Waterways Experiment Sta
New England Division
Corps of Engineers, U.S. Army

DEPARTMENT OF THE ARMY CORPS OF ENGINEERS

BEACH EROSION BOARD
OFFICE OF THE CHIEF OF ENGINEERS

THE RAYLEIGH DISK AS A WAVE DIRECTION INDICATOR

TECHNICAL MEMORANDUM NO. 18

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PREFACE

The principles of operation of the Rayleigh disk in stream flow and wave systems are presented and its erratic behavior when used as a wave direction gage is discussed.

The principle of operation of the disk is explained in accordance with Konig's theory of the Rayleigh Disk, given in the appendix. The erratic behavior of the disk when used as a wave direction indicator under natural conditions was noted during the use of two units operated for about six months at Long Branch, New Jersey, and Huntington Beach, California.

This paper was prepared by Jay V. Hall, Jr., Chief, Field Research Section, Engineering and Research Branch, Beach Erosion Board, Corps of Engineers.

The field installations of the instruments and observations were made by the Beach Erosion Boards' Field Research Groups under the direction of Mr. William J. Herron and Mr. Robert L. Harris.

Konig's theory of the Rayleigh Disk was translated from the German by Mrs. Karin Melchior, Studies and Reports Branch, Beach Erosion Board, Corps of Engineers.

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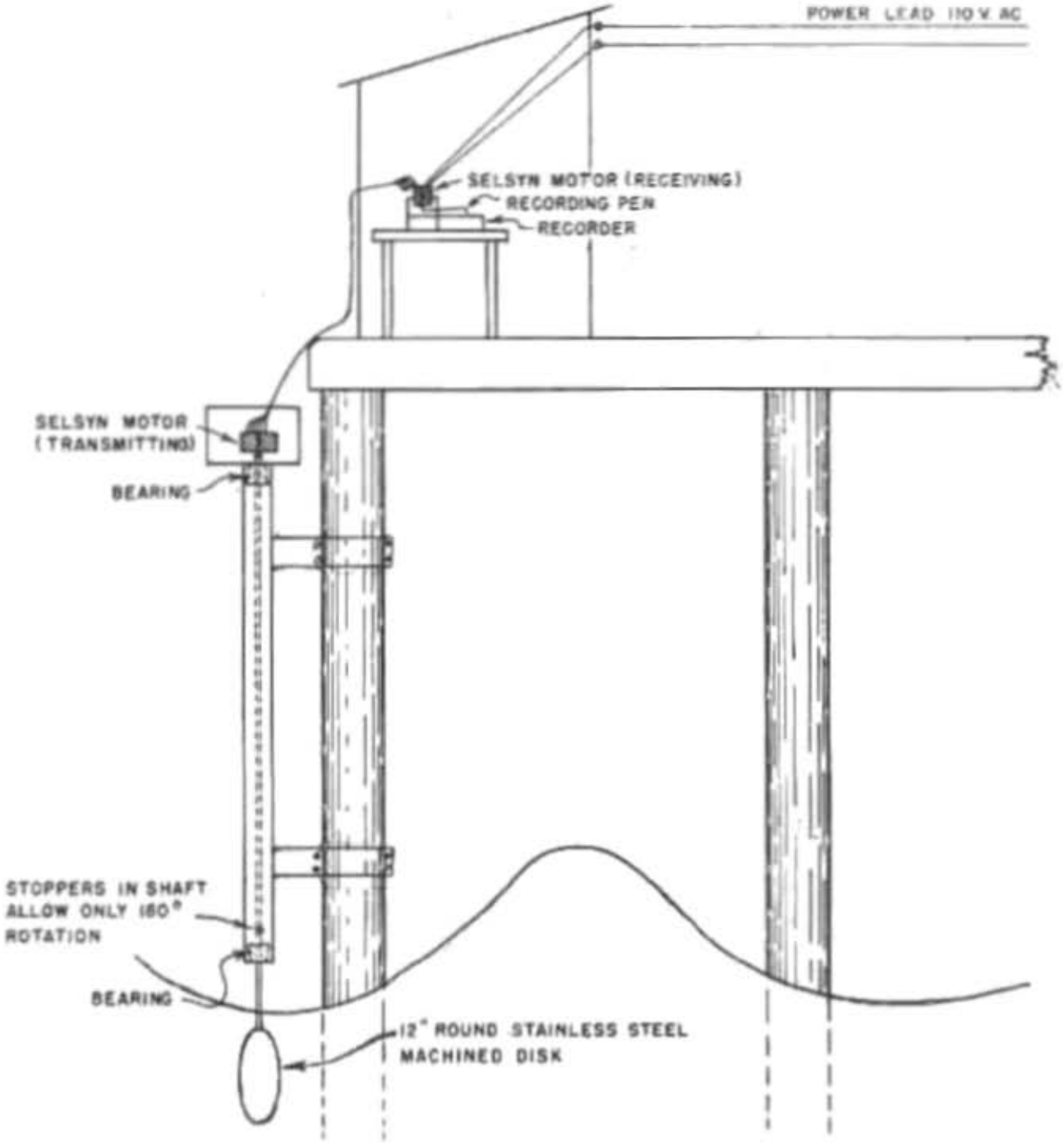


RAYLEIGH DISK WAVE DIRECTION GAGE—
PIER MOUNTED

PLATE I

THE RAYLEIGH DISK AS A WAVE DIRECTION INDICATOR

Introduction

The study of the character of the movement of sands along any shore line requires information on the characteristics of the waves which produce the movement. The quantities which must be measured for a comprehensive study include the wave height, wave period and wave direction. Automatic recording wave gages of various types are available for the measurement of wave height and period but no satisfactory means of observing and recording wave direction has been developed.

In an attempt to develop a satisfactory wave direction indicator various investigators have employed a Rayleigh Disk with inconclusive results.

The study was made to evaluate the possibilities of the Rayleigh Disk for this application.

Principles of Operation of the Rayleigh Disk

Walter Konig developed the theory of the Rayleigh Disk by studying the pressures resulting from the flow around an ellipsoid of revolution. In the following paragraphs the behavior of an ellipsoid and a disk in steady and alternating flow will be explained by Konig's theory.

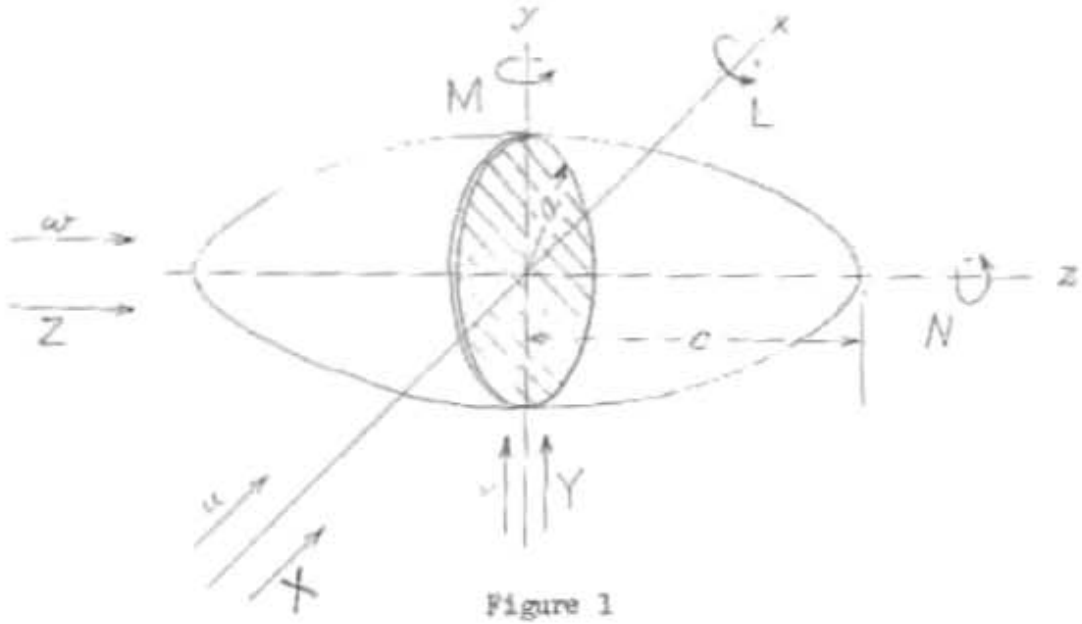


Figure 1

Referring to Figure 1 let

$x, y, \& z$ = the coordinate axes of the ellipsoid

$L, M, \& N$ = the torques acting around the x, y, and z axes respectively

$X, Y, \& Z$ = the forces acting along the x, y, and z axes respectively

$u, v, \& w$ = the velocity components along the x, y, and z axes respectively

$a \; b \; c$ = the semi-minor and semi-major axes of the ellipsoid.

