US Army Corps of Engineers®

Adaptive Hydraulics (ADH) Control Volume and Mass Conservation

by Gaurav Savant, Charlie Berger and Jennifer N. Tate

PURPOSE: This paper demonstrates that the conservative finite element methods (FEM) are finite volume methods (FVM) with the edge fluxes derived from the interior of the elements. The element is the control volume. A set of algebraic equations is used to force the edge fluxes derived from surrounding elements to be unique.

PREFACE: Berger and Howington (*2002)* demonstrated that the ADH model conserves mass globally and locally, as well as the fact that a control volume exists. This paper further simplifies the mathematics of the FEM implemented in ADH presented to just one dimension (1-D) for ease of understanding. This paper also extends the testing of ADH to large physical scales and long time scales.

INTRODUCTION: In this technical note we will show that the FEM in conservative form 1) has a control volume 2) is locally conservative over the control volume, and 3) the set of equations solved in Adaptive Hydraulics (ADH) force the uniqueness of fluxes on an edge. We will demonstrate this using a Petrov-Galerkin finite element scheme and a One-Dimensional (1D) conservation equation. We will utilize the US Army Corps of Engineers unstructured finite element code Adaptive Hydraulics (ADH) as the numerical code to illustrate the above. A description of ADH is beyond the scope of this paper and the interested reader is referred to the ADH website at http://www.adh.usace.army.mil, or the works by *Tate et.al (2006)* and *Savant et.al (2010).*

One-Dimensional Equation:

Consider the conservative form convection diffusion equation in 1D

$$\frac{\partial c}{\partial t}+\frac{\partial uc}{\partial x}-\frac{\partial}{\partial x}\left(K\frac{\partial c}{\partial x}\right)=0 \tag{1}$$

where c = constituent, t = time, x = distance, u = velocity, and K = elemental constant.

We wish to derive the fluxes at the ends of an element. We will assume a form of u and c for which the flux at any point is $F \equiv uc - K\frac{\partial c}{\partial x}$ within the interior of the element, and u and c are linear within the element . On an element c and u are represented as shown in figure 1. The distribution of c is linear (figure 1). Note that the definition approaches the end points but does not include them (represented by dashed lines in figure 2).

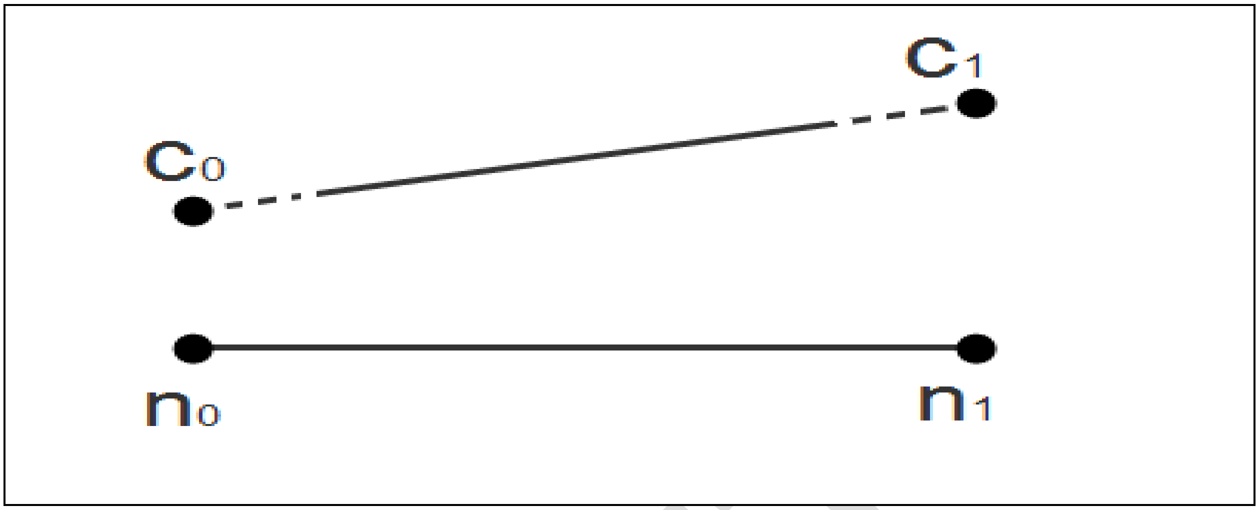

Figure 1: Linear Definition of c

Therefore, the evaluation of $u_h c_h$ (where the subscript h means the numerical approximation of the variable) might look like the representation in figure 2.

