U.S. ARMY ENGINEER WATERWAYS EXPERIMENT STATION
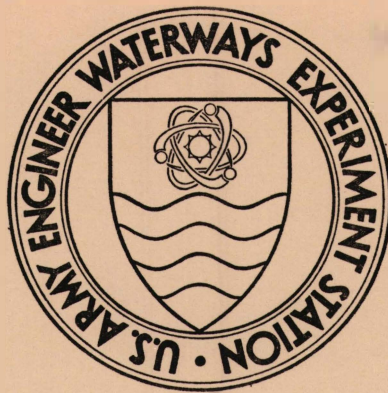
MISCELLANEOUS PAPER M-69-8

A GENERAL THEORY OF STRESSES AND DISPLACEMENTS IN ELASTIC AND VISCOELASTIC LAYERED SYSTEMS

by

Y. T. Chou

December 1969

Sponsored by U. S. Army Materiel Command

Conducted by U. S. Army Engineer Waterways Experiment Station, Vicksburg, Mississippi

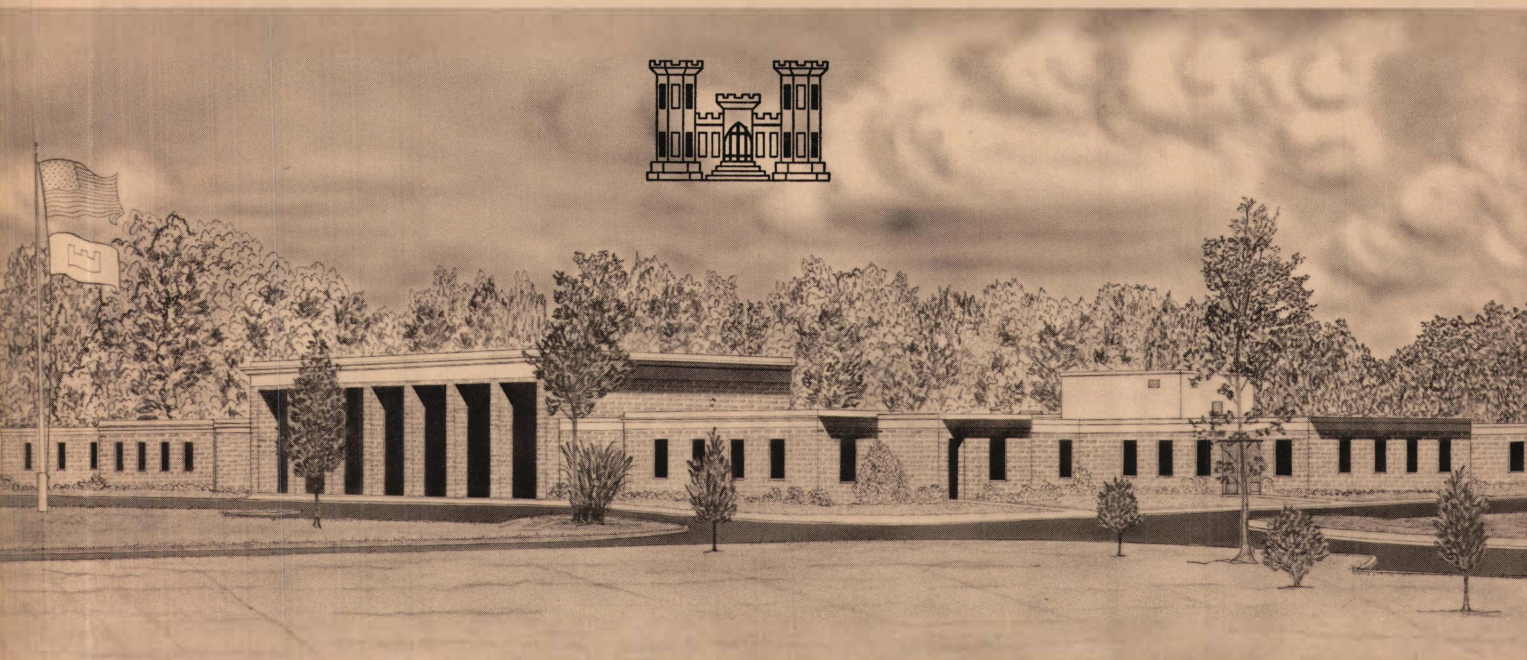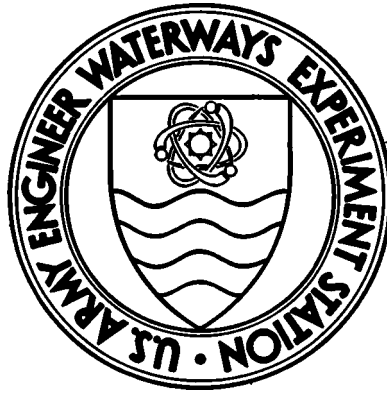APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

MISCELLANEOUS PAPER M-69-8

A GENERAL THEORY OF STRESSES AND DISPLACEMENTS IN ELASTIC AND VISCOELASTIC LAYERED SYSTEMS

by

Y. T. Chou

December 1969

.

Foreword

This paper was prepared for presentation at the Fifteenth Conference of Army Mathematicians in June 1969 at St. Louis, Missouri, under DA Project 1T061102B52A, "Research in Military Aspects of Terrestrial Sciences," Task 01, "Military Aspects of Off-Road Mobility," under the sponsorship and guidance of the Directorate of Development and Engineering, U. S. Army Materiel Command.

The study was performed and the paper was written by Dr. Y. T. Chou of the Mobility Research Branch (MRB), Mobility and Environmental (M&E) Division, U. S. Army Engineer Waterways Experiment Station (WES) under the general supervision of Messrs. W. G. Shockley and S. J. Knight, Chief and Assistant Chief, respectively, of the M&E Division, and Dr. D. R. Freitag, Chief, MRB, and under the direct supervision of Dr. K. W. Wiendieck, MRB.

COL Levi A. Brown, CE, was Director of WES during the course of this study. Messrs. J. B. Tiffany and F. R. Brown were Technical Directors.

Contents

A GENERAL THEORY OF STRESSES AND DISPLACEMENTS

IN ELASTIC AND VISCOELASTIC LAYERED SYSTEMS

Y. T. Chou
U. S. Army Engineer Waterways Experiment Station
Vicksburg, Mississippi

ABSTRACT. A multilayered, linear, elastic, and viscoelastic half space under stationary and moving axisymmetric loads is analyzed. Solutions are presented for normal stress, radial stress, tangential stress, shear stress, vertical deflection, and radial displacement at any point within the half space. The elastic solutions are obtained by Love's stress function and the Fourier-Hankel transform; the viscoelastic solutions, based on the elastic-viscoelastic correspondence principle, are obtained by applying the Laplace transformation to replace the time variable with a transformed variable. The time-dependent problem is then changed to an associated elastic problem. Inversion of the solution of the associated elastic problem into the real time variable solves the viscoelastic problem. By neglecting the inertia effect, the static viscoelastic solution has been extended to the moving load case.

Numerical examples are given to illustrate the response of the materials to a normal point load. Such an analysis is believed to be an essential step in the development of a rational method for designing airport and highway flexible pavements.

INTRODUCTION. A rational design method for airport and highway flexible pavements requires knowledge of the stresses and displacements within the layered systems. Once these quantities are numerically determined, other design criteria, such as fatigue and failure considerations, can then be taken into account.

This paper presents an analysis of linear, elastic and viscoelastic, multilayered systems under both stationary and moving axisymmetric loads. These loads were comprised of normal point load, uniform and parabolic normal loads, and tangential load, which are commonly found on pavements. Expressions for stresses and displacements under stationary loads at points within the systems were developed by Love's stress function[1] and the Fourier-Hankel transform. Expressions were developed for viscoelastic cases under stationary and moving loads, based on the elastic-viscoelastic correspondence principle.[2] This was accomplished by applying the Laplace transform to replace the time variable with a transformed variable p. The time-dependent problem was thus changed to an associated elastic problem. Inversion of the solution for the associated elastic problem into the real time variable solved the viscoelastic problem. Inertial effects were not considered in the derivation of equations for moving loads.

In deriving equations, a "rough" interface was assumed; i.e., the displacement in the horizontal direction and the shearing stress were continuous across the interface. These conditions were felt to be better descriptions of the actual conditions of flexible pavement structures.