Konig assumed potential flow of a perfect fluid and determined the torques L, M, and N about the x, y, and z axes respectively (Figure 1) produced by the fluid pressures acting on the ellipsoid and the forces X, Y, and Z acting along their respective axes. See Appendix. His analysis shows that for flow in the x-z plane all of the torques except M are equal to zero.

Konig's expression for the torque exerted on the ellipsoid of revolution is:

$$M = 4\pi\mu u w \frac{a^2}{(2-A)(2-C)} \frac{a^2-c^2}{c^2} \int_{-c}^{+c} \frac{(c^2-z^2)z^2}{c^4+(a^2-c^2)z^2} dz \qquad \ldots\ldots (1)$$

where M is the torque acting around the y axes

μ is the density of the fluid

$u \; \& \; w$ are velocity components along the x and z axes, and A, C

are functions involving the dimensions of the ellipsoid only.

By reducing the semi-major axis of the ellipsoid, "c", until it becomes smaller than "a" the ellipsoid approximates a disk. For this case Konig found that the torque exerted on the disk is,

$$M = \frac{8}{3} \mu a^3 u w \left(1 - \frac{\pi^2 - 8}{2\pi} \frac{c}{a}\right) \qquad \ldots\ldots (2)$$

It can be seen therefore that if the disk is arranged in either of the positions shown in Figures II and III, "u" or "w" will become zero, the torque M will become zero, and the disk will be in equilibrium.

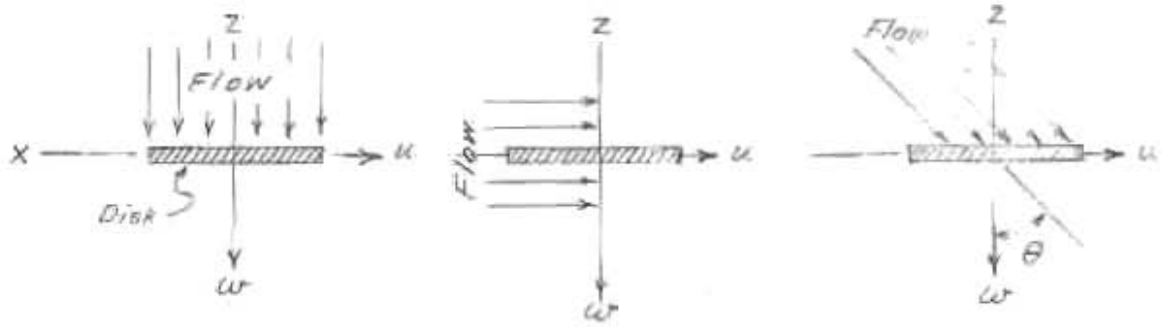


Figure II Figure III Figure IV

If however, the disk is in some intermediate position, for example, Figure IV, both components "u" and "w" exert a torque on the disk.

Figure II represents a condition of stable equilibrium and Figure III one of unstable equilibrium. It can be seen that if the disk were placed in a perfect fluid under potential flow conditions as shown in Figure III it would remain in that position. It can be seen further however, that if the disk was moved slightly from this position it would take the position shown in Figure II. Under natural conditions there usually will be enough turbulence in the flow to prevent the disk from assuming the unstable position shown in Figure III. Consideration of the torques defined by equation 2 shows that the disk will align itself normal to the flow direction, i.e., the position for which M equals zero, and therefore a freely supported disk can be employed to indicate direction of a steady flow.

If we assume that the velocity of flow is reversing, but otherwise is identical in magnitude and follows the same stream lines, the disk will behave in the same manner as in unidirectional flow. This is easily seen since the disk is symetrical about its axes of rotation and the direction and magnitude of the velocity in one direction is the mirror effect of that in the opposite direction.

Behavior of the Rayleigh Disk in a Single Wave System

The use of the Rayleigh Disk to determine the direction of travel of a single wave system appears to be possible if it can be shown that the internal flow in the waves of the system may be resolved into a current flowing along the line of the direction of travel of the wave. An examination of the characteristics of the wave of oscillation shows that the water particles revolve in a vertical plane having the direction of the line of propagation of the wave. If the effect of the vertical components of the wave orbital motion can be eliminated, a disk should align itself normal to the direction of the propagation of the wave; this condition is satisfied by a disk mounted in a vertical position and free to move only

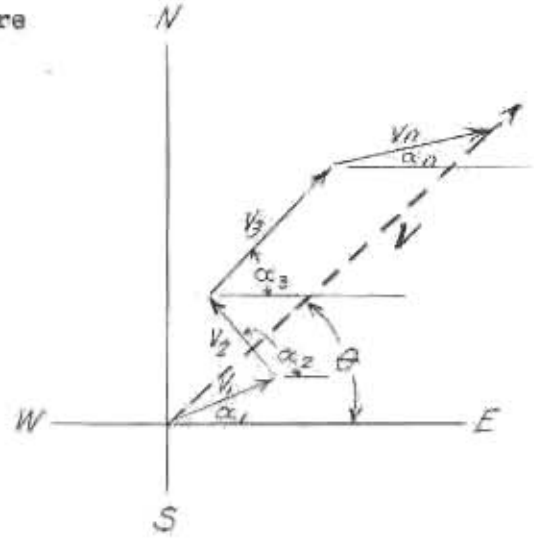
about a vertical axis. Although the horizontal components during the passage of a wave act in diametrically opposite directions as in alternating flow, one produces the mirror effect of the other on the disk. In view of the above it can be said for all practical purposes that the oscillatory currents of a single wave system will act as a single current which will orient the Rayleigh Disk normal to the direction of wave motion.

Behavior of the Rayleigh Disk in a Multiple Wave System

Unfortunately the problem of determining wave direction cannot be confined to the consideration of a single wave system since such a system seldom if ever, is found to exist in the open sea. It will be shown that when two or more wave systems approach a given point from different directions the horizontal velocity components combine vectorially to produce a resultant current of variable direction and magnitude to which the disk will orient itself.

In considering the general problem of "n" trains of waves of different height and period, approaching a given point, from different directions and displaced in phase, all current directions and hence disk positions are referred to an orbitrary datum line.

In the following figure



Let WE be the datum line from which all angles are reckoned

$\alpha_1, \alpha_2, \alpha_3, \alpha_n$ = the angles of approach of n wave systems

V_1, V_2, V_3, V_n = the instantaneous horizontal velocities of water motion in each system of waves

V = the resultant instantaneous horizontal velocity of all wave systems considered

θ = the angle V makes with the datum line.

The resolution of vector velocities shows that the component of the resultant along the line NS in the above figure is:

$$V \sin\theta = \sum_{1}^{n} V_n \sin\alpha_n = V_1 \sin\alpha_1 + V_2 \sin\alpha_2 + V_3 \sin\alpha_3 \quad + V_n \sin\alpha_n \quad \ldots\ldots (3)$$

In like manner the component of the resultant along the line WE is

$$V \cos\theta = \sum_{1}^{n} V_n \cos\alpha_n = V_1 \cos\alpha + V_2 \cos\alpha_2 + V_3 \cos\alpha_3 \quad + V_n \cos\alpha_n \quad \ldots\ldots (4)$$

Dividing equation 3 by 4

$$\tan\theta = \frac{V_1 \sin\alpha_1 + V_2 \sin\alpha_2 + V_3 \sin\alpha_3 \quad \ldots \quad V_n \sin\alpha_n}{V_1 \cos\alpha_1 + V_2 \cos\alpha_2 + V_3 \cos\alpha_3 \quad \ldots \quad V_n \cos\alpha_n} \quad \ldots\ldots (5)$$

Then, in general

$$\tan\theta = \frac{\sum_{1}^{n} V_n \sin\alpha_n}{\sum_{1}^{n} V_n \cos\alpha_n} \quad \ldots\ldots\ldots (6)$$

The expression for the instantaneous horizontal velocity of water motion at any depth z due to a surface wave with a given period and height is

$$V = \left(\frac{\pi H}{T} e^{\frac{-2\pi z}{5.12 T^2}}\right) \sin \frac{2\pi(t-E)}{T} \quad \ldots\ldots\ldots (7)$$