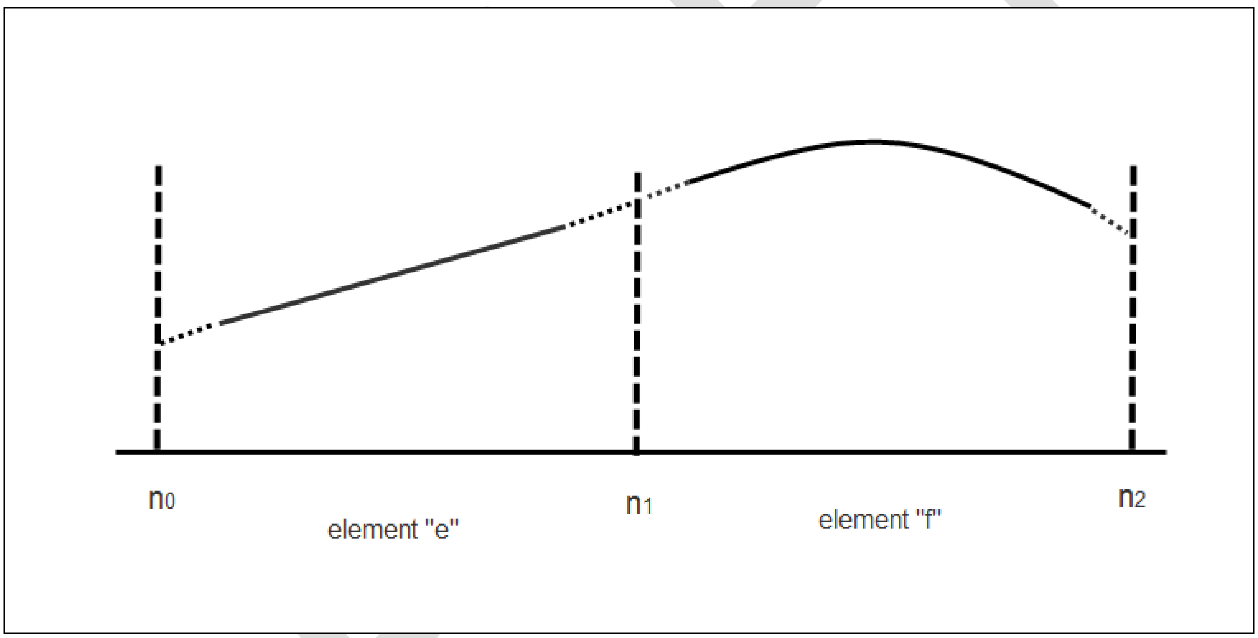

Figure 2: Representation of Variables

We want to derive the flux on each end of the element. If we define our weight function "W" to be a constant value of "1" on the element and zero everywhere else, we can write

$$\int_{\Omega} W_e \left(\frac{\partial c_h}{\partial t} + \frac{\partial u_h c_h}{\partial x} - \frac{\partial}{\partial x} K \frac{\partial c_h}{\partial x} \right) dx = 0 \qquad (2)$$

where Ω is the entire domain, the subscript h indicates approximation within element e, and W_e is the step function on element e as represented in figure 3. We note that W_e is zero (0) outside of the element, therefore the contribution outside the element is also 0. We will hereafter only perform this integration on Ω_e i.e. on the element "e".