STATIC ELASTIC SOLUTION. The elastic solution for two-and three-layered systems was developed by Burmister[3] and later extended to multilayer systems by Mehta and Veletsos.[4] An n-layer elastic system, subjected to an axisymmetrically distributed load, is shown in Figure 1. Cylindrical coordinates are used to facilitate the solution of the problem, since axial symmetry exists. The general method of analysis involves the determination of a stress function that satisfied the governing differential equation

$$\nabla^4 \phi = 0 \tag{1}$$

in which

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

for each of the layers. The stresses and displacements for the various layers are expressed in terms of the stress function ϕ.[1]

Consider a stress function of the form:

$$\phi_j = G_j(z)\, J_o(mr)$$

where

$G_j(z)$ = a function of depth z only,

$J_o(mr)$ = the Bessel function of the first kind and order of 0

m = a parameter

Subscript j = the quantities corresponding to the jth layer

Substitutions can easily verify that

$$\phi_j = \frac{H^3 J_o(m\rho)}{m^2}\left\{ A_j \exp[-m(\lambda_j-\lambda)] - B_j \exp[-m(\lambda-\lambda_{j-1})] \right.$$
$$\left. + C_j m\lambda \exp[-m(\lambda_j-\lambda)] - D_j m\lambda \exp[-m(\lambda-\lambda_{j-1})] \right\} \tag{2}$$

is a stress function for the jth layer that satisfies equation 1, where

A, B, C, D = integration constants

ρ = r/H

λ = z/H

H = the distance from the surface to the upper boundary of the lowest layer

The stresses and displacements determined from the stress functions are:

$$(\sigma_z^{**})_j = -m\, J_o(m\rho)\ \{[A_j - C_j(1-2\nu_j - m\lambda)]\ \exp[-m(\lambda_j - \lambda)]$$
$$+ [B_j + D_j(1-2\nu_j + m\lambda)]\ \exp[-m(\lambda - \lambda_{j-1})]\} \tag{3a}$$

$$(\sigma_r^{**})_j = [mJ_o(m\rho) - \frac{J_1(m\rho)}{\rho}]\ \{[A_j + C_j(1+m\lambda)]\ \exp[-m(\lambda_j - \lambda)]$$
$$+ [B_j - D_j(1-m\lambda)]\ \exp[-m(\lambda - \lambda_{j-1})]\} + 2\nu_j mJ_o(m\rho) \tag{3b}$$
$$\{C_j \exp[-m(\lambda_j - \lambda)] - D_j \exp[-m(\lambda - \lambda_{j-1})]\}$$

$$(\sigma_\theta^{**})_j = \frac{J_1(m\rho)}{\rho}\ \{[A_j + C_j(1+m\lambda)]\ \exp[-m\lambda_j - \lambda)]$$
$$+ [B_j - D_j(1-m\lambda)]\ \exp[-m(\lambda - \lambda_{j-1})]\} + 2\nu_j mJ_o(m\rho) \tag{3c}$$
$$\{C_j \exp[-m(\lambda_j - \lambda)] - D_j \exp[-m(\lambda - \lambda_{j-1})]\}$$

$$(\tau_{rz}^{**})_j = mJ_1(m\rho) \cdot \{[A_j + C_j(2\nu_j + m\lambda)]\ \exp[-m(\lambda_j - \lambda)]$$
$$- [B_j - D_j(2\nu_j - m\lambda)]\ \exp[-m(\lambda - \lambda_{j-1})]\} \tag{3d}$$

3

$$(w^{**})_j = -\frac{1+\nu_j}{E_j} J_0(m\rho) \cdot \{[A_j - C_j(2-4\nu_j-m\lambda)] \exp[-m(\lambda_j-\lambda)]$$

$$- [B_j + D_j(2-4\nu_j+m\lambda)] \exp[-m(\lambda-\lambda_{j-1})]\} \tag{3e}$$

$$(u^{**})_j = -\frac{1+\nu_j}{E_j} J_1(m\rho) \{[A_j + C_j(1+m\lambda)] \exp[-m(\lambda_j-\lambda)]$$

$$+ [B_j - D_j(1-m\lambda)] \exp[-m(\lambda-\lambda_{j-1})]\} \tag{3f}$$

where

σ_z, σ_r, and σ_θ = stresses in z, r, and θ directions, respectively.

w and u = vertical and horizontal displacements, respectively.

J_1 = Bessel function of the first kind and order of 1.

The double asterisks indicate that these quantities are not the actual stresses and displacements due to the actual load, but are those due to a vertical load, $-mJ_0(m\rho)$, and a horizontal shearing force, $mJ_1(m\rho)$. From equations 3a and 3d, at the upper surface (j = 1, λ = 0), $\sigma_z = -mJ_0(m\rho)$ and $\tau_{rz} = mJ_1(m\rho)$, when the respective expression in the brackets is taken as unity.

The boundary and interface conditions are:

a. __At the upper surface,__ j = 1 and λ = 0

$$(\sigma_z^{**})_1 = -mJ_0(m\rho) \tag{4a}$$

$$(\tau_{rz}^{**})_1 = mJ_1(m\rho) \tag{4b}$$

If the influence of normal loads is considered alone, then $(\tau_{rz}^{**})_1 = 0$ in equation 4b; if only the horizontal force exists, then $(\sigma_z^{**})_1 = 0$ in equation 4a. In the following discussion, the effects of normal and horizontal loads are considered separately.

b. __At the interface,__

$$(\sigma_z^{**})_j = (\sigma_z^{**})_{j+1} \tag{5a}$$

$$(\tau_{rz}^{**})_j = (\tau_{rz}^{**})_{j+1} \tag{5b}$$

4

$$(w^{**})_j = (w^{**})_{j+1} \qquad\qquad (5c)$$

$$(u^{**})_j = (u^{**})_{j+1} \qquad\qquad (5d)$$

Since the stresses and displacements must vanish as λ approaches infinity, it can be concluded from equation 2 that for the lowermost layer ($i = n$), $A_n = C_n = 0$.

The remaining $4n-2$ integration constants can be evaluated from the $4n-2$ simultaneous equations obtained by satisfying equations 4 and 5 for each of the $n-1$ interfaces. After the integration constants are determined, they can be substituted into equation 3 to get the stresses and displacements due to loadings $-mJ_o(m\rho)$ or $mJ_1(m\rho)$.

To find the stresses and displacements due to various types of surface loading, the Hankel transform is used. For a normal load of constant intensity q distributed over a circular area of radius a:

$$\overline{f}(m) = \int_0^\alpha q\rho \; J_o(m\rho)d\rho = \frac{q\alpha}{m} J_1(m\alpha)$$

where

$$\alpha = \frac{a}{H}$$

The Hankel inversion of $\overline{f}(m)$ is

$$q(\rho) = \int_0^\infty \overline{f}(m) \; mJ_o(m\rho)dm = q\alpha \int_0^\infty J_o(m\rho) \; J_1(m\alpha)dm$$

If Y* is the stress or displacement in equation 3 due to the vertical loading $-mJ_o(m\rho)$ and Y is that due to the load $-q$, then:

$$Y = q\alpha \int_0^\infty \frac{Y*}{m} J_1(m\alpha)dm \qquad\qquad (6)$$

For a concentrated normal load $-q_o$:

$$\overline{f}(m) = \lim_{\alpha \to 0} \left[\frac{-q\alpha}{m} J_1(m\alpha) \right] = \frac{-q_o}{2\pi H^2}$$

$$q_o(\rho) = \frac{-q_o}{2\pi H^2} \int_0^\infty m J_o(m\rho) \, dm$$

$$Y = \frac{q_o}{2\pi H^2} \int_0^\infty Y^* \, dm \tag{7}$$

For a parabolically distributed normal load over a circular area of radius a with $q(r) = -q(1 - \frac{r^2}{a^2})$:

$$\overline{f}(m) = \int_0^\alpha - \frac{q}{a^2} (a^2 - r^2)\rho \; J_o(m\rho) \, d\rho$$

$$= \frac{-2q}{m^2} \left[\frac{2}{\alpha m} J_1(m\alpha) - J_o(m\alpha) \right]$$

$$q(\rho) = \int_0^\infty \overline{f}(m) m J_o(m\rho) \, dm$$

$$= -2q \int_0^\infty \frac{1}{m} \left[\frac{2}{\alpha m} J_1(m\alpha) - J_o(m\alpha) \right] J_o(m\rho) \, dm$$

$$Y = 2q \int_0^\infty \frac{1}{m^2} \left[\frac{2}{\alpha m} J_1(m\alpha) - J_o(m\alpha) \right] Y^* \, dm \tag{8}$$

For a horizontal shear force of constant intensity T distributed over a circular area of radius a:

$$\overline{f}(m) = \int_0^\alpha T \rho \, J_1(m\rho) \, d\rho$$

6

For a given value of m, $\overline{f}(m)$ can be found by numerical integration.

$$T(\rho) = \int_0^\infty \overline{f}(m) \, mJ_1(m\rho) \, dm$$

$$Y = \int_0^\infty \overline{f}(m) \, Y* \, dm \qquad (9)$$

Similarly, the same procedure may be employed for other types of axisymmetric loads.