Where

H = the height of the wave in feet

T = the period of the wave in seconds

z = the depth of the water in feet

E = the phase displacement = $\frac{\Delta}{360} T$ in seconds.

t = the time in seconds

combining equations 6 and 7 results in the following general expression

$$\tan\theta = \frac{\sum_{1}^{n} \left[\left(\frac{\pi H_n}{T_n} e^{\frac{-2\pi z}{5.12 T^2}}\right) \sin \frac{2\pi(t-E_n)}{T_n}\right] \sin\alpha_n}{\sum_{1}^{n} \left[\left(\frac{\pi H_n}{T_n} e^{\frac{-2\pi z}{5.12 T^2}}\right) \cos \frac{2\pi(t-E_n)}{T_n}\right] \cos\alpha_n} \quad \ldots\ldots\ldots (8)$$

If it is assumed that the disk is to be used at the water surface, z = 0 then $e^{\frac{-2\pi z}{5.12 T^2}} = 1$ and the general expression (8) takes the following special form.

$$\tan\theta = \frac{\sum_{1}^{n}\left[\frac{\pi H_n}{T_n}\sin\frac{2\pi(t-\varepsilon_n)}{T_n}\right]\sin\alpha_n}{\sum_{1}^{n}\left[\left(\frac{\pi H_n}{T_n}\sin\frac{2\pi(t-\varepsilon_n)}{T_n}\right]\cos\alpha_n} \qquad \text{---------(9)}$$

This resultant current is that which controls the position assumed by the disk.

To further clarify the behavior of the disk when acted on by two wave trains of dissimilar period and height, shifted in phase, a computation of disk positions resulting from composition of two hypothetical wave systems follows:

The characteristics for the two hypothetical wave systems are:

H_1 = 12 feet	H_2 = 6 feet
T_1 = 16 seconds	T_2 = 8 seconds
$\alpha_1 = 75^\circ$	$\alpha_2 = 15^\circ$
Phase displacement $\Delta = 40^\circ$	
$\varepsilon_1 = 0$	$\varepsilon_2 = \frac{40}{360} T_1 = 1.78$ seconds

Using equation 9 and substituting the values selected for the two hypothetical wave systems, computations were made for the angle θ, or disk position reckoned from the line EW for 0.25 second intervals from 0 to 20.00 seconds. The results of the computations are given in Table I.

It is seen that during one complete wave cycle the disk rotates through 270 degrees. An even wider variation in rotation is possible with other combinations of height and phase. It is obvious that a set of forces can combine to produce a full rotation of 360 degrees. Although the analysis given in this paper is based on sinusoidal waves it is believed to be applicable to ocean wave situations.

Field Test Employing Rayleigh Disk

Studies to verify the predicted behavior of the Rayleigh Disk in multiple systems of waves were made at Long Branch, New Jersey; Huntington Beach, California; and El Segundo, California; with an instrument designed by the Board. The unit employed includes a 12-inch circular disk mounted to rotate freely about a vertical axis and a pair of selsyn motors to define the motion of the disk. The instrument is shown schematically on Plate I.

If the disk were not restrained the recording selsyn could turn through 360 degrees and break the recording pen; therefore, the motion of the disk was confined to the 180 degree arc covering all directions from down coast waves through on coast waves to up coast waves.

TABLE I

Reaction of Disk to Resultant Horizontal Orbital Forces of Two Hypothetical Waves

Time	θ	Time	θ	Time	θ	Time	θ	Time	θ
0.00	195°	4.00	45°	8.00	195°	12.00	314°	16.00	195°
0.25	189°	4.25	46°	8.25	201°	12.25	312°	16.25	189°
0.50	182°	4.50	47°	8.50	206°	12.50	308°	16.50	182°
0.75	171°	4.75	49°	8.75	212°	12.75	302°	16.75	171°
1.00	154°	5.00	52°	9.00	218°	13.00	293°	17.00	154°
1.25	127°	5.25	58°	9.25	228°	13.25	282°	17.25	127°
1.50	97°	5.50	65°	9.50	239°	13.50	270°	17.50	97°
1.75	78°	5.75	76°	9.75	253°	13.75	257°	17.75	78°
2.00	64°	6.00	88°	10.00	268°	14.00	244°	18.00	64°
2.25	58°	6.25	117°	10.25	283°	14.25	234°	18.25	58°
2.50	54°	6.50	138°	10.50	295°	14.50	226°	18.50	54°
2.75	51°	6.75	154°	10.75	303°	14.75	219°	18.75	51°
3.00	47°	7.00	167°	11.00	309°	15.00	213°	19.00	47°
3.25	46°	7.25	176°	11.25	312°	15.25	208°	19.25	46°
3.50	46°	7.50	184°	11.50	314°	15.50	205°	19.50	46°
3.75	46°	7.75	190°	11.75	315°	15.75	201°	19.75	46°
								20.00	45°

Note:
The angle θ is the azimuth of the normal to the Rayleigh Disk, referred to a given line at a given time.

The unit was mounted vertically on the seaward end of a pier. The disk was located slightly below the elevation of the trough of the highest wave. The installation on the Huntington Beach Pier differed from the others in that the unit was mounted on a tide adjusting apparatus which kept the disk in a constant depth of water with respect to the surface. This arrangement was considered to offer a distinct advantage since the disk remained in the influence of the surface conditions through a complete tidal cycle.

Results of Field Test

Typical wave direction records made in Long Branch, New Jersey, and Huntington Beach, California, are shown on Plates II and III respectively. These records were selected from a group on which the observed wave direction had been written. An examination of all Long Branch records showed those given on Plate II to be typical. As a matter of fact considering all Long Branch records on which the observed wave direction had been written only in one instance (see record d) did the Rayleigh Disk confirm the observation. It is interesting to note in examining the remainder of the records shown that all have a tendency to record a southeasterly direction although the waves were observed to be approaching the gage from the north through northeast to east. This lack of agreement between the observed and recorded direction was noted during these tests. In the belief that the disk might be restrained it was turned manually to the position it should have assumed to indicate the true wave direction and released. Upon being released it assumed its former position.

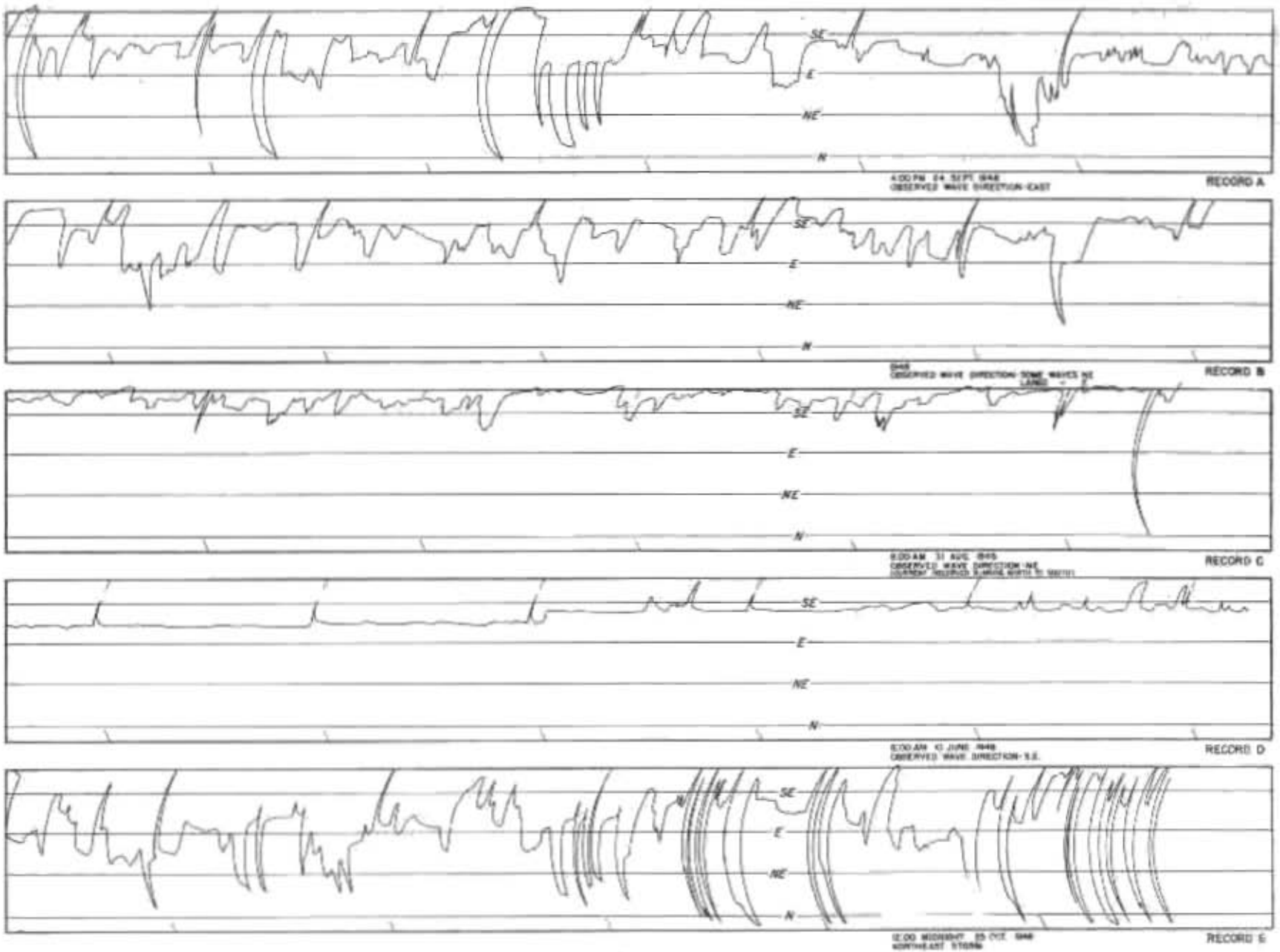
The exact reason for this phenomenon could not be ascertained but since the disk is known to be sensitive to all currents it may have been reacting to currents other than those generated by the observed wave system.