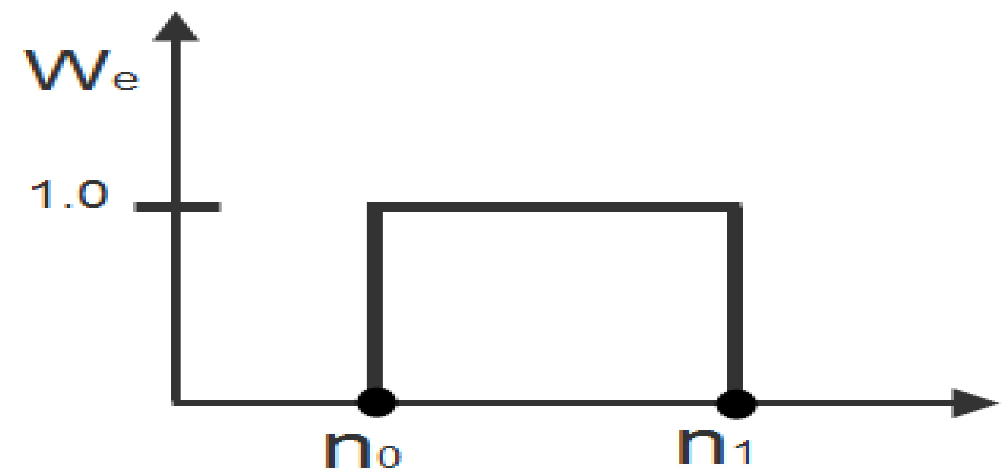

Figure 3: Step Function W_e

Integrating equation 2 by parts we obtain

$$\int_{\Omega_e} \left(W_e \frac{\partial c_h}{\partial t} - \frac{\partial W_e}{\partial x} \left(u_h c_h - K \frac{\partial c_h}{\partial x} \right) \right) dx \equiv -F_1 + F_0 \qquad (3)$$

Where F_0 and F_1 are fluxes at node 0 and 1 respectively. Note that $\frac{\partial W_e}{\partial x} \equiv 0$ and $W_e \equiv 1$ on element e. Therefore we can write

$$\int_{\Omega_e} \frac{\partial c_h}{\partial x} dx \equiv -F_1 + F_0 \qquad (4)$$

Equation 4 is a statement of local mass conservation over element e. Stabilization such as the one utilized in the Streamline-Upwind Petrov-Galerkin (SUPG) (*Hughes and Brooks, 1982*) has no net impact on local mass conservation due to the fact that $\sum \frac{\partial W_e}{\partial x} \equiv 0$ over the entire element. The stabilization involves elemental constants times the gradients of the test function integrated against the discrete equations. Under these conditions local mass conservation is not changed, although the fluxes are modified.

Notice that equation 4 provides the sum of fluxes but we desire the individual flux at each end. We can get this decomposing our weight function into two weight functions as represented in figure 4.

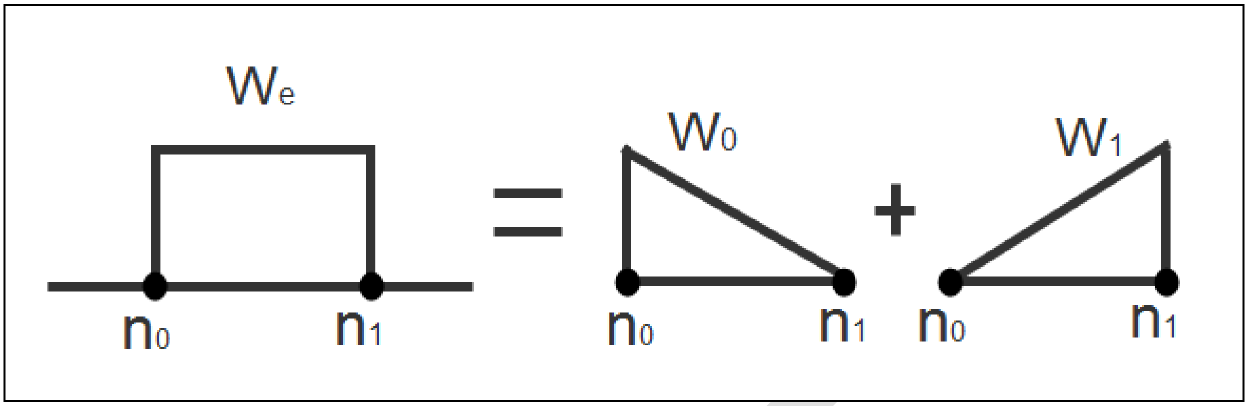

Figure 4: Decomposition of W_e

When we apply these one at a time we get the individual end fluxes.

$$\int_{\Omega_e} \left\{ W_0 \frac{\partial c_h}{\partial t} - \frac{\partial W_0}{\partial x}\left(u_h c_h - K \frac{\partial c_h}{\partial x} \right) \right\} dx \equiv \mathrm{F}_0$$

and (5)

$$\int_{\Omega_e} \left\{ W_1 \frac{\partial c_h}{\partial t} - \frac{\partial W_1}{\partial x}\left(u_h c_h - K \frac{\partial c_h}{\partial x} \right) \right\} dx \equiv -\mathrm{F}_1$$

F_0 and F_1 are fluxes as defined on element "e", hereafter we will refer to them as $\mathrm{F}_0^{\mathrm{e}}$ and $\mathrm{F}_1^{\mathrm{e}}$, where "e" signifies as calculated by element "e".