The analysis of elastic layered systems can, therefore, be summarized in the following steps: (a) assign successifve values of m to any one of equations 6 to 9, depending upon the surface loading conditions, starting from zero to a rather large positive number until Y converges; (b) for each value of m, establish the simultaneious equations from equations 3, 4, and 5, and solve for the constants of integration; and, (c) substitute the constants into equation 3 to obtain Y* and determine Y from one of equations 6 to 9 by numerical integration.

To illustrate - the vertical displacement w_o at the surface, the vertical displacement w_B, and the vertical stress σ_B at a point $2h_1$ below the surface for an incompressible, two-layer elastic system due to a vertical load $-mJ_o(m\rho)$ are

$$W_o^{**} = \frac{1}{2G_1} \, J_o(m\rho) \, V_o \qquad (10a)$$

$$W_B^{**} = \frac{1}{2G_2} \, J_o(m\rho) \, V_B \qquad (10b)$$

$$\sigma_B^{**} = -J_o(m\rho) \, V_\sigma \qquad (10c)$$

where

$$V_o = \frac{1 + 4Nm \, \exp(-2m) - N^2 \, \exp(-4m)}{1 - 2N(1 + 2m^2) \, \exp(-2m) + N^2 \, \exp(-4m)}$$

$$V_B = (1-N) \frac{(1+2m) \exp(-2m) - N(1-2m^2) \exp(-4m)}{1-2N(1+2m^2) \exp(-2m) + N^2 \exp(-4m)}$$

$$V_\sigma = mV_B$$

$$N = \frac{G_1 - G_2}{G_1 + G_2}$$

The corresponding expressions w_o, w_B, and σ_B due to a normal point load $-q_o$ can be obtained from equation 7:

$$w_o = \frac{q_o}{4\pi h_1^2 G_1} \int_0^\infty J_o(m\rho) V_o \, dm \qquad (11a)$$

$$w_B = \frac{q_o}{4\pi h_1^2 G_2} \int_0^\infty J_o(m\rho) V_B \, dm \qquad (11b)$$

$$\sigma_B = - \frac{q_o}{2\pi h_1^2} \int_0^\infty J_o(m\rho) V \, dm \qquad (11c)$$

where V_o, V_B, and V_σ are the same as in equation 10.

The expressions for stresses and displacements at any point in the layered system induced by other axisymmetric normal loads have the same form as shown in equation 11. For example, equation 11 for a uniform circular load $-q$ can be written:

$$w_o = \frac{q\alpha}{2G_1} \int_0^\infty \frac{J_o(m\rho)}{m} J_1(m\alpha) V_o \, dm \qquad (12a)$$

$$w_B = \frac{q\alpha}{2G_2} \int_0^\infty \frac{J_o(m\rho)}{m} J_1(m\alpha) V_B \, dm \qquad (12b)$$

$$\sigma_B = -q\alpha \int_0^\infty \frac{J_o(m\rho)}{m} J_1(m\alpha) V_\sigma \, dm \qquad (12c)$$

where V_o, V_B, and V_σ are the same as in equation 10.

8

Note that singularity exists in surface displacement w_o under a normal point load, as shown in equation 11a. The value of the Bessel function becomes 1 when the value of ρ is zero, and the value of V_o converges to 1 when m becomes larger than 10.

When the number of layers is increased, the equations for stresses and displacements have the same form as equation 11; but the terms V_o, V_B, V_σ, which contain expressions for material properties of the components of each layer, become more complicated.

STATIC VISCOELASTIC SOLUTIONS. The viscoelastic stress-strain relations used throughout this paper are based on the principle of elastic-viscoelastic correspondence developed by Lee[2] and are defined by linear differential operators with respect to time. For a linear, homogeneous, and isotropic material, the stress-strain relation can be represented by two pairs of operators, one of which relates the deviatoric stress to the deviatoric strain, the other the mean normal dilatational stress to the mean normal strain. Thus:

$$P'S_{ij} = 2Q'e_{ij} \qquad (13a)$$

$$P''\sigma_{ii} = 3Q''\epsilon_{ii} \qquad (13b)$$

where

S_{ij} = deviatoric stress = $\sigma_{ij} - \frac{1}{3}\sigma_{KK}\delta_{ij}$

e_{ij} = deviatoric stress = $\epsilon_{ij} - \frac{1}{3}\epsilon_{KK}\delta_{ij}$

σ_{ij} = stress tensors

ϵ_{ij} = strain tensors

δ_{ij} = Kronecker delta; i.e., $\delta_{ij} = 1$, $i = j$, $\delta_{ij} = 0$, $i \neq j$

The analogous equations for the elastic solid are:

$$S_{ij} = SGe_{ij} \qquad (14a)$$

$$\sigma_{ii} = 3K_{ii} \qquad (14b)$$

where G is the shear modulus and K the bulk modulus.

9

Compare equations 13 and 14

$$G = \frac{Q'}{P'} \tag{15a}$$

$$K = \frac{Q''}{P''} \tag{15b}$$

where Q', P', Q'', and P'' are linear operators of the form $\sum\limits_{r=0}^{n} a_r \dfrac{\partial^r}{\partial t^r}$.

The coefficients a_r and the integer n are, in general, different for each operator. Linearity requires that a_r be independent of stress and strain, and the assumption of homogenity and isotropy eliminates the dependence of a_r on location and orientation of stress within the system. The Young's modulus E and the Poisson's ratio ν are related to G and K by the formulas

$$E = \frac{9KG}{3K + G} \tag{16a}$$

$$\nu = \frac{3K - 2G}{6K + 2G} \tag{16b}$$

In elastic problems, K, G, E, and ν are time-independent constants; but in viscoelastic problems, they are not constants, but are linear differential operators, as shown in equation 15.

To eliminate the time variable t, the Laplace transform with respect to t can be applied to equation 13. The transforms of stresses and strains are denoted by an asterisk on the corresponding function. If the system is initially undisturbed; i.e., with initial stress and strain conditions equal to zero, and the operator in $\partial/\partial t$ merely becomes the same function of the transformed variable p,[2] equation 13 becomes:

$$P'(p)S_{ij}^{*} = 2Q'(p) e_{ij}^{*} \tag{17a}$$

$$P''(p)\sigma_{ii}^{*} = 3Q''(p) \epsilon_{ii}^{*} \tag{17b}$$

where P'(p), Q'(p), P''(p), and Q''(p) are polynomials of the form $\sum\limits_{r=0}^{n} a_r p^r$.

If the Laplace transforms of the stresses and strains are considered, and if G* and K* are defined as follows:

$$G* = \frac{S_{ij}*}{2e_{ij}*} = \frac{Q'(p)}{P'(p)} \tag{18a}$$

$$K* = \frac{\sigma_{ii}*}{3\epsilon_{ii}*} = \frac{Q''(p)}{P''(p)} \tag{18b}$$

then by replacing G, K, and load q_o (or q, T) in the elastic equations for stresses and displacements with G*, K*, and q_o*, respectively, the equations for the Laplace transforms of stresses and displacements are obtained. These equations are functions of the transformed variable, p, and their inversions give the stresses and displacements in terms of the time variable, t.

The corresponding equations in the transformed domain for equation 11 are:

$$w_o* = \frac{1}{4\pi h_1^2} \int_0^\infty J_o(m\rho) \, V_o* \, dm \tag{19a}$$

$$w_B* = \frac{1}{4\pi h_1^2} \int_0^\infty J_o(m\rho) \, V_B* \, dm \tag{19b}$$

$$\sigma_B = -\frac{1}{2\pi h_1^2} \int_0^\infty J_o(m\rho) \, V_\sigma* \, dm \tag{19c}$$

where

$$V_o* = \frac{q_o*}{G_1*} \; \frac{1+4N*m \exp(-2m) - (N*)^2 \exp(-4m)}{1-2N* (1+2m^2) \exp(-2m) + (N*)^2 \exp(-4m)} \tag{20a}$$

$$V_B* = \frac{q_o*(1-N*)}{G_2*} \; \frac{(1+2m) \exp(-2m) - N*(1-2m^2) \exp(-4m)}{1-2N* (1+2m^2) \exp(-2m) + (N*)^2 \exp(-4m)} \tag{20b}$$

11

$$V_\sigma^* = q_o^* \; m(1-N^*) \; \frac{(1+2m) \exp(-2m) - N^*(1-2m^2) \exp(-4m)}{1-2N^* (1+2m^2) \exp(-2m) + (N^*)^2 \exp(-4m)} \tag{20c}$$

$$N^* = \frac{G_1^* - G_2^*}{G_1^* + G_2^*} \tag{20d}$$

$$q_o^* = \frac{q_o}{P} \tag{20e}$$

G_1^* and G_2^*, which characterize the material properties, can be obtained from the linear differential operators as shown in equation 18a. For a given mechanical model that properly characterizes the shear behavior of the material, the transformed shear modulus G^* can be substituted into equation 20. V_o^*, V_B^*, and V_σ^* will be found to be the ratios of two polynomials, the polynomial in the denominator having degrees higher than that in the numerator. Consequently, V_o^*, V_B^*, and V_σ^* can be partial-fractioned, and the direct Laplace inversion can be applied. By substituting V_o, V_B, and V_σ into equation 11, the viscoelastic solution can be obtained. The resulting integrals can be evaluated numerically by the Gaussian quadrature formula.[3] However, when the polynomial is of such a high degree that finding its roots becomes very tedious, the collocation method is suggested, which will be explained later. For other types of surface loads, similar procedures can be used to obtain solutions.