Plate II shows two series of records made during a test of the Huntington Beach gage. At the time each test was made two wave systems were running. Their directions are indicated on the records by dashed lines.

A study of the records in test A show some indications of a two wave system with peaks in the curve falling alternately in the south and west-southwest directions. There are also strong indications that the two wave systems are approaching from the west-southwest and west and that the disk is recording the resultant currents of the two wave systems.

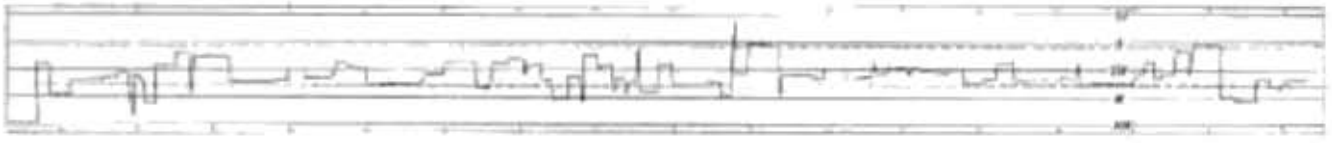
A study of the records in test B shows peaks in the curve which might indicate the directions of the two wave systems observed. There are also sections of the record which indicate that the instrument is reacting to the resultant current of the two systems. In the final record, taken when wind chop had died down, there appears to be an indication that the instrument is reacting to the combined currents or a southwest wave system.

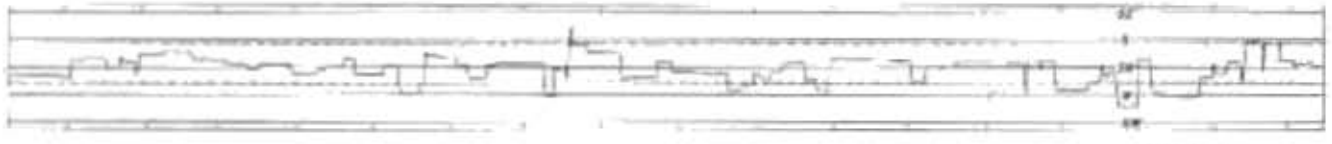
If it was not known that two wave systems were present in the above tests it would be difficult to discern the true direction of the system or systems of waves. A test conducted to compare the results of a number



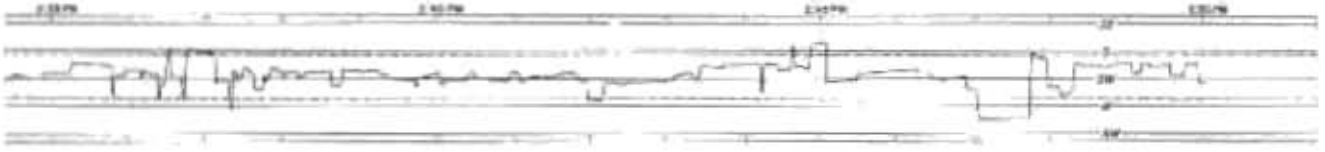
WAVE DIRECTION RECORD
LONG BRANCH, NEW JERSEY

PLATE 2





TEST A



TEST B

WAVE DIRECTION RECORDS
HUNTINGTON BEACH, CALIFORNIA

PLATE III

of individual analyses of the same record showed a wide variation in interpretation.

Concluding Remarks

It appears that the records made under natural conditions confirm as nearby as might be expected the theoretical analysis of the behavior of the disk when acted upon by two sine wave systems.

It also appears from the study that the presence of two or more wave systems; the presence of strong currents other than wave currents such as rip tides, and the uncertainty in analysis of the records proves the Rayleigh Disk to be unreliable as a wave direction indicator.

A P P E N D I X

Theory of the Rayleigh Disk

by

Walter Konig

ANNALEN DER PHYSIK

Physikalisches Institute der
Universitaet Leipzig
Maerz 1891

On the Torque Conveyed to an Ellipsoid of Revolution at Rest in an Oscillating fluid. The Theory of the Rayleigh Disk.

Apart from the dynamic forces discussed heretofore, an oscillating fluid conveys torque to bodies suspended in it, if these are not spherical. The most simple case of this problem presents itself, if we assume only a solid body at rest in the fluid of infinite volume, which oscillates back and forth in straight lines to infinity and if we assume, furthermore, that this body shows the simplest deviation from a spherical body, e.g., an ellipsoid of revolution. Let the body be suspended in the fluid in an optional position toward the flow of the fluid and let the components of the forces resulting from the fluid pressure, which is irregularly distributed over the surface of the body due to the flow of the fluid, be first computed for the case of steady flow of the fluid.

Thomson and Tait[1] gave in their "Theory of Physics", a general term for this torque in the case of a body of revolution of any form and they drew the conclusion that any one body of revolution tends to adopt a position whereby its longest axis is at right angles to the flow of the fluid. In the above mentioned case, therefore, a prolate ellipsoid will adopt a position with its axis crosswise to the flow, while an oblate ellipsoid will establish its axis in the direction of the flow as is exactly the case in the instance of the Rayleigh Disk. For the specific computation of the torque, we must go back to the form of flow taking place in the fluid, i.e., to the velocity potential. The case of an ellipsoid at rest in an oscillating fluid has been frequently discussed.[2] I refer to the presentation of the problem as given by Kirchhoff in his Mechanics.[3] Let the center of the coordinate system be situated at the center of the ellipsoid of revolution, and the z axis coinside with the axis of revolution. The lengths of the semi-axes be designated a and c, then the equation of the ellipsoid will be:

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1 \qquad \text{--- ---- (1)}$$

The angles of elevation of the diverging normal of a surface element dS, of the ellipsoid be marked α and p be the pressure which the fluid exerts on this surface element.

Since the pressure acts normally on the surface in a frictionless fluid, as assumed in this case, the components of the total force exerted by the oscillating fluid on the ellipsoid are the following:

1 Thomson and Tait, Theor. Physics Brounschweig, 1871. Part 1, p. 303
2 With respect to pertinent literature I refer to Auerbach, Theoretical Hydrodynamics. Braunschweig, 1881, p. 83
3 Kirchhoff, Mechanics, 1877, p. 219.

$$\left.\begin{aligned} X &= -\int p \cos\alpha \, dS \\ Y &= -\int p \cos\beta \, dS \\ Z &= -\int p \cos\gamma \, dS \end{aligned}\right\} \qquad \text{.....} (2)$$

and the components of torque are:

$$\left.\begin{aligned} L &= -\int p(y\cos\gamma - z\cos\beta)\, dS \\ M &= -\int p(z\cos\alpha - x\cos\gamma)\, dS \\ N &= -\int p(x\cos\beta - y\cos\alpha)\, dS \end{aligned}\right\} \qquad \text{.....} (3)$$

where the integral is to be extended over the entire surface of the ellipsoid. The signs of the torque are chosen in such a way that a positive value of M for instance signifies a torque around the y axis from the z axis to the x axis. The pressure, p, is computed from the velocity potential ϕ according to the formula:

$$p = const - \mu \frac{\partial \phi}{\partial t} - \frac{1}{2}\mu\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] \quad \text{.....} (4)$$

provided that no other forces act on the fluid. Since the constant is neglected in integration and $\partial\phi/\partial t$ is zero with a nonoscillating movement, only the last term of the above equation will be considered for the time being.