We are assured of being locally conservative on element "e". However, we are not guaranteed that the elements surrounding element "e" will calculate the same flux at the same node. If we consider an element "f" adjacent to "e" (figure 5) we get the following definition of $\mathrm{F}_1^{\mathrm{f}}$

$$\int_{\Omega_f} \left\{ W_1 \frac{\partial c_h}{\partial t} - \frac{\partial W_1}{\partial x}\left(u_h c_h - K \frac{\partial c_h}{\partial x} \right) \right\} dx \equiv \mathrm{F}_1^{\mathrm{f}} \tag{6}$$

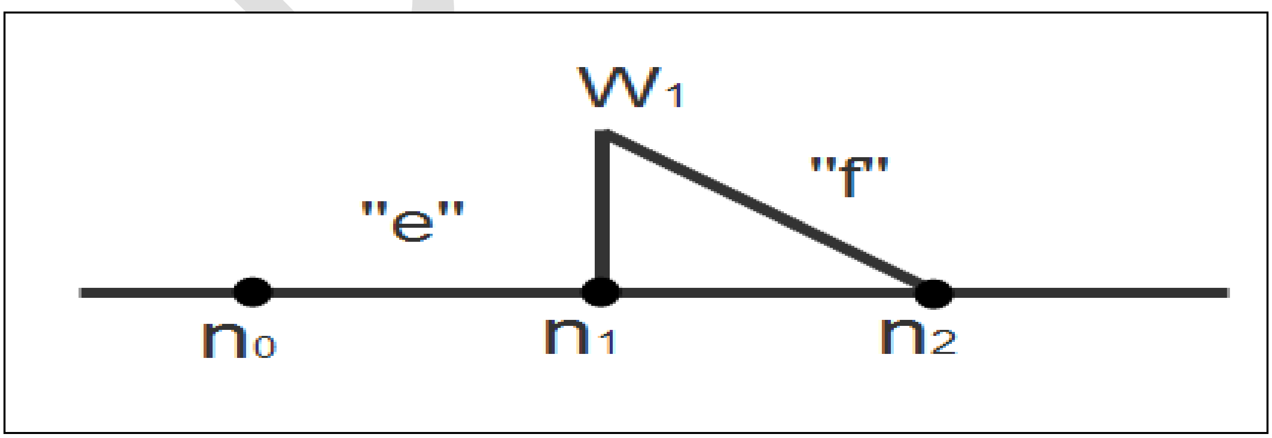

Figure 5: Element f with the decomposed $W_{1.}$

If we add the two definitions of F_1 from elements "e" and "f" we get

$$\int_{\Omega_e+\Omega_f} \left\{ W_1 \frac{\partial c_h}{\partial t} - \frac{\partial W_1}{\partial x}\left(u_h c_h - K\frac{\partial c_h}{\partial x}\right)\right\} dx \equiv \mathrm{F}_1^{\mathrm{f}} - \mathrm{F}_1^{\mathrm{e}} \tag{7}$$

We require $\mathrm{F}_1^{\mathrm{e}}$ and $\mathrm{F}_1^{\mathrm{f}}$ to be identical. Therefore we solve for c_{h} such that this holds true as follows

$$\int_{\Omega_e+\Omega_f} \left\{ W_1 \frac{\partial c_h}{\partial t} - \frac{\partial W_1}{\partial x}\left(u_h c_h - K\frac{\partial c_h}{\partial x}\right)\right\} dx = 0 \tag{8}$$

Attention is now drawn to some salient points indicated by equation 8:

1) We have unique fluxes at ends of the elements. If we compute the temporal and convective mass rate of change (to include diffusive terms if utilized) around an element, they will balance. This is a direct result of the weight function "W" around an element being a constant value of "1",
2) We are locally conservative; that is, we have a control volume representing each individual element, and
3) While there are some additional considerations for application to higher dimension, the approach and result are the same. The mathematical equations hold true for two-dimensional (2D) and three-dimensional (3D) cases as well, with or without stabilization.

Application: We now apply the 2D ADH code to a practical large scale test case involving Lake Ponchartrain (Louisiana) without wetting-drying to demonstrate local as well as global mass conservation. The domain is enclosed to make the accounting obvious. We will do this in two ways. We will demonstrate local mass conservation on two typical elements over a few days during which a single wind event takes place. The second test is a year-long simulation in which the winds are applied cyclically every day. Water and solute mass will be shown to remain conserved over the entire domain during this extended period. Figure 6 illustrates the finite element ADH mesh and the direction of the wind forcing for the first test.