MOVING LOADS. Since the speeds of the loads of vehicles traveling on pavements are much less than the speeds of the shear and compression waves propagated in the pavement structures, inertial effects can be assumed negligible. The system is assumed to be at rest initially. For time greater than zero, the surface of the system is subject to an axisymmetrical loading moving along the x-axis with constant velocity v. A point load $-q_o(t)$ is shown in Figure 2. At any instant the stresses and displacement at the point $(x, 0, z)$ directly under the path of the load in the elastic problem with this point load are given in equation 11 with r replaced by $(x-\alpha t)$, since the directions of x and r are the same at the point in question, and the radians of the point $(x, 0, z)$ relative to cylindrical coordinates based on the load axis is $(x-vt)$. Equation 19 for the case of a moving load can be written:

$$w_o^* = \frac{1}{4\pi h_1^2} \int_0^\infty \{q_o J_o [m(X-V\tau')]\}^* V_o^* \; dm \tag{21a}$$

12

$$w_B^* = \frac{1}{4\pi h_1^2} \int_0^\infty \{q_o J_o[m(X-V\tau')]\}^* V_B^* \, dm \qquad (21b)$$

$$\sigma_B^* = \frac{-1}{2\pi h_1^2} \int_0^\infty \{\alpha q_o J_o[m(X-V\tau')]\}^* V_\sigma^* \, dm \qquad (21c)$$

where $X = \frac{x}{H}$, V_o^*, V_B^*, and V_σ^* are the same as shown in equation 20, except that q_o^* does not appear in the expression.

Moreover, the viscoelastic solution can be obtained by using the convolution integral. After the exchange of integrals we have:

$$w_o = \frac{4\pi}{q_o} w_o(t) = \frac{1}{h_1^2} \int_0^t \int_0^\infty J_o[m(X-V\tau')] V_o(t-\tau') dm d\tau' \qquad (22a)$$

$$w_B = \frac{4\pi}{q_o} w_B(t) = \frac{1}{h_1^2} \int_0^t \int_0^\infty J_o[m(X-V\tau')] V_B(t-\tau') dm d\tau' \qquad (22b)$$

$$\sigma_B = -\frac{2\pi}{q_o} \sigma_B(t) = \frac{1}{h_1^2} \int_0^t \int_0^\infty \alpha J_o[m(X-V\tau')] V_\sigma(t-\tau') dm d\tau' \qquad (22c)$$

where τ' is time varying from zero to t, and the expression $(t - \tau')$ means that V_o, V_B, and V_σ are functions of $(t - \tau')$.

Mathematically, the exchange of integrals is possible because the inner integrals of equation 22 are uniformly convergent in the range $0 \leq \tau' \leq t$.[6] This exchange is desirable, since it greatly reduces the computation time.

The value of V_o, V_B, and V_σ in equation 22 can be obtained by inverting V_o^*, V_B^*, and V_σ^*, as in the case of a stationary load. For cases of incompressible two-layer systems, where the material in each layer is characterized by simple models, the direct inversion method may also be used. To solve the double integral shown in equation 22, the inner integral can be evaluated by the Gaussian quadrature formula, and the outer integral by Simpson's rule. In equation 22, the dimensionless factor $X(=x/H)$ cannot be factored out of the Bessel function $J_o[m(X-V_\tau')]$, as in the case of the load q_o; therefore, a value should be assigned to X in numerical computations.

The singularity that exists in equation 22a can be treated by separate consideration of the conditions where the load approaches the point and those where it moves away from the point. Equation 22a can be written:

$$w_o = \int_0^{\frac{X}{v}(1-\epsilon)} \int_0^{\infty} J_o[m(X-V\tau')] \, V_o(t-\tau') \, dm \, d\tau$$
$$+ \int_{\frac{X}{v}(1+\epsilon)}^{t} \int_0^{\infty} J_o[m(X-V\tau')] \, V_o(t-\tau') \, dm \, d\tau' \tag{23}$$

where ϵ is any positive number less than unity.

Equation 22 is for two-layer systems under point load; the same procedure can be applied to obtain stresses and displacements for other types of surface loads.

COLLOCATION METHOD. For multilayer systems or compressible material, the mathematical work required for the direct inversion method becomes very tedious, and the polynomial that results is of such a high degree that its roots are almost impossible to find. Therefore, the direct inversion method is used mainly when equations can be reduced to a sum of simple partial fractions. For multilayer or compressible material, an approximate method is developed. This method was originally proposed by Schapery,[7,8] based on results obtained from irreversible thermodynamics and variational principles. Schapery concluded that the class of problems to which the elastic-viscoelastic analogy may be applied has time-dependent solutions of the form:

$$\psi(t) = \psi' + \psi''t + \Delta\psi(t) \tag{24}$$

where ψ' and ψ'' are constants with respect to time, and $\Delta\psi(t)$ is the transient component of the solution defined as

$$\Delta\psi(t) = \int_0^{\infty} \phi(\tau') \exp(-\frac{t}{\tau'}) \, d\tau' \tag{25}$$

The function $\phi(\tau')$, referred to as a spectral function of the variable τ', may consist either entirely or partly of Dirac delta functions. If $\Delta\psi(t)$

14

is expressed approximately by a Dirichlet series of decaying exponentials,

$$\Delta\psi(t) \simeq \Delta\psi_o = \sum_{i=1}^{n} S_i \exp\left(-\frac{t}{\gamma_i}\right) \tag{26}$$

Since viscoelastic materials have exponential stress-relaxation characteristics, equation 26 allows a wide spectrum of relaxation times; e.g., a reasonably large number of springs and dashpots may be incorporated in the material representation. Values of γ_i are positive constants prescribed in such a way as to provide adequate coverage of the time spectrum, and the S_i are unspecified coefficients to be evaluated by minimizing the total square error between the actual $\Delta\psi(t)$ and $\Delta\psi_D$, as given by the series. The minimization process will result in the expression:

$$\int_0^\infty \Delta\psi \exp\left(-\frac{t}{\gamma_i}\right) dt = \int_0^\infty \Delta\psi_D \exp\left(-\frac{t}{\gamma_i}\right) dt \quad i=1,2,\ldots n \tag{27}$$

Equation 27 essentially means that, for the mean square error of the approximation to be a minimum, the Laplace transform of the approximation must equal the Laplace transform of the exact function, at least at the n points $p = 1/\gamma_i$, $i=1,2,\ldots n$. Therefore, n relations are available for the Laplace transforms of $\Delta\psi(t)$ and evaluated at $p = 1/\gamma_i$; i.e.,

$$\Delta\psi^*(1/\gamma_i) = \Delta\psi_D^*(1/\gamma_i) \qquad i=1,2,\ldots n \tag{28}$$

A more convenient form is obtained by multiplying these by $1/\gamma_i$ which yields:

$$(p\Delta\psi^*(p))_{p=1/\gamma_i} = (p\Delta\psi_D^*(p))_{p=1/\gamma_i} \qquad i=1,2,\ldots n \tag{29}$$

Since $\Delta\psi(t)$ was assumed to be of the form indicated by equation 26, equation 29 reduces to the series of equations:

$$(p\Delta\psi^*(p))_{p=1/\gamma_i} = \sum_{j=1}^{n} \frac{S_i}{1 + \dfrac{\gamma_i}{\gamma_j}} \qquad i=1,2,\ldots n \tag{30}$$

which constitutes a set of simultaneous equations for different values of p, which are solved for S_i. Thus, the collocation consists of matching the summation on the left-hand side of equation 30 with the calculated values of $p\Delta\psi^*(p)$ for various values of real, positive p. Hence, the inversion procedure consists essentially in being able to determine the values of the transform all along the positive real p-axis.

The constants ψ' and ψ'' in equation 24 are determined from given initial and boundary conditions. For the special case of a moving load, the viscoelastic pavement system is assumed to be initially undistrubed, and at infinite time; i.e., the load has passed over the station for a very long time, the stresses and displacements induced by the moving load approach zero. Therefore, the constants ψ' and ψ'' are equal to zero.