If the fluid flows with the velocity ω in a direction opposed to that of the +z axis, the velocity potential, according to Kirchhoff, is given by the following formula:

$$\phi_1 = \frac{\omega}{2\pi(2-C)} \frac{\partial \Omega}{\partial z} - \omega z$$

$$\Omega = \pi a^2 c \int_S^\infty d\lambda \cdot \frac{1 - \frac{x^2+y^2}{a^2+\lambda} - \frac{z^2}{c^2+\lambda}}{(a^2+\lambda)\sqrt{c^2+\lambda}}$$

$$C = a^2 c \int_0^\infty \frac{d\lambda}{(a^2+\lambda)(c^2+\lambda)^{\frac{3}{2}}} \qquad \text{.....} (5)$$

and if s is the positive root of the equation:

$$\frac{x^2+y^2}{a^2+s} + \frac{z^2}{c^2+s} = 1 \qquad \text{.....} (6)$$

If the fluid flows in the direction of the negative x axis with the velocity u, the velocity potential, accordingly has the value:

$$\phi_2 = \frac{u}{2\pi(2-A)} \frac{\partial \Omega}{\partial x} - ux$$

where

$$A = a^2 c \int_0^\infty \frac{d\lambda}{(a^2+\lambda)^2 \sqrt{c^2+\lambda}} \qquad \text{------- (7)}$$

and Ω has the same meaning as above. Also the summation of both solutions represents a possible flow of the fluid and obviously satisfies the requirement that the fluid flow in straight lines at infinity and parallel to the xz plane, however, with a velocity:

$$W = \sqrt{u^2 + w^2} \qquad \text{------- (8)}$$

and in a direction at an angle with the z axis which is determined by the equation:

$$\tan \theta = \frac{u}{w} \qquad \text{------- (9)}$$

This is the case for which we want to compute the effect of the fluid flow on the ellipsoid of revolution. Since all points are arranged symmetrically around the z axis, the y axis may be put, without any restriction of the general conditions, in such a way that it is normal to the direction of the fluid flow. Therefore:

$$\phi = \phi_1 + \phi_2$$

If one computes the values for $\frac{\partial \Omega}{\partial x}$ and $\frac{\partial \Omega}{\partial z}$, introducing for further simplification the symbols:

$$A_s = a^2 c \int_s^\infty \frac{d\lambda}{(a^2+\lambda)^2 \sqrt{c^2+\lambda}} \qquad \text{------- (10)}$$

$$C_s = a^2 c \int_s^\infty \frac{d\lambda}{(a^2+\lambda)(c^2+\lambda)^{\frac{3}{2}}} \qquad \text{------- (11)}$$

ϕ may be written in the following form:

$$\phi = \frac{ux A_s}{2-A} - \frac{wz C_s}{2-C} - ux - wz \qquad \text{------- (12)}$$

and consequently;

$$\left.\begin{aligned}
\frac{\partial \phi}{\partial x} &= -\frac{ux}{2-A}\frac{\partial A_s}{\partial x} - \frac{wz}{2-C}\frac{\partial C_s}{\partial x} - \left(\frac{A_s}{2-A}+1\right)u \\
\frac{\partial \phi}{\partial y} &= -\frac{ux}{2-A}\cdot\frac{\partial A_s}{\partial y} - \frac{wz}{2-C}\frac{\partial C_s}{\partial y} \\
\frac{\partial \phi}{\partial z} &= -\frac{ux}{2-A}\frac{\partial A_s}{\partial z} - \frac{wz}{2-C}\frac{\partial C_s}{\partial z} - \left(\frac{C_s}{2-C}+1\right)w
\end{aligned}\right\} \qquad \text{------- (13)}$$

For a computation of the fluid pressure on the ellipsoid, we need those values of the velocity components that occur on the surface of the

ellipsoid, that is when $s = 0$. The differential quotients of A_s and C_s must therefore, be computed by means of the values for $\partial S/\partial x$, $\partial S/\partial y$, $\partial S/\partial z$ which are determined by equation (6) and in the terms of general application thus obtained, $s = 0$. In this way the following values are obtained for the velocity components at the surface of the ellipsoid.

$$\left.\begin{aligned}
\frac{\partial \phi}{\partial x} &= 2c^4 \frac{x}{c^4+(a^2-c^2)z^2}\left(\frac{ux}{a^2(z-A)}+\frac{wz}{c^2(z-C)}\right)-\frac{2u}{z-A}\\
\frac{\partial \phi}{\partial y} &= 2c^4 \frac{y}{c^4+(a^2-c^2)z^2}\left(\frac{ux}{a^2(z-A)}+\frac{wz}{c^2(z-C)}\right)\\
\frac{\partial \phi}{\partial z} &= 2a^2c^2 \frac{z}{c^4+(a^2-c^2)z^2}\left(\frac{ux}{a^2(z-A)}+\frac{wz}{c^2(z-C)}\right)-\frac{2w}{z-C}
\end{aligned}\right\} \quad \cdots\cdots\cdots (14)$$

and accordingly the pressure, p, becomes the following expression, neglecting the two terms of equation (4) as before:

$$p = -2\mu\left(\frac{u^2}{(z-A)^2}+\frac{w^2}{(z-C)^2}\right)+2\mu a^2c^4\frac{1}{c^4+(a^2-c^2)z^2}\left(\frac{ux}{a^2(z-A)}+\frac{wz}{c^2(z-C)}\right)^2$$

Of these terms for p, the first is constant, thus being negligible in integration. Only the latter term must be integrated. For the integration, polar coordinates may be introduced for x and y:

$$x = r\cos\phi \qquad y = r\sin\phi$$

Then r is exclusively dependent on z:

$$r^2 = \frac{a^2}{c^2}\left(c^2-z^2\right)$$

and the integration needs to be carried out for ϕ from 0 to 2π and for z from c to - c. If the direction cosines of the normal to the surface element, S, are computed according to the known formulas, as functions of ϕ and z, $r\,d\phi$ and dz respectively, the following equations are obtained for the values to be inserted into integral (2) and (3):

$$\left.\begin{aligned}
\cos\alpha\, dS &= \frac{a}{c}\sqrt{c^2-z^2}\; dz\cos\phi\, d\phi\\
\cos\beta\, dS &= \frac{a}{c}\sqrt{c^2-z^2}\; dz\sin\phi\, d\phi\\
\cos\gamma\, dS &= \frac{a^2}{c^2}\; z\,dz\, d\phi
\end{aligned}\right\}
\qquad
\left.\begin{aligned}
(y\cos\gamma - z\cos\beta)\,dS &= \frac{a}{c}\,\frac{a^2-c^2}{c^2}\sqrt{c^2-z^2}\; z\,dz\sin\phi\, d\phi\\
(z\cos\alpha - x\cos\gamma)\,dS &= -\frac{a}{c}\,\frac{a^2-c^2}{c^2}\sqrt{c^2-z^2}\; z\,dz\cos\phi\, d\phi\\
(x\cos\beta - y\cos\alpha)\,dS &= 0
\end{aligned}\right\} \cdots (15)$$

and for the variable factors in the term for p:

$$\frac{4\mu c^2u^2}{(z-A)^2}\;\frac{c^2-z^2}{c^4+(a^2-c^2)z^2}\cos^2\phi + \frac{2\mu a^2w^2}{(z-C)^2}\;\frac{z^2}{c^4+(a^2-c^2)z^2} = \frac{4uacuw}{(z-A)(z-C)}\;\frac{\sqrt{c^2-z^2}\cdot z}{c^4+(a^2-c^2)z^2}\cos\phi$$