The ADH simulation was forced by applying a uniform wind over the domain for one (1) day and thereafter letting the domain slosh. The domain bottom elevation was specified as zero (0) meters and had an initial water depth of three (3) meters everywhere. An initial constituent concentration of one (1) mg/l was specified everywhere in the domain.

The simulation was run for a period of five (5) days at time steps of six hundred (600) seconds. The global mass was computed every day and the local mass conservation was checked at every successive time step. Table 1 lists model behavior in terms of global mass conservation for the first test.

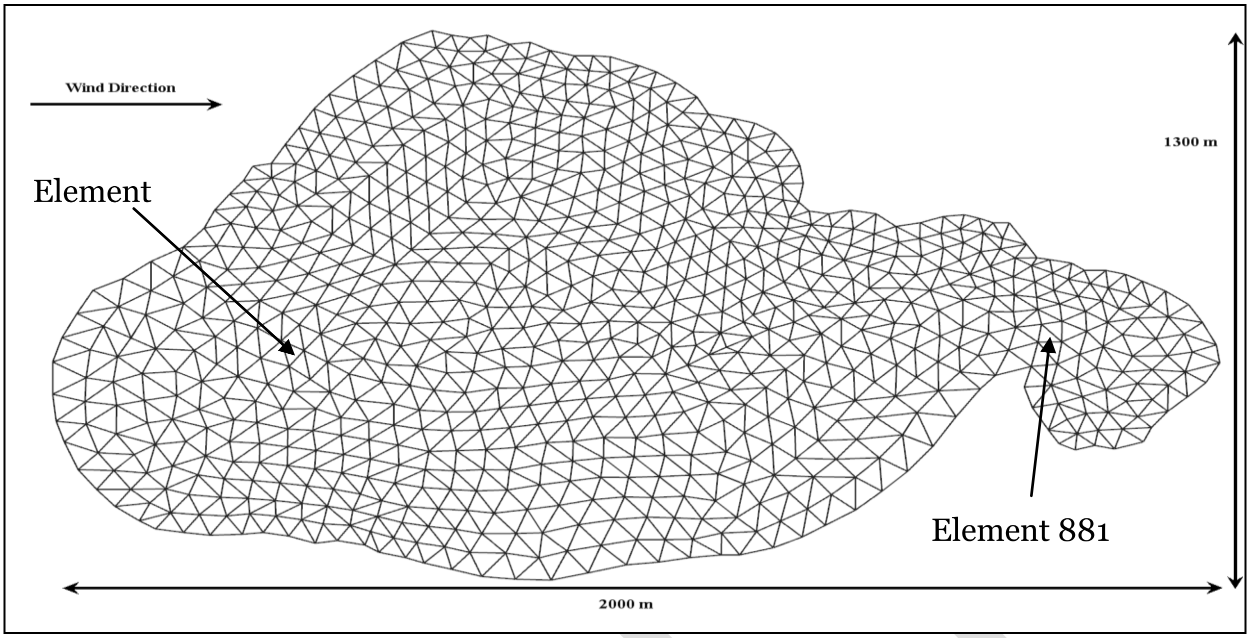

Figure 6: ADH mesh for Lake Pontchartrain showing dimensions and wind direction

Local mass conservation is guaranteed, as described by equation 4. To illustrate this in the test problem the authors picked two (2) random elements inside of the domain. These were elements eight hundred eighty-one (881) and seventy-seven (77). Also as stated earlier, within the control volume (i.e. the element) the temporal and convective fluxes balance. This is illustrated in figures 7 and 8.

The second test consisted of a year long run forced by winds alternating every day from the positive "+" to the negative "-" x- direction. Table 2 lists model behavior in terms of global solute and fluid mass conservation. Figure 9 shows the local solute mass conservation for element 77, element 881 shows similar behavior. For the sake of clarity we only show the first 100 days of simulation.