If we are interested in the time interval between zero to 1000 seconds for the moving point load case, 10 values of p, i.e., 100, 0.1, 0.055, 0.031, 0.018, 0.01, 0.0055, 0.0031, 0.0018, and 0.001, may be used; so equation 11 with ρ replaced by (X-Vt) becomes

$$\psi(t) = S_1 \exp(-100t) + S_2 \exp(-0.1t) + S_3 \exp(-0.055t)$$

$$+ \ldots + S_{10} \exp(-0.001t) \tag{31}$$

Take the Laplace transform of equation 31 and then multiply by p:

$$p\psi(p)^* = \frac{S_1}{1 + \frac{100}{p}} + \frac{S_2}{1 + \frac{0.1}{p}} + \frac{S_3}{1 + \frac{0.055}{p}} + \ldots + \frac{S_{10}}{1 + \frac{0.001}{p}} \tag{32}$$

By assigning successively p = 100, 0.1, 0.055, 0.031, 0.018, 0.01, 0.0055, 0.0031, 0.0018, and 0.001 in equation 32, 10 equations are obtained, which will give a solution to the 10 unknowns, S_1 through S_{10}.

$$
\left[
\begin{array}{ccccc}
\dfrac{1}{1+\dfrac{100}{100}} & \dfrac{1}{1+\dfrac{0.1}{100}} & \dfrac{1}{1+\dfrac{0.055}{100}} & ---- & \dfrac{1}{1+\dfrac{0.001}{100}} \\[3ex]
\dfrac{1}{1+\dfrac{100}{0.1}} & \dfrac{1}{1+\dfrac{0.1}{0.1}} & \dfrac{1}{1+\dfrac{0.055}{0.1}} & ---- & \dfrac{1}{1+\dfrac{0.001}{0.1}} \\[3ex]
\dfrac{1}{1+\dfrac{100}{0.055}} & \dfrac{1}{1+\dfrac{0.1}{0.055}} & ---- & & \\[3ex]
\dfrac{1}{1+\dfrac{100}{0.001}} & ---- & & & \dfrac{1}{1+\dfrac{0.001}{0.001}}
\end{array}
\right]
\left\{
\begin{array}{c}
S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_{10}
\end{array}
\right\}
=
\left\{
\begin{array}{c}
p\psi(p)*_{p=100} \\ p\psi(p)*_{p=0.1} \\ \vdots \\ p\psi(p)*_{p=0.001}
\end{array}
\right\}
\tag{33}
$$

The values of p used in the example are quarter decades, spaced in the time interval between 10 and 1000 seconds. Basically, the accuracy of the inversion can be improved by adding more terms to the series in equation 26, but "ill condition" may be encountered in solving the simultaneous equations if the p values selected are too close to each other.

The most basic aspect of viscoelastic response is that the complete time history of the event must be taken into consideration to determine the present state of the material. Therefore, the criterion for the selection of an effective solution range is that the p-multiplied transform, i.e. $p\Delta\psi*(p)$, should go from a constant value (before any actual response takes place), through its "transition" range, to a final (long-time) constant value. Since the solution at any given time is basically a function of the value of the Laplace transform throughout its entire range, the time span of the solution should be as wide as possible.

NUMERICAL EXAMPLES AND DISCUSSIONS. The effect of E_1/E_2 on vertical stress and vertical displacement for a two-layer incompressible elastic system under a stationary normal point load is shown in Figure 3; E_1 and E_2

are the Young's moduli of layers 1 and 2, respectively. When E_1/E_2 is less than unity, the stress and displacement decreases slowly with increasing values of E_1/E_2; but when E_1/E_2 is greater than unity, the stress and displacement decrease rapidly, and the stress decreases faster than the displacement.

The stress factors in the same elastic system are listed in Table 1. The stress factors, S_B, are compressive when the distance, R, is less than 7.5, and oscillate between compression and tension when the values of R are greater than 7.5.

The stress and displacement in a viscoelastic two-layer incompressible system are shown in Figure 4. The response of the material in each layer to deviatoric stress is characterized by a simple Kelvin model. The displacement has a very small value initially, but the value increases slowly with time. The stress, on the other hand, has a value about one-third its maximum at 0.1 second; the value then increases slowly with time. At longer times, the stress and displacement factors reach the elastic solutions having the ratio $G_1/G_2 = 10$.

The values of the stress factor, S_B, in an incompressible viscoelastic two-layer system at a point 2 ft. below the surface are shown in Table 2. T, a dimensionless time factor, is the ratio of the real time, t, to the retardation time, τ_2, of the bottom layer. For longer periods of time, i.e., T = 100, values of the stress factor, S_B, for a viscoelastic case approach the elastic case values. Also, interesting is the fact that the stress factor, S_B, for a given value of R may either oscillate in sign between compression and tension, or decrease in value when the time factor, T, increases from 0 to 100.

The effect of the speed of a moving point load on the displacement at a point 2 ft. below the surface of a viscoelastic half space is shown in Figure 5. The displacement decreases as the speed of the load increases. At low speed of the load, the displacement curves rebound as soon as the load moves away from the station; as the speed of the load increases, the displacement subsequently increases as the load moves away from the station.

The surface displacement for an incompressible two-layer viscoelastic system under a moving point load is shown in Figure 6. The discontinuities shown in the figure are caused by the singularity that exists when the load is acting directly over the point under consideration. The shapes of the curves are similar to those in Figure 5.

In multilayer systems, the collocation method replaces the direct inversion method. The former method requires long computer time to

evaluate the stresses or displacements in their transformed domain. The collocation method was used to obtain the displacement at the second interface in a viscoelastic incompressible three-layer system under a moving point load, with each layer having the same material properties. The solutions obtained were then compared with those for a viscoelastic semi-infinite solid. The computed results, with five collocated points, are shown in Figure 7. The collocation method is applicable to moving loads on multilayer systems.

SUMMARY. An analysis of a multilayered, linear, elastic, and viscoelastic half space under stationary and moving axisymmetric loads has been made. Solutions have been presented for the normal stress, σ_z, radial stress σ_r, tangential stress, σ_θ, shear stress, τ_{rz}, vertical deflection, w, and radial displacement u, at any point within the half space, in terms of integral equations. Numerical examples have been given for illustration.

REFERENCES

1. Love, A. E. H., "A Treatise on the Mathematical Theory of Elasticity," 4th ed, Dover Publications, New York, 1944.

2. Lee, E. H., "Stress Analysis in Viscoelastic Bodies," Quarterly of Applied Mathematics, Vol. 13, 1955, pp. 183-190.

3. Burmister, D. M., "The General Theory of Stresses and Displacements in Layered Soil Systems," Journal of Applied Physics, Vol. 16, 1945, pp. 89-94, 126-127, 296-302.

4. Mehta, M. R., and Veletsos, A. S., "Stresses and Displacements in Layered Systems," Structural Research Series No. 178, 1959, University of Illinois, Champaign, Ill.

5. Hilderbrand, F. P., Introduction to Numerical Analysis, 1956, McGraw-Hill, New York, pp. 323-325.

6. Titchmarsh, E. C., The Theory of Functions, 1939, Oxford University Press, England, pp. 53.

7. Schapery, R. A., "Approximate Methods of Transform Inversion for Visco-Elastic Stress Analysis," Proceedings of the Fourth U. S. National Congress of Applied Mechanics, 1962, pp. 1075-1085.