It is evident from the form of these factors and of the terms to be multiplied therewith, that of the integrals to be computed all, except one, become zero. The following is obtained:

$$X = Y = Z = L = N = 0$$

$$M = \frac{4\pi a^2 \mu w}{(2-A)(2-C)} \; \frac{a^2-c^2}{c^2} \int_{-c}^{+c} \frac{(c^2-z^2)z^2 dz}{c^4+(a^2-c^2)z^2} \int_0^{2\pi} \cos^2\phi \, d\phi$$

$$M = 4\pi\mu u w \frac{a^2}{(2-A)(2-C)} \; \frac{a^2-c^2}{c^2} \int_{-c}^{+c} \frac{(c^2-z^2)z^2}{c^4+(a^2-c^2)z^2} \, dz$$

From these equations it is obvious that a constant flow of an ideal fluid does not convey a translatory action to an ellipsoid placed in it, but it exerts a torque around an axis that is normal to the axis of revolution of the ellipsoid and to the direction of the flow. For further computation of this torque we put:

$$\frac{z}{c} = t \quad \text{and} \quad \frac{a^2-c^2}{c^2} = \pm \varepsilon^2$$

according to whether the ellipsoid is oblate or prolate. The torque then becomes:

$$M = \pm 4\pi\mu u w \frac{a^2 c}{(2-A)(2-C)} \; \varepsilon^2 \int_{-1}^{+1} \frac{(1-t^2)t^2}{1 \pm \varepsilon^2 t^2} \, dt$$

Prolate Ellipsoid of Revolution, $a < c$

$$\int_{-1}^{+1} \frac{(1-t^2)t^2}{1-\varepsilon^2 t^2} \, dt = +\frac{2}{3\varepsilon^2} + 2\frac{1-\varepsilon^2}{\varepsilon^4} - \frac{1-\varepsilon^2}{\varepsilon^3} \log_e \frac{1+\varepsilon}{1-\varepsilon}$$

When integrating equations (5) and (7), the following equations are obtained:

$$\left.\begin{aligned} A &= \frac{1}{\varepsilon^2} - \frac{1}{2}\frac{1-\varepsilon^2}{\varepsilon^3} \log_e \frac{1+\varepsilon}{1-\varepsilon} \\ C &= -2\frac{1-\varepsilon^2}{\varepsilon^4} + \frac{1-\varepsilon^2}{\varepsilon^3} \log_e \frac{1+\varepsilon}{1-\varepsilon} \end{aligned}\right\} \cdots (16)$$

$$M = -4\pi\mu u w a^2 c \frac{\varepsilon^3}{\frac{1}{2}(1-\varepsilon^2)\log_e \frac{1+\varepsilon}{1-\varepsilon} - \varepsilon} \; \frac{\frac{1}{3}\varepsilon^3 - \frac{1}{2}(1-\varepsilon^2)\log_e \frac{1+\varepsilon}{1-\varepsilon} - \varepsilon}{2\varepsilon^3 + \frac{1}{2}(1-\varepsilon^2)\log_e \frac{1+\varepsilon}{1-\varepsilon} - \varepsilon} \cdots (17)$$

The two extreme cases of the prolate ellipsoid are then sphere, on the one side, and the infinitely long circular cylinder, on the other. For the former extreme case is $c = a$, and $\varepsilon = 0$, and proof may be rendered without difficulty that the above term for M becomes zero, as is the case for a sphere. For the latter extreme case is $c = \infty$ and $\varepsilon = 1$. Considering that for this value of ε the product $(1 - \varepsilon) \log [(1+\varepsilon)/(1-\varepsilon)]$

becomes zero, it is clear from the above equation that M becomes infinite as of the same order of magnitude as the axis of the cylinder, or that M/c remains finite, while this quotient, which might be defined as that portion of the total torque that acts on the longitudinal unit obtains the value:

$$\lim_{c=\infty} \frac{M}{c} = -\frac{4}{3}\pi\mu a^3 uw$$

or, when introducing the total velocity W and its angle θ with the z axis for the components u and w

$$\lim_{c=\infty} \frac{M}{c} = -\frac{2}{3}\pi\mu a^2 W^2 \sin 2\theta$$

It must, however, be remarked that this limiting case is no longer in accord with the basic assumption of our problem inasmuch as the fluid can no longer flow parallel to the z axis at infinity, if the body at rest in it extends to infinity. However, the same limiting case is also obtained, when c is assumed limited and infinitely small. The body then becomes a rod of the limited length $l = 2c$ and the infinitely small circular cross section $q = \pi a^2$; the torque exerted on it by the fluid is then infinitely small and is represented in the first approximation, i.e., neglecting the second and higher powers of a/c, by the following equation:

$$M = -\tfrac{1}{3}\mu q l W^2 \sin 2\theta$$

For a prolate ellipsoid of revolution the torque is always negative, having a rotation from the +z to the -x axis. According to our assumptions, the fluid flows with the velocity components u and w in the direction of the -x and the -z axis, the direction of the flow thus falling into the quadrants between the -x and -z or +x and +z axes respectively. The sense of the torque, therefore, is such that the ellipsoid tries, with its axis of revolution, to adopt a position normal to the flow of the fluid. The torque becomes zero for $\theta = 0$ and for $\theta = \frac{\pi}{2}$; but the condition of equilibrium is unstable in the first case, while it is stable in the latter.

Oblate Ellipsoid of Revolution, where a > c

$$\int_{-1}^{+1} \frac{(1-t^2)t^2}{1+\varepsilon^2 t^2}\,dt = -\frac{2}{3\varepsilon^2} + 2\frac{1+\varepsilon^2}{\varepsilon^4} - 2\frac{1+\varepsilon^2}{\varepsilon^3}\arctan\varepsilon$$

$$\left.\begin{aligned} A &= -\frac{1}{\varepsilon^2} + \frac{1+\varepsilon^2}{\varepsilon^3}\arctan\varepsilon \\ C &= +2\frac{1+\varepsilon^2}{\varepsilon^2} - 2\frac{1+\varepsilon^2}{\varepsilon^3}\arctan\varepsilon \end{aligned}\right\} \text{---------- (19)}$$

By means of these values, the following equation is obtained for M:

$$M = 4\pi\mu uwa^2c\,\frac{\varepsilon^3}{(1+\varepsilon^2)\arctan\varepsilon - \varepsilon} \times \frac{(1+\varepsilon^2)\arctan\varepsilon - \varepsilon - \frac{2}{3}\varepsilon^3}{(1+\varepsilon^2)\arctan\varepsilon - \varepsilon - 2\varepsilon^3} \text{ ----- (20)}$$

where ε lies between 0 and ∞. The first of these boundary values corresponds to the case of a sphere and results in $M = 0$ as required. The second boundary value applies to the case of an infinitely thin circular disk of the radius, a. In order to obtain the value for M for this limiting case, the above term should preferably be written as follows:

$$M = 4\pi\mu u w a^3 \sqrt{a^2 - c^2} \; \frac{1}{\frac{1+\varepsilon^2}{\varepsilon^2} \operatorname{arctan} \varepsilon - \frac{1}{\varepsilon}} \times \frac{\frac{1+\varepsilon^2}{\varepsilon^3} \operatorname{arctan} \varepsilon - \frac{1}{\varepsilon^2} - \frac{2}{3}}{\frac{1+\varepsilon^2}{\varepsilon^3} \operatorname{arctan} \varepsilon - \frac{1}{\varepsilon^2}} \; 2$$

When $c = 0$, and $\operatorname{arctg} \varepsilon = \pi/2$, the following torque is found for the boundary value in question:

$$\left.\begin{aligned} M &= \frac{8}{3} \mu u w a^3, \\ &= \frac{4}{3} \mu a^3 W^2 \sin 2\theta \end{aligned}\right\} \cdots (21)$$

The sign of the torque is positive in this case and always for an oblate ellipsoid of revolution, i.e., the axis of the ellipsoid tends to place itself parallel to the direction of the flow, or to establish a position where its oblate side is normal to the flow.

It is remarkable that the computation results in a finite value for the torque even when the ellipsoid passes into an infinitely thin disk, although in this limiting case the assumptions of the problem are no longer completely fulfilled. An inspection of the terms given in the equations for the velocity components makes clear that at the edge of the disk the velocity of the fluid would be infinitely great. Pressure, however, is negative and infinite, a consequence which does not seem to be compatible with the assumption of a uniform continuity of the fluid. In nature this theoretical consideration would not be taken into account, because the ideal case as treated here could not be realized for two reasons: firstly because no infinitely thin disks exist, and secondly because frictionless fluids do not exist. These two facts should be taken into account in order to find a formula applicable to natural phenomena.

