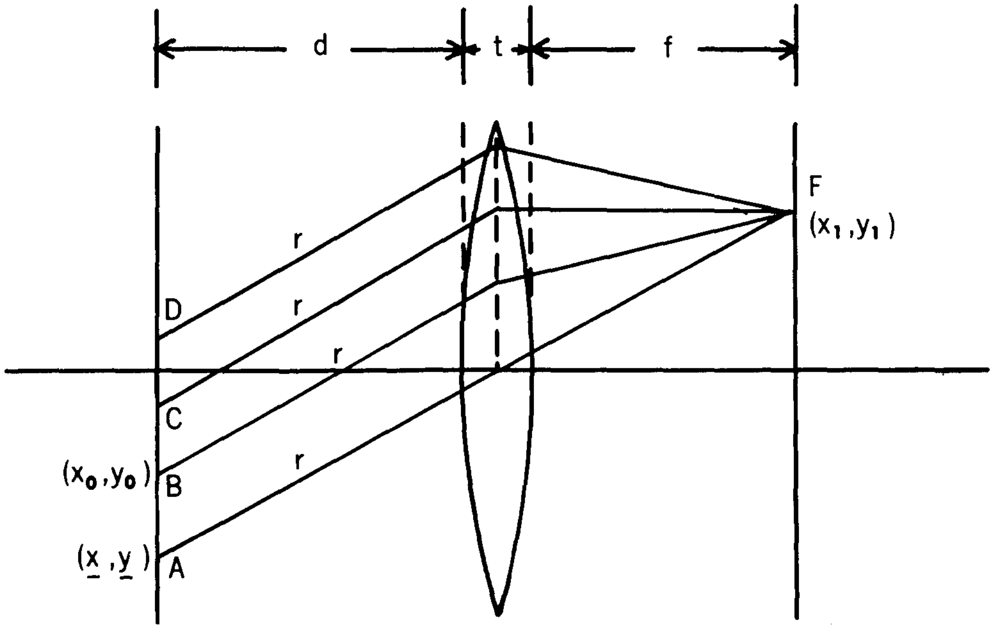

Figure B1. Collimated Light Incident on a Thin Lens.

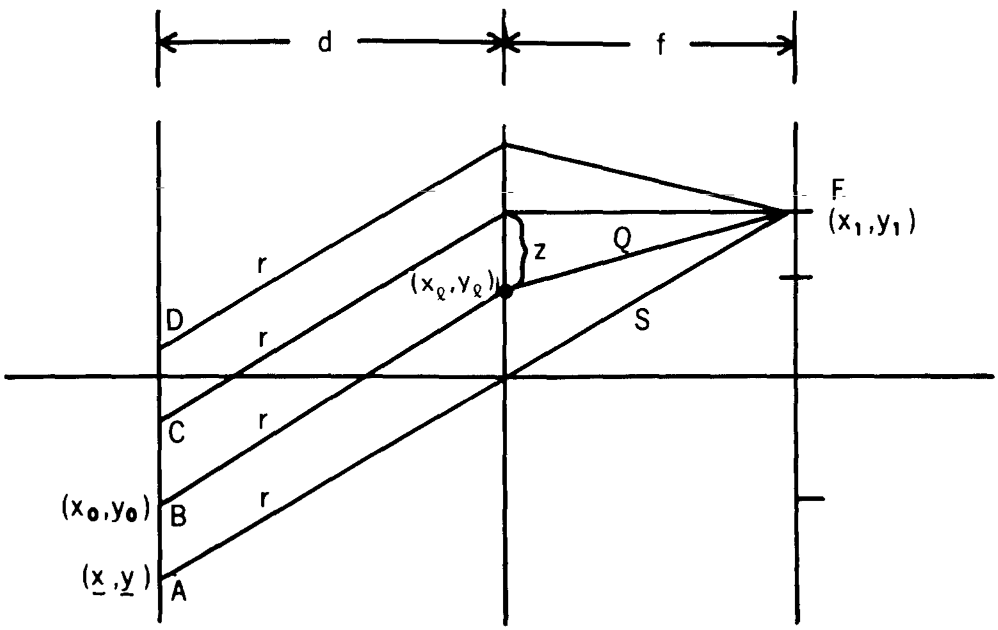

Figure B2. Thin Lens Approximation for Figure B1.

but s can be written as

$$s = (x_1{}^2 + y_1{}^2 + f^2)^{1/2}$$

Thus,

$$r = \frac{d}{f}(x_1{}^2 + y_1{}^2 + f^2)^{1/2}$$

Substituting into equation (B2) for r, yields

$$BF = \frac{d}{f}(x_1{}^2 + y_1{}^2 + f^2)^{1/2} + nt - \frac{1}{2f}\left[(x_o - \underline{x})^2 + (y_o - \underline{y})^2\right] + \left[f^2 + (x_1 - x_o + \underline{x})^2 + (y_1 - y_o + \underline{y})^2\right]^{1/2} \qquad (B3)$$

Assuming paraxial conditions, the terms raised to the ½ power may be expanded using the binomial theorem to give

$$BF = d\left(1 + \frac{x_1{}^2 + y_1{}^2}{2f^2}\right) + nt - \frac{1}{2f}\left[(x_o - \underline{x})^2 + (y_o - \underline{y})^2\right] + f + \frac{(x_1 - x_o + \underline{x})^2}{2f} + \frac{(y_1 - y_o + \underline{y})^2}{2f} \qquad (B4)$$

Writing out the squared terms, we get

$$BF = d\left(1 + \frac{x_1{}^2 + y_1{}^2}{2f^2}\right) + nt - \frac{1}{2f}\left[x_o{}^2 + \underline{x}^2 - 2x_o\underline{x} + y_o{}^2 + y^2 - 2y_o\underline{y}\right] + f + \frac{1}{2f}\left[x_1{}^2 + \underline{x}^2 + x_o{}^2 + 2x_1\underline{x} - 2x_o\underline{x} - 2x_1x_o\right] + \frac{1}{2f}\left[y_1{}^2 + \underline{y}^2 + y_o{}^2 + 2y_1\underline{y} - 2y_o\underline{y} - 2y_1y_o\right] \qquad (B5)$$

Collecting like terms and summing

$$BF = d\left(1 + \frac{x_1^{\,2} + y_1^{\,2}}{2f^2}\right) + nt + f + \frac{1}{2f}\left[x_1^{\,2} + 2x_1\underline{x} - 2x_1x_o\right] + \frac{1}{2f}\left[y_1^{\,2} + 2y_1\underline{y} - 2y_1y_o\right] \qquad \text{(B6)}$$

It is obvious from figure B2 that if d is equal to f (as is the case for an equiconvex lens),

$$\underline{x} = -x_1 \text{ and } \underline{y} = -y_1$$

and

$$BF = f + nt + f - \frac{x_1x_o + y_1y_o}{f}$$

If we now write the diffraction equation, equation (1) Section 1, substituting 2f for the attenuation factor and BF for the phase term, we have

$$E\,(x_1,y_1) = \frac{e^{jk\,(nt + 2f)}}{2j\lambda f}\int_{-\infty}^{+\infty}\!\!\int_{-\infty}^{+\infty} E\,(x_o,y_o)\, e^{\frac{-2\pi j}{\lambda f}(x_1x_0 + y_1y_o)}\, dx_o dy_o$$

where the term outside the integral is a constant for any point (x_1,y_1), and the integral itself is the Fourier Transform of the input $E\,(x_o,y_o)$. Thus, for an equiconvex thin lens, the field at the back focal plane is equal to a constant amplitude and phase factor times the Fourier Transform of the field at the front focal plane.