8. Schapery, R. A., "Application of Thermodynamics to Thermomechanical, Fracture, and Birefringement Phenomena in Viscoelastic Media," Journal of Applied Physics, Vol. 35, 1964.
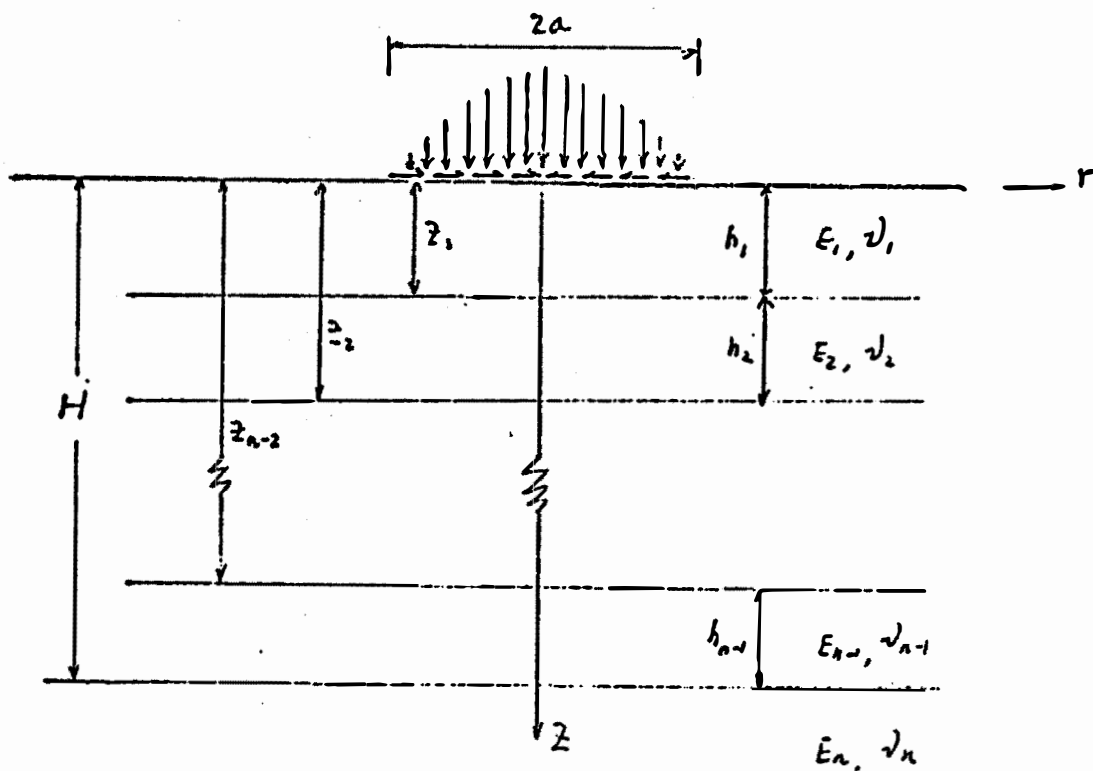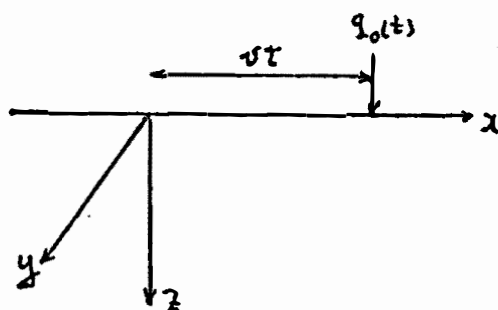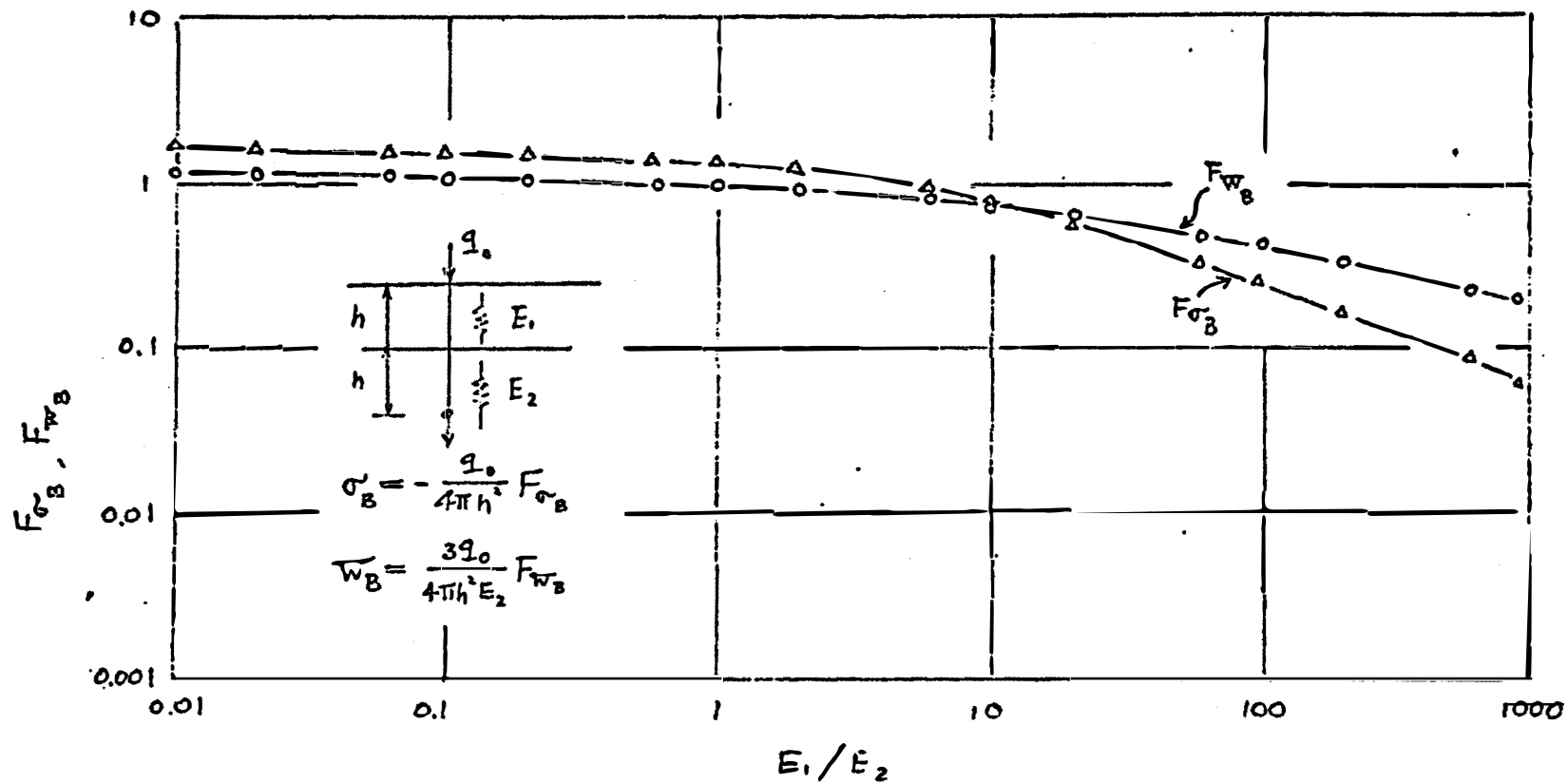
Fig. 1. An n-layer elastic system.

Fig. 2

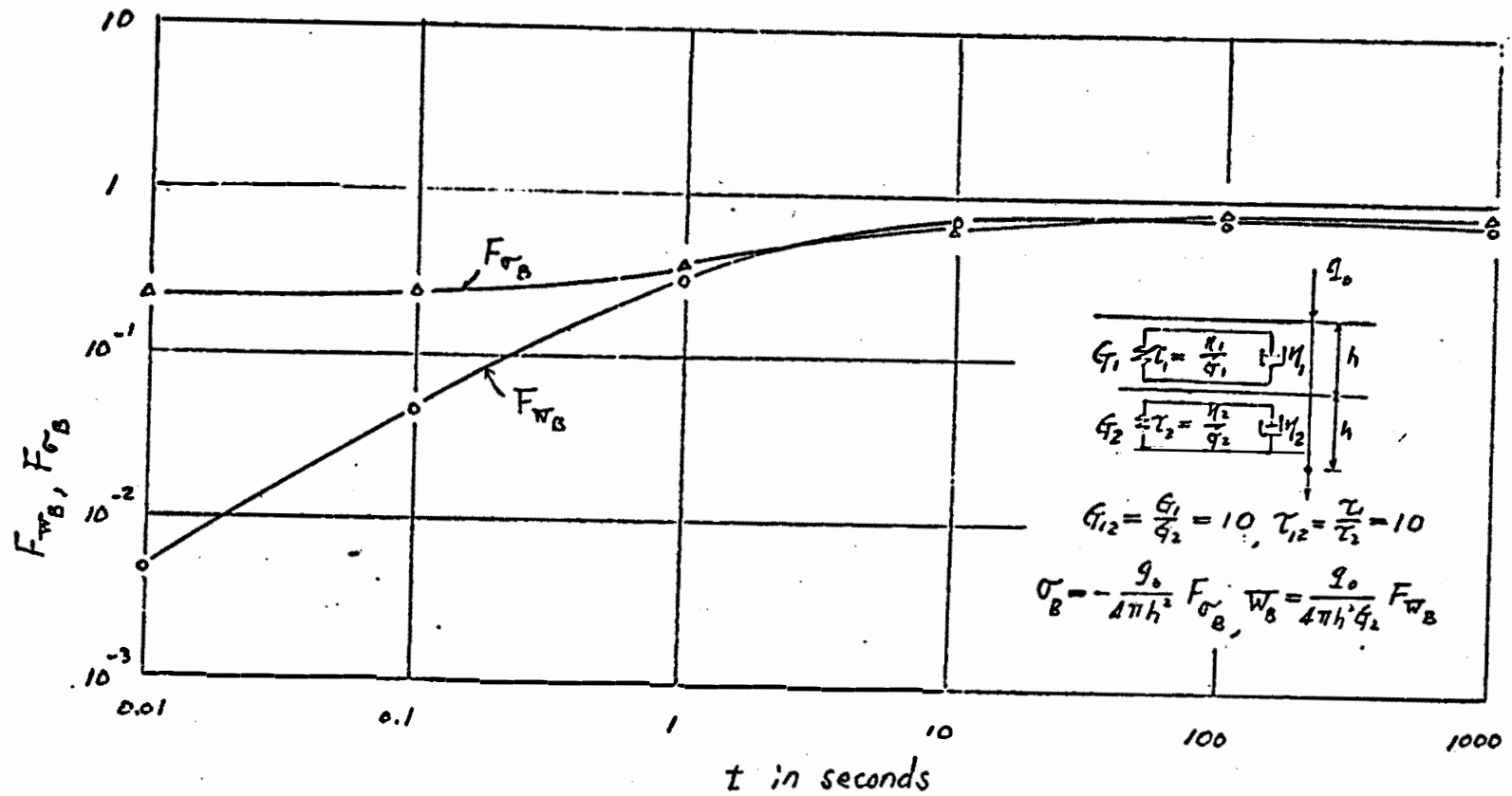Fig. 3. Effect of E_1/E_2 on stress and displacement factors.