In order to obtain a correction for the thickness of the disk, it is advisable to consider, in a first approximation, a disk of finite thickness as an extremely oblate ellipsoid and thus compute M maintaining the first power of the ratio c/a. This computation gives

$$\left.\begin{aligned} M &= \frac{4}{3} \mu a^3 W^2 \sin 2\theta \left(1 - \frac{\pi}{2\pi} \frac{c}{a}\right) \\ &= \frac{4}{3} \mu a^3 W^2 \sin 2\theta \left(1 - 0.20 \frac{c}{a}\right) \end{aligned}\right\} \cdots (22)$$

where a designates the radius and c half the thickness of the disk.

For the case of friction in fluid corresponding to this problem of the flow of an ideal fluid, Oberbeck established a solutions within the usual limits of infinitesimal velocities. Owing to this limitation, only such effects of the fluid may be obtained as are dependent on the first power of the velocity so that affects, as computed above, which are proportionate to the square of the velocity, are no longer considered here. Oberbeck gives the values for the velocity components u, v, and w of a fluid in friction for the special case where an ellipsoid is at rest in the fluid and where the flow of the fluid at infinity is parallel to one of the axes of the ellipsoid. In order to obtain the form of the problem in accordance with previous assumptions, it must be assumed that the ellipsoid be one of revolution and the fluid as flowing in one direction at infinity, where the direction of the flow forms an angle θ with the axis of rotation of the ellipsoid. The solution for this general case may then be established again by the summation of the solutions for the two special cases of a flow parallel and normal to the axis of the ellipsoid of revolution. The solutions for these special cases may be found in Oberbeck's paper. Assume the axis of rotation as the z axis, and designate the velocity components and the pressure of the fluid as u_z, v_z, w_z, p_z when the fluid at infinity flows parallel to the z axis with the velocity w; and u_x, v_x, w_x and p_x, when the fluid at infinity flows parallel to the x axis with the velocity u. Then follow the solutions:

$$u' = u_x + u_z \qquad v' = v_x + v_z \qquad w' = w_x + w_z \qquad p' = p_x + p_z$$

to satisfy the differential equations and represent a flow, where the fluid at infinity flows with the velocity $W = \sqrt{u^2 + w^2}$ is one direction at angle with the axis of rotation of the ellipsoid as defined by the equation:

$$\tan \theta = \frac{u}{w}$$

The terms for u', v', w' and p' may be found by applying Oberbeck's solutions; utilizing these values the pressure components of X_x, X_y, X_z, etc. may then be computed by means of the known formulas:

$$\begin{aligned} X_x &= p' - 2k\frac{\partial u'}{\partial x} & Y_z &= Z_y = -k\left(\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right) \\ Y_y &= p' - 2k\frac{\partial v'}{\partial y} & Z_x &= X_z = -k\left(\frac{\partial w'}{\partial x} + \frac{\partial u'}{\partial z}\right) \\ Z_z &= p' - 2k\frac{\partial w'}{\partial z} & X_y &= Y_x = -k\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) \end{aligned} \qquad \ldots (23)$$

and finally, in the integrals (2) and (3) the terms

$$\left.\begin{aligned} X_n &= X_x \cos\alpha + X_y \cos\beta + X_z \cos\gamma \\ Y_n &= Y_x \cos\alpha + Y_y \cos\beta + Y_z \cos\gamma \\ Z_n &= Z_x \cos\alpha + Z_y \cos\beta + Z_z \cos\gamma \end{aligned}\right\} \qquad \ldots (24)$$

* A. Oberbeck, Crelle's Journ., No. 81, page 62, 1876.

may be substituted for the previously applied pressure components $p \cos \alpha$, $p \cos \beta$, $p \cos \gamma$. The complicated procedure of the computations can be simplified by representing the components X_x, X_y, X_z, etc. in the more general solution of the u', v', w', p', simply as the summation of the stress components for the two special cases (u_x, v_x, w_x, p_x,) and (u_z, v_z, w_z, p_z,), because the equation (23) are linear with respect to these terms or their derivations. Therefore, the final values for the total pressures and torques as computed by the equations (2) and (3) will, in the general case, simply equal the summation of the values for the total pressures and torques as computed for the two special cases. As a consequence of this conclusion, the resultant value for the torques is very simple. For reasons of symmetry it is readily evident that the torque must be zero when the flow parallels the axis of rotation as well as when the flow is normal to the axis. Since with any one direction of the flow, the torque is obtained by summation of the values in the two special cases, it must be zero in any case. Thus taking into consideration only the influence of the friction, a flowing fluid does not exert any torque on an ellipsoid but only a translatory action. For the sake of completeness, the components of this latter effect, in the case of a non-oscillating flow of fluid, remain to be investigated. These components may be readily derived from Oberbeck's paper. If the flow of fluid at infinity parallels the x-axis then

$$X = \frac{16\pi a^2 c k u}{Q_0 + a^2 A} \; ; \quad Y = 0 \; ; \quad Z = 0$$

and if it parallels the z axis at the velocity w, then

$$X = 0 \; ; \quad Y = 0 \; ; \quad Z = \frac{16\pi a^2 c k w}{Q_0 + c^2 C}$$

where k designates the constant of friction; A and C defined by equations (5) and (7) and Q_0 defined as:

$$Q_0 = a^2 c \int_0^\infty \frac{d\lambda}{(a^2+\lambda)\sqrt{c^2+\lambda}}$$

For the case when the fluid flows obliquely to the axis of rotation at the angle θ, the summation of the two solutions results in:

$$X = \frac{16\pi a^2 c}{Q_0 + a^2 A} k u \; ; \qquad Y = 0 \qquad Z = \frac{16\pi a^2 c}{Q_0 + c^2 C} k w \quad \text{------} (25)$$

Prolate Ellipsoid of Revolution where $a < c$

When applying the previous method of designation

$$Q_0 = a^2/\varepsilon \log (1+\varepsilon)/(1-\varepsilon)$$

If this value for Q and the values for A and C, previously given, are inserted in the equations for the forces, it then follows that:

$$\left.\begin{aligned} X &= 16\pi ck u \frac{2\varepsilon^3}{(3\varepsilon^2-1)\log_e \frac{1+\varepsilon}{1-\varepsilon} + 2\varepsilon} \quad ; \quad Y = 0 \; ; \\ Z &= 16\pi ck w \frac{\varepsilon^3}{(\varepsilon^2+1)\log_e \frac{1+\varepsilon}{1-\varepsilon} - 2\varepsilon} \end{aligned}\right\} \cdots\cdots (26)$$

For a = c, ε becomes zero, and the force terms result in the known values for a sphere:

$$X = 6\pi aku, \quad Y = 0; \quad Z = 6\pi akw$$

Only in this case the resultant of the components coincides with the direction of the flow of the fluid. In general, the direction toward which the fluid tends to displace the ellipsoid of revolution forms the angle ϕ with the axis of rotation as defined by the equation:

$$\tan\phi = \tan\theta \frac{2(\varepsilon^2+1)\log_e \frac{1+\varepsilon}{1-\varepsilon} - 4\varepsilon}{(3\varepsilon^2-1)\log_e \frac{1+\varepsilon}{1-\varepsilon} + 2\varepsilon} \quad \cdots\cdots\cdots (27)$$

and therefore it is obvious that ϕ is always greater than θ. For an ellipsoid of revolution where c is infinitely great as compared to a,

$$\tan\phi = 2\tan\theta$$

Oblate Ellipsoid of Revolution where a > c

$$\left.\begin{aligned} Q_0 &= 2\frac{a^2}{\varepsilon} \arctan\varepsilon \\ X &= 16\pi ck u \frac{\varepsilon^3}{(1+3\varepsilon^2)\arctan\varepsilon - \varepsilon} \\ Y &= 0 \\ Z &= 16\pi ckw \frac{\varepsilon^3}{-2(1-\varepsilon^2)\arctan\varepsilon + 2\varepsilon} \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots (28)$$

Similar considerations are applicable here as in the case of a prolate ellipsoid. The resultant of the above components, with respect to the axis of rotation, forms the angle ϕ as is defined by the equation:

$$\tan\phi = \tan\theta \frac{2\varepsilon - 2(1-\varepsilon^2)\arctan\varepsilon}{(1+3\varepsilon^2)\arctan\varepsilon - \varepsilon} \quad \cdots\cdots\cdots\cdots\cdots (29)$$

Only for the sphere the direction of this force coincides with the direction of the flow. In general, it forms a smaller angle with the axis of rotation than with the direction of the flow. In the boundary case of a disk, ε, and the components become:

$$X = \frac{32}{3} aku; \quad Y = 0; \quad Z = 16\, akw$$

and:

$$\tan\phi = \frac{2}{3}\tan\theta$$

Provided only the first power of velocity is taken into consideration, a linear non-oscillating flow of a fluid subject to friction does not tend

to rotate an ellipsoid of revolution at rest, but to displace it in a direction which does not generally coincide with the direction of the flow of the fluid, but establishes a definite angle to the direction.