Time, Days	Constituent Mass
0.00	5432580.570000
1.00	5432580.570000
2.00	5432580.570000
3.00	5432580.570000
4.00	5432580.570000
5.00	5432580.570000

Table 1: Global Constituent Mass Conservation for test 1

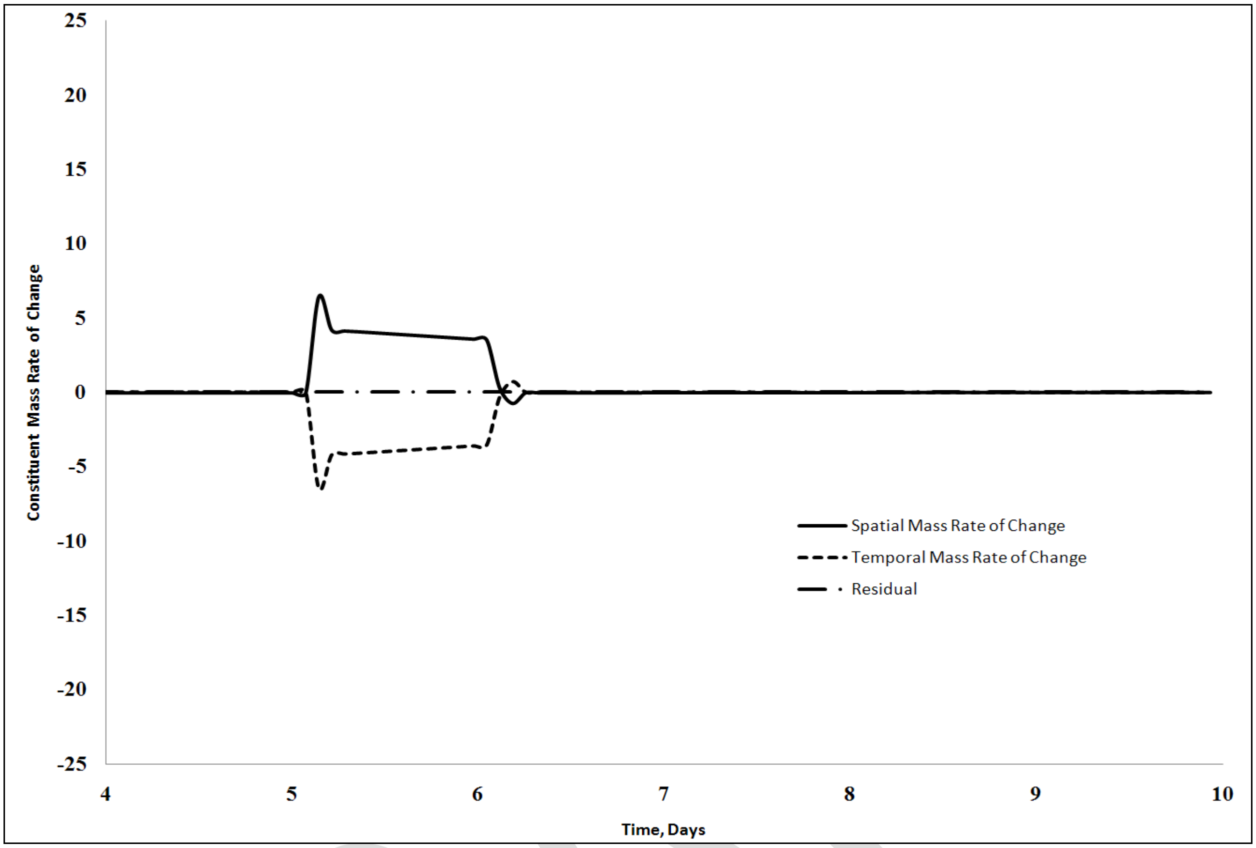

Figure 7: Local Mass Conservation for Element 881.