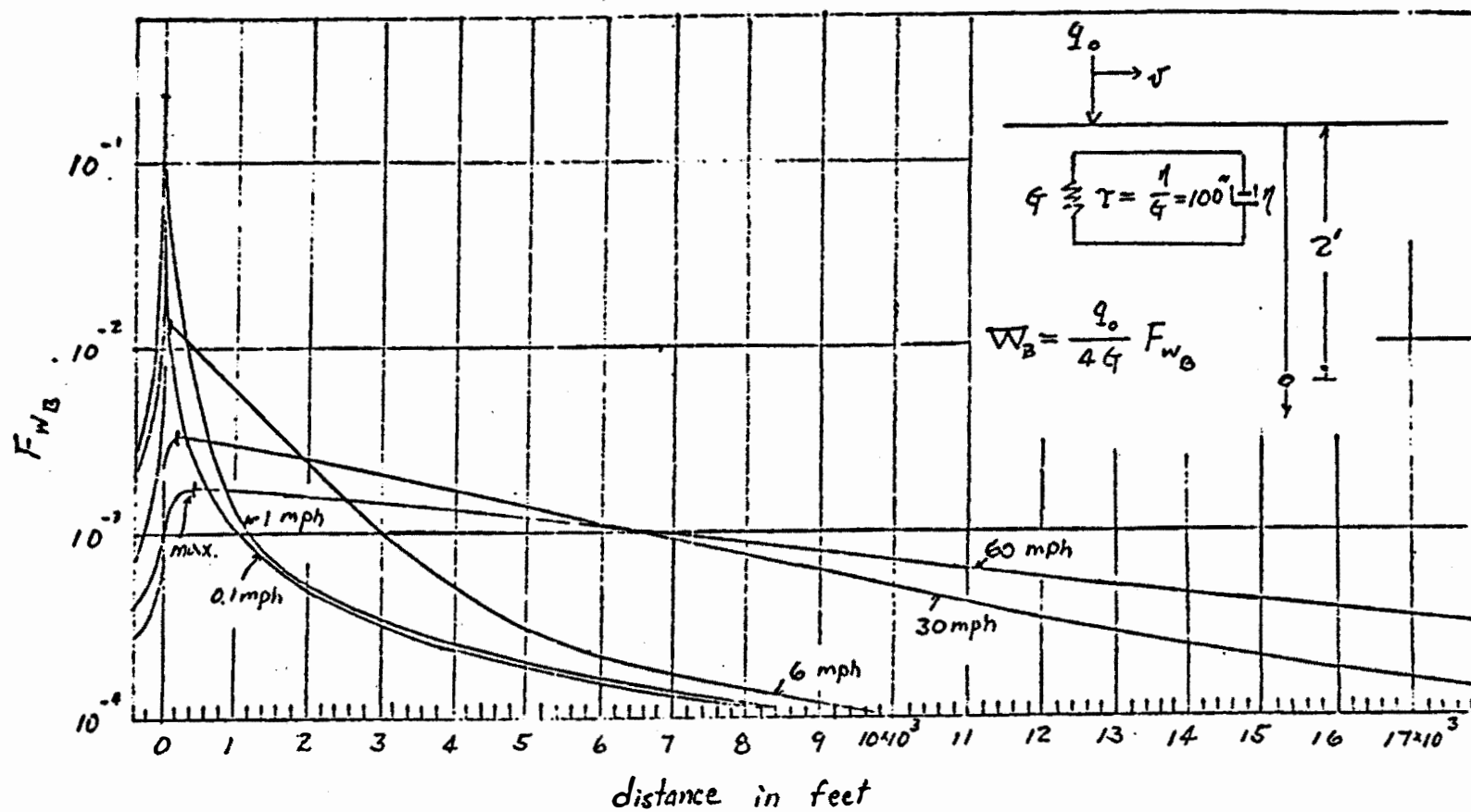**Fig. 4.** Stress and displacement factors under a stationary point load.

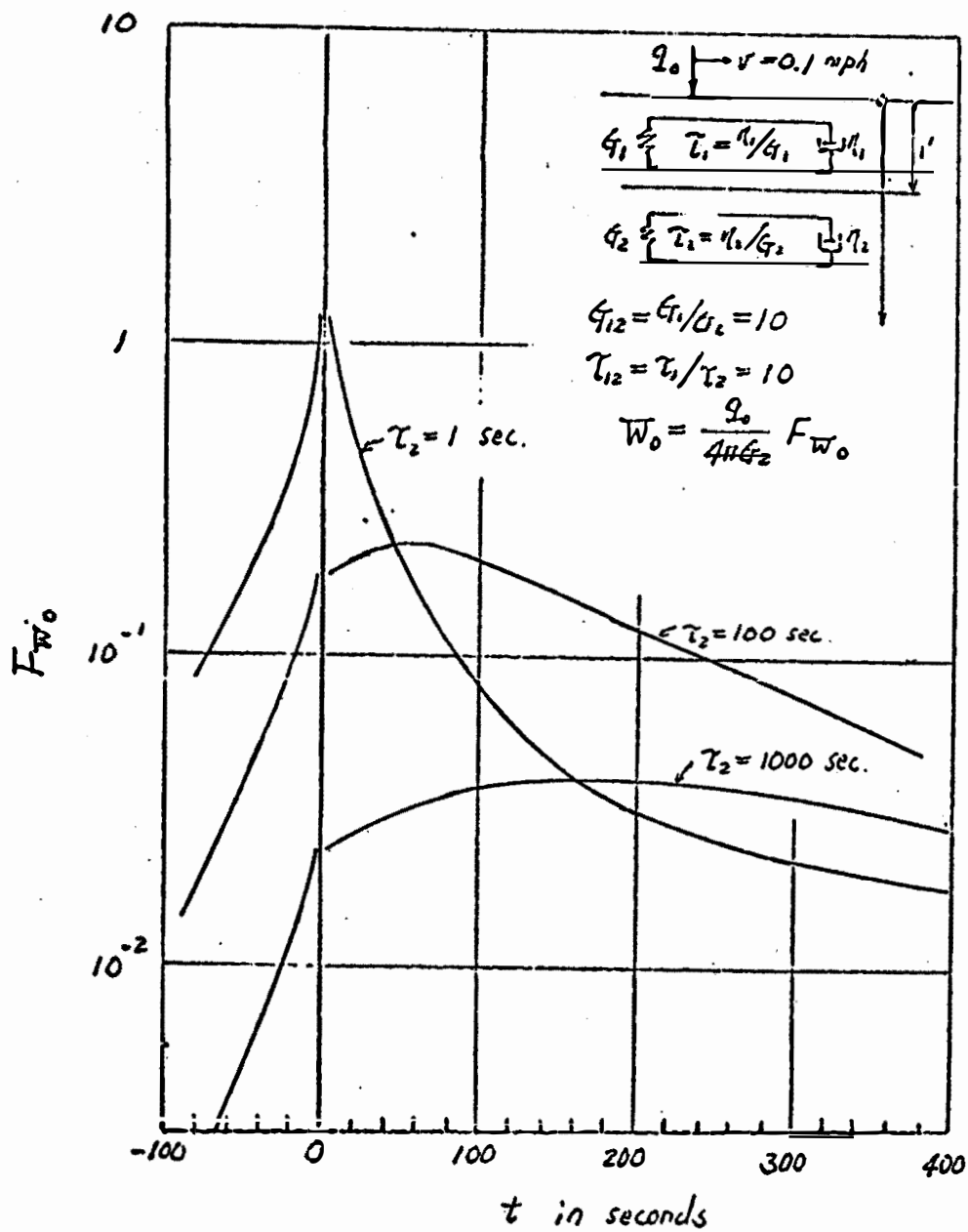Fig. 5. Effects of speed on the displacement factor F_{W_B}