What would be the conditions in a fluid subject to friction at greater velocities, remains to be investigated. No solution has yet been found to this general problem. Since the case of an ideal fluid represents the boundary case of a fluid subject to friction at an infinitely small value of friction, it may be expected that the term derived above for the torque in fluids subject to insignificant friction should at least represent the first approximation to reality. How far this approximation goes, will have to be clarified by an experimental investigation.

In the above considerations, a non-oscillating flow of fluid has been assumed. The question still to be answered is what are the accelerating forces and the torques, if the flow, neglecting friction, varies with time. In order to establish this case, it suffices to assume u and w as functions of time in the term for the velocity potential as defined by equation (12). Then the previously computed value of p must, in accordance with equation (4), be augmented by $-\mu(\partial\phi/\partial t)$, and this item attains the following value for the surface of the ellipsoid of revolution:

$$\mu \frac{\partial u}{\partial t} \frac{2}{2-A} x + \mu \frac{\partial w}{\partial t} \frac{2}{2-C} z$$

This additional term inserted in the integral (2) leads to the following terms for the accelerating forces:

$$\left.\begin{aligned} X &= \frac{8}{3}\pi\mu \frac{a^2c}{2-A} \frac{\partial u}{\partial t} \\ Y &= 0 \\ Z &= \frac{8}{3}\pi\mu \frac{a^2c}{2-C} \frac{\partial w}{\partial t} \end{aligned}\right\} \quad \text{------} \quad (30)$$

Assuming that the resultant velocity W has constant direction but variable intensity, where

$$\frac{\partial u}{\partial t} = \frac{\partial W}{\partial t} \sin\theta, \quad \frac{\partial w}{\partial t} = \frac{\partial W}{\partial t} \cos\theta$$

it is clear from the above formulas that the resultant accelerating force establishes a different angle with the axis of rotation than the direction of the flow, this fact being similar to that found in the resultant of the frictional effects. The equation defining this angle is:

$$\tan\phi = \tan\theta \frac{2-C}{2-A} \quad \text{----------} \quad (31)$$

where the terms for A and C derived above be substituted. For a prolate ellipsoid ϕ is greater than θ , and this quantity would be defined by $\tan\phi = 2\tan\theta$ for an ellipsoid the crosswise dimension of which is infinitely small as compared with its length. For an oblate ellipsoid ϕ is smaller than θ ; $X=0$, $Z=\frac{8}{3}\mu a^2 \frac{\partial w}{\partial t}$ and thus $\phi = 0$. Only for a sphere, $\phi = \theta$, because

in this case $A = E = 2/3$. Utilizing these values for the sphere, the term for the accelerating forces that has been previously applied in the first paper, becomes:

$$X = \frac{3}{2} m' \frac{\partial u}{\partial t} , \qquad Z = \frac{3}{2} m' \frac{\partial w}{\partial t}$$

If the respective computations are carried out with the integrals (3) that express the torque, it is found that those terms of the torque that depend on the acceleration of the fluid become zero. Therefore, the value of M is not only applicable to a non-oscillating flow, but it is the complete term of the torque present at any moment at the respective velocity W. Assuming that the movement has the form of a simple harmonic motion, the velocity becomes:

$$W = W_0 \cos 2\pi \frac{t}{T}$$

and for the case of a disk of the radius a, and half thickness c, at the time t, the torque becomes:

$$M = \frac{4}{3} \mu a^3 \left(1 - 0.2977 \frac{c}{a}\right) W_0^2 \sin 2\theta \cdot \cos^2 2\pi \frac{t}{T}$$

The torque thus varies between zero and a maximum value, but moving in the same direction in all phases of the movement inasmuch as the disk always tends to adopt a position with its oblate side normal to the flow. Therefore, if the period of oscillation is small, as is the case of acoustic oscillations, the disk can react slowly to the active components of forces and the rapidly varying torque M gives the same effect as a constant torque. This constant torque becomes equal to the mean value of the variable torque for one oscillation period or

$$[M] = \frac{2}{3} \mu a^3 \left(1 - 0.2977 \frac{c}{a}\right) W_0^2 \sin 2\theta$$

The phenomenon of a disk suspended in an oscillating air column has first been observed by Lord Rayleigh. His suggestion to utilize it in the measuring of sound intensity was carried out in more detail by Grimsehl. An application of the formula evolved above would make this procedure a very simple method of absolute intensity measurement, provided that the formula satisfies the demands of the natural circumstances. This method would not only be applicable to the measurement of continued sounds, but theoretically it should be possible to measure the intensity of a sound of short duration in its absolute value under application of the ballistic principle, like continued alternating currents as well as individual spontaneous currents or oscillations of short duration may be measured by an electro-dynometer. In practice, however, this procedure would be infeasible (impracticable) because of the difficulty to avoid all other air movements.

Whether the evolved formula suffices for an interpretation of the phenomena, will have to be left to an experimental investigation. The measurements available at this time, as carried out by Rayleigh and Grimsehl are unfortunately insufficient for the purpose of an absolute computation and would not permit a judgment on the correctness of the absolute values, because the intensity of the sounds was measured only by this one method. In conclusion of these theoretical considerations, one remark may

** Compare with the foregoing volume, page 360.

be added that the phenomenon explained herein has furthermore a certain importance in the pronounced formation of Knudt's dust ribs. For, if the dust figures are produced by cork free light, the particles of which are very irregular, the tendency is noticeable that all of them take a position with their lengthwise axis normal to the direction of the oscillation of the air, i.e., normal to the axis of the tube. It is evident that in this way the wall-like character of the ribs consisting of these particles is further stressed. Apart from this external coincidence, it follows from the above that both phenomena, the rib formation of Knudt's dust figures and the torques of Rayleigh's disks, must be regarded as manifestations of one and the same principle, if one tries to interpret them as phenomena of a purely hydrodynamic nature, as attempted here.

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The Institute of Physics of the Leipzig University, March 1891.

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$$[M] = \frac{2}{3} \mu a^3 \left(1 - 0.2977 \frac{c}{a}\right) W_0^2 \sin 2\theta$$

The phenomenon of a disk suspended in an oscillating air column has first been observed by Lord Rayleigh. His suggestion to utilize it in the measuring of sound intensity was carried out in more detail by Grimsehl. An application of the formula evolved above would make this procedure a very simple method of absolute intensity measurement, provided that the formula satisfies the demands of the natural circumstances. This method would not only be applicable to the measurement of continued sounds, but theoretically it should be possible to measure the intensity of a sound of short duration in its absolute value under application of the ballistic principle, like continued alternating currents as well as individual spontaneous currents or oscillations of short duration may be measured by an electro-dynometer. In practice, however, this procedure would be infeasible (impracticable) because of the difficulty to avoid all other air movements.

Whether the evolved formula suffices for an interpretation of the phenomena, will have to be left to an experimental investigation. The measurements available at this time, as carried out by Rayleigh and Grimsehl are unfortunately insufficient for the purpose of an absolute computation and would not permit a judgment on the correctness of the absolute values, because the intensity of the sounds was measured only by this one method. In conclusion of these theoretical considerations, one remark may

** Compare with the foregoing volume, page 360.

be added that the phenomenon explained herein has furthermore a certain importance in the pronounced formation of Knudt's dust ribs. For, if the dust figures are produced by cork free light, the particles of which are very irregular, the tendency is noticeable that all of them take a position with their lengthwise axis normal to the direction of the oscillation of the air, i.e., normal to the axis of the tube. It is evident that in this way the wall-like character of the ribs consisting of these particles is further stressed. Apart from this external coincidence, it follows from the above that both phenomena, the rib formation of Knudt's dust figures and the torques of Rayleigh's disks, must be regarded as manifestations of one and the same principle, if one tries to interpret them as phenomena of a purely hydrodynamic nature, as attempted here.

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