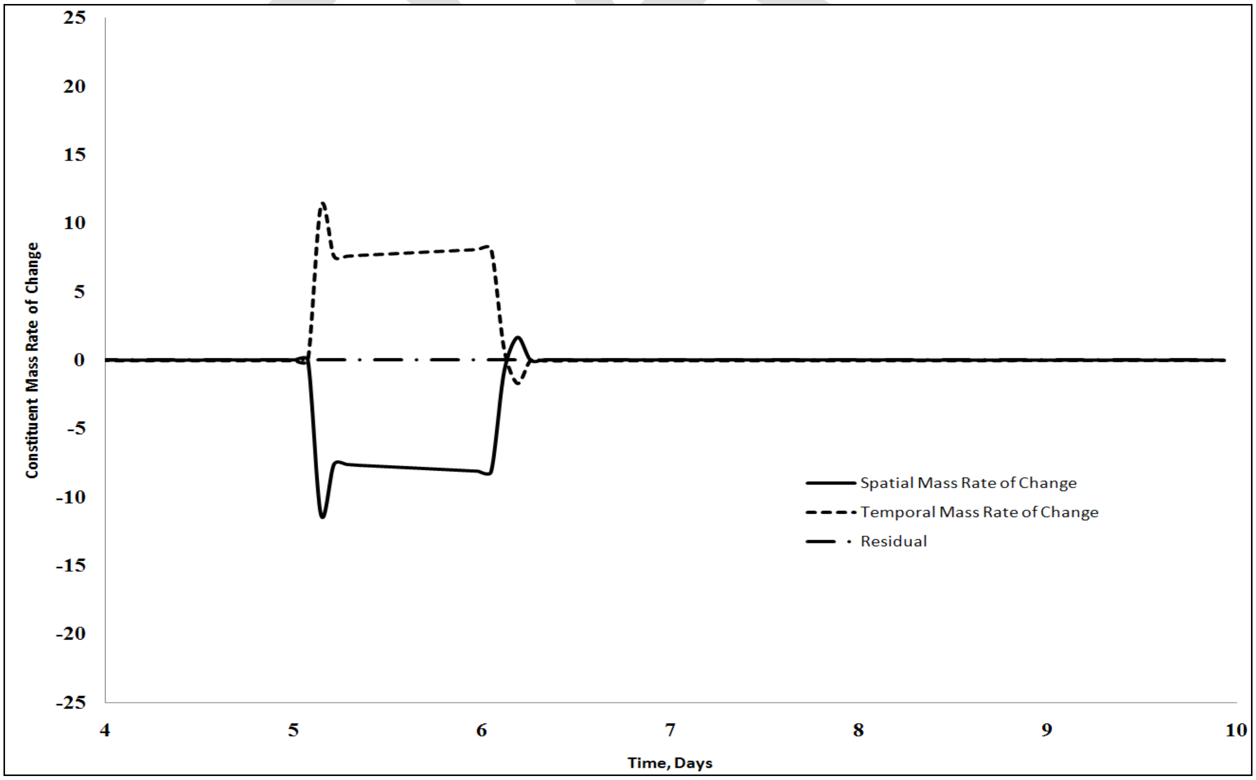

Figure 8: Local Mass Conservation for Element 77.

Time, Days	Constituent Mass	Fluid Mass	Constituent Mass Error	Fluid Mass Error
0	5432580.570000	5432580570.000000	0.000000	0
5	5432580.570000	5432580569.999990	0.000000	1.75547E-13
10	5432580.570000	5432580569.999990	0.000000	1.75547E-13
15	5432580.570000	5432580569.999990	0.000000	1.75547E-13
20	5432580.570000	5432580570.000010	0.000000	-1.75547E-13
50	5432580.570000	5432580570.000000	0.000000	0
100	5432580.570000	5432580570.000010	0.000000	-1.75547E-13
150	5432580.570000	5432580570.000000	0.000000	0
200	5432580.570000	5432580569.999990	0.000000	1.75547E-13
250	5432580.570000	5432580569.999990	0.000000	1.75547E-13
300	5432580.570000	5432580570.000000	0.000000	0
350	5432580.570000	5432580570.000030	0.000000	-5.44196E-13
365	5432580.570000	5432580570.000020	0.000000	-3.68649E-13

