SR 104

Special Report 104

DEPTH OF FROST PENETRATION IN NON-UNIFORM SOIL

by

Harl P. Aldrich. Jr. and Henry M. Paynter

OCTOBER 1966

Conducted for
CORPS OF ENGINEERS, U. S. ARMY
by
U.S. ARMY MATERIEL COMMAND
COLD REGIONS RESEARCH & ENGINEERING LABORATORY
HANOVER, NEW HAMPSHIRE

Contract DA-19-016-ENG-2641

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PREFACE

Authority for the investigation reported herein is contained in FY 1953 Instructions and Outline, Military Construction Investigations, Engineering Criteria and Investigations and Studies, Studies of Construction in Areas of Seasonal Frost; Verification of Design Criteria. The investigation was conducted to augment improved techniques, being developed under a concurrent contract,* for predicting depths of freeze and thaw in soils and is directed to the application of these techniques to special design problems in both frost and permafrost areas.

The study was conducted for the Office, Chief of Engineers, Directorate of Military Construction, Civil Engineering Branch (Mr. T. B. Pringle, Chief), by Dr. Harl P. Aldrich, Jr., and Dr. Henry M. Paynter, of the Massachusetts Institute of Technology, under a contract awarded by the former Arctic Construction and Frost Effects Laboratory (ACFEL) of the U.S. Army Engineer Division, New England. The functions of ACFEL were transferred to the U. S. Army Cold Regions Research and Engineering Laboratory (USA CRREL) in 1961.†

This report was prepared by the Construction Engineering Branch, Mr. E.F. Lobacz, Chief (former Coordinator, ACFEL) as a project of the Experimental Engineering Division, Mr. K. A. Linell, Chief (former Director, ACFEL), USA CRREL. Mr. W. F. Quinn, Research Civil Engineer of the Liaison and Technical Publications Branch, USA CRREL, finalized the contract report for publication.

Colonel Dimitri A. Kellogg was Director of the U.S. Army Cold Regions Research and Engineering Laboratory during the publication of this report, and Mr. W.K. Boyd was Chief Engineer.

USA CRREL is an Army Materiel Command laboratory.

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*"Frost Penetration in Multilayer Soil Profiles," Massachusetts Institute of Technology, Department of Civil and Sanitary Engineering, Soil Engineering Division, June 1957 (ACFEL Technical Report 67).

†ACFEL was merged with the former Snow Ice and Permafrost Research Establishment (USA SIPRE) in 1961 to form the U.S. Army Cold Regions Research and Engineering Laboratory (USA CRREL), Hanover, New Hampshire.

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SUMMARY

This report presents the results of an investigation relative to the collection and improvement of techniques developed for the prediction of frost penetration depth and rate into non-uniform (multilayered) soil.

The approximate techniques encompassing latent heat only, latent heat plus volumetric specific heat, and an adaptation of the modified Berggren equation are reviewed and compared. A sample problem is treated mathematically using each technique.

Results indicate that an adaptation of the modified Berggren equation is the best technique for determining the depth of frost penetration in a non-uniform soil.

NOTATION

Symbol	Description	Units
C	Volumetric specific heat	Btu/cu ft °F
C_{wt}	Weighted average of volumetric specific heat	Btu/cu ft °F
d	Distance between centers of adjacent layers	ft
d'	Thickness of frozen portion within the lowest layer experiencing frost penetration	ft
d_n	Thickness of layer "n"	ft
F	Surface-freezing index	degree-days
k	Thermal conductivity	Btu/ft hr °F
L	Latent heat	Btu/cu ft
L_{wt}	Weighted average of volumetric latent heat	Btu/cu ft
R	Thermal resistance $= \frac{d}{k}$	sq ft hr °F/Btu
t	Time required to freeze soil	days
U_n	Thermal energy removed from layer "n"	Btu/sq ft
v	Temperature	°F
v_o	No. of degrees by which mean annual temperature exceeds freezing point of soil moisture	°F
v_s	No. of degrees by which effective surface temperature is less than freezing point of soil moisture during freezing period = F/t	°F
X	Depth of frost penetration	ft
α	Thermal ratio	dimensionless
λ	Correction coefficient, step-change in surface temperature	dimensionless
λ'	Correction coefficient, sinusoidal variation in surface temperature	dimensionless
μ	Fusion parameter	dimensionless

CONVERSION TABLE

Multiply	by	To Obtain
Btu/cu ft °F	16.018	kg cal/m^3 °C
ft	30.48	cm
Btu/ft hr °F	1.488	kg cal/m hr °C
Btu/cu ft	8.899	kg cal/m^3
sq ft hr °F/Btu	1.761	sq cm °C/mw
Btu/sq ft	2.34×10^{-2}	kg cal/m^2
°F	5/9 (°F-32)	°C or °K

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INTRODUCTION

Background

On 24 March 1953, Drs. Harl P. Aldrich, Jr. and Henry M. Paynter of the Massachusetts Institute of Technology entered into research contract DA-19-016-ENG-2641, "Analytical Studies Related to Freezing and Thawing of Ground" with the Arctic Construction and Frost Effects Laboratory of the U.S. Army Engineer Division, New England.

This report describes results of investigations included within the following paragraph of Appendix "A" of the contract:

> "1. Adapt the formulas and methods, developed under an investigational contract during Fiscal Year 1953, for predicting the depth and rate of frost penetration to the prediction of the depth and rate of thaw beneath surfaces kept free of snow."

Scope

The investigation described herein is directed primarily at reviewing and clarifying computational techniques available for solving frost penetration problems in non-uniform soil. The use of λ values in the modified Berggren equation, both for step-change and gradual variation in surface temperature, is given particular attention.

Sample problems are included to illustrate the variation in results obtained by the available techniques.

GENERAL