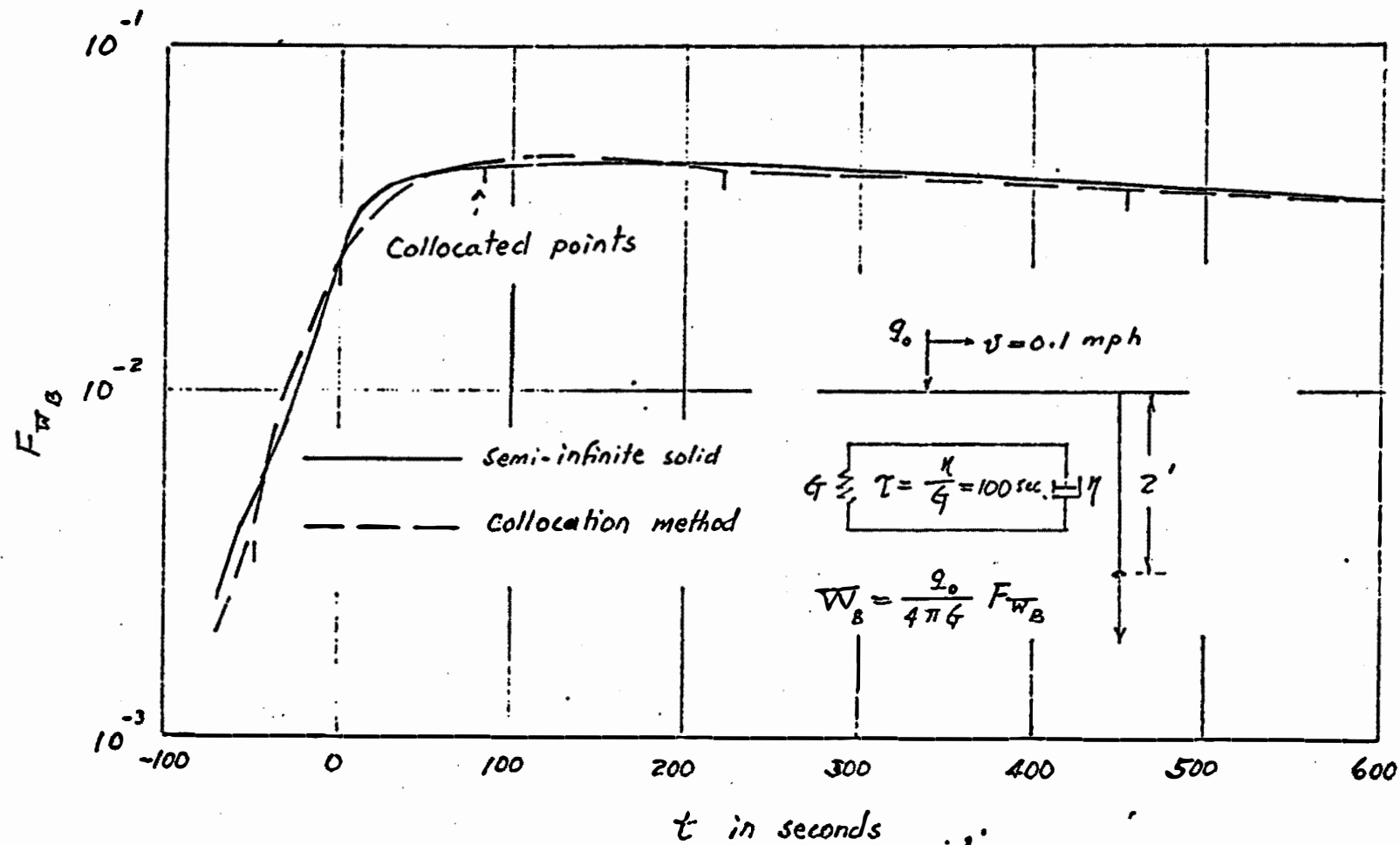Fig. 6. Effect of τ_2 on surface displacement factor F_{W_0}

Fig. 7. Collocation Method - Comparison of results
(displacement factor 2 ft below the surface)

Table 1

Stress Factors in a Two-Layer Incompressible Elastic System
Under a Stationary Point Load, $\sigma_B = \dfrac{-q_o}{2\pi} S_B$

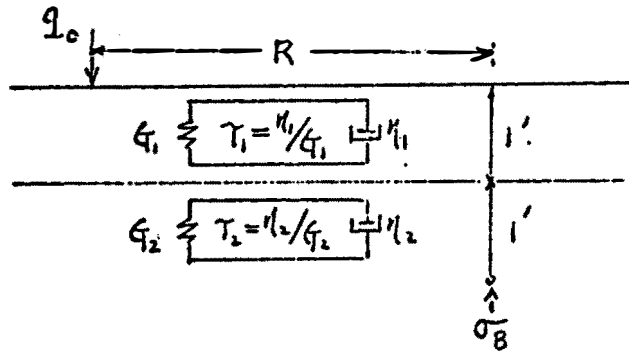| R	S_B	R	S_B	R	S_B	R	S_B	Sketch
0.0	3.91 @ -1	0.4	3.71 @ -1	1.0	2.88 @ -1	1.6	1.99 @ -1	
2.4	1.11 @ -1	3.2	0.57 @ -1	4.0	2.81 @ -2	5.0	1.01 @ -2	
6.0	2.89 @ -3	7.5	-0.95 @ -4	9.0	-3.33 @ -4	13 5	-2.72 @ -5	
15.0	-3.02 @ -5	18.0	-2.00 @ -5	22.0	2.07 @ -6	30.0	1.71 @ -5	

Table 2

Stress Factors S_B in a Two-Layer Incompressible Viscoelastic System Under a Stationary Point Load

$$G_{12} = G_1/G_2 = 10$$
$$\tau_{12} = \tau_1/\tau_2 = 10$$
$$T = t/\tau_2$$

$$\sigma_B = -\frac{q_o}{2\pi} S_B$$

R	Time Factor $T = t/\tau_2$				
	0.01	0.1	1	10	100
0.0	1.29 @ -1	1.35 @ -1	1.82 @ -1	3.49 @ -1	3.91 @ -1
1.0	1.12 @ -1	1.17 @ -1	1.54 @ -1	2.68 @ -1	2.88 @ -1
2.0	0.84 @ -1	0.87 @ -1	1.07 @ -1	1.51 @ -1	1.50 @ -1
5.0	2.68 @ -2	2.67 @ -2	2.49 @ -2	1.13 @ -2	1.01 @ -2
7.5	0.81 @ -2	0.77 @ -2	4.50 @ -3	-0.84 @ -3	-0.95 @ -4
13.5	-3.24 @ -4	-3.66 @ -4	-0.52 @ -3	3.89 @ -5	-2.72 @ -5
22.0	-0.76 @ -4	-0.66 @ -4	-0.51 @ -5	1.00 @ 6	2.07 @ -6
30.0	-3.89 @ -6	-2.96 @ -6	1.61 @ -6	1.21 @ -5	1.71 @ -5

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
U. S. Army Engineer Waterways Experiment Station Vicksburg, Mississippi	Unclassified
	2b. GROUP

3. REPORT TITLE

A GENERAL THEORY OF STRESSES AND DISPLACEMENTS IN ELASTIC AND VISCOELASTIC LAYERED SYSTEMS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Final report

5. AUTHOR(S) (First name, middle initial, last name)

Yu-Tang Chou

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 1969	30	8

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO. 1T061102B52A	Miscellaneous Paper M-69-8
c. Task 01	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	U. S. Army Materiel Command Washington, D. C.

13. ABSTRACT

The analysis of linear viscoelastic layered systems under any axially symmetrical, time-dependent surface traction is presented. Inertial effects are disregarded, and solutions are obtained for the normal, radial, and shear stresses, vertical deflection, and radial displacements at any point within the half space in multilayered systems. Solutions in layered elastic systems first are obtained by using the Love's stress function and the Fourier-Henkel transformation. Solutions in viscoelastic cases then are obtained by using the elastic-viscoelastic correspondence principle, in which the Laplace transformation is applied to replace the time variable with a transformed variable, and thus change the viscoelastic problem into an associated elastic one. The solution of the associated elastic problem, when transformed into the real time variable, will give the desired viscoelastic solution. Sample numerical results are presented. The analysis is an essential step in the development of a rational method of design for flexible pavements, since such pavement systems respond in a markedly time-dependent fashion.

DD FORM **1473** REPLACES DD FORM 1473, 1 JAN 64, WHICH IS NOV 65 OBSOLETE FOR ARMY USE.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Displacement						
Elasticity						
Flexible pavements						
Layered systems						
Stresses						
Viscoelasticity						