Table 2: Global Constituent and Fluid Mass Conservation for 365 day test

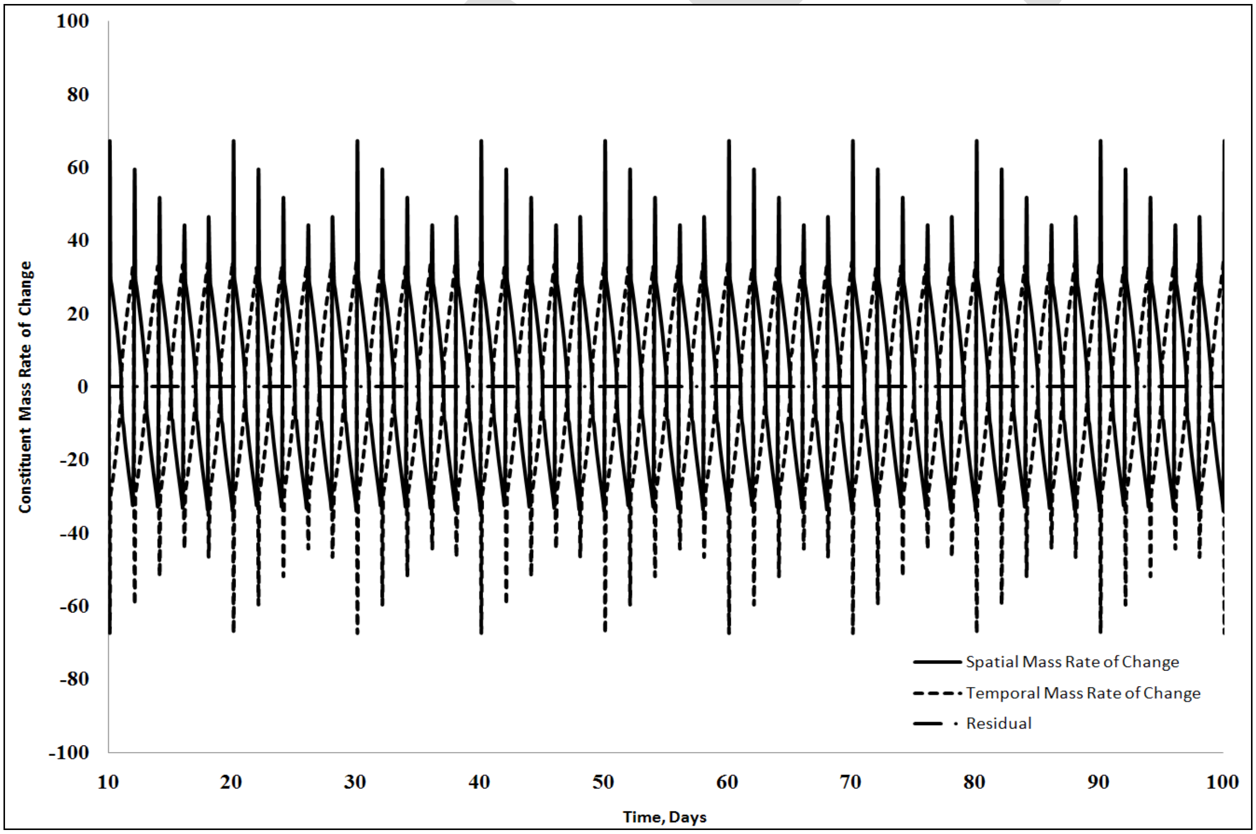

Figure 9: Long Term Local Solute Mass Conservation for element 77.

SUMMARY: This technical note provides mathematical proof that the Finite Element Method (FEM) are Finite Volume Methods (FVM) with consistent edge fluxes derived from the interior of the elements.

This technical note also demonstrates that the Adaptive Hydraulics model is locally and globally mass conservative for fluid and solute transport, this is demonstrated on a large scale test problem over long time scales. ADH is shown to conserve fluid and solute mass to machive precision.

ADDITIONAL INFORMATION: For additional information, contact Dr. G. Savant, or Dr. R.C. Berger at Gaurav.Savant@usace.army.mil or Charlie.R.Berger@usace.army.mil respectively. This CHETN should be cited as follows:

> Savant, G., Berger, R.C. and Tate J.N. (2012). *Adaptive Hydraulics (ADH) Control Volume and Mass Conservation. Coastal and Hydraulics Engineering Technical Note,* ERDC/CHL CHETN-XI-16. Vicksburg, MS: U.S. Army Engineer Research and Development Center. An electronic copy of this CHETN is available from *http://chl.erdc.usace.army.mil/chetn*.

REFERENCES:

Berger, R. C. and Howington, S. E. (2002). "Discrete Fluxes and Mass Balance in Finite Elements," ASCE Journal of Hydraulic Engineering, Vol 128, No. 1 (Jan) 87-92.

Hughes, T. J. R., and Brooks, A. N. (1982). ''*A Theoretical Framework for Petrov-Galerkin Methods with Discontinuous Weighting Functions: Applications to the Streamline-Upwind Procedure.*'' Finite Elements in Fluids, Gallagher, R. H., ed., Vol. 4, Wiley, London, 47– 64.

Savant, Gaurav; Berger, Charlie; McAlpin, Tate O. and Tate, Jennifer N. (2011) "*An Efficient Implicit Finite Element Hydrodynamic Model for Dam and Levee Breach".* Journal of Hydraulic Engineering,Vol. 137, No. 9 (Sep) 1005-1018.

Tate, J. N., Berger, R. C., and Stockstill, R. L. (2005). "Refinement indicator for mesh adaption in shallow-water modeling." *Journal of Hydraulic Engineering*, 132(8), 854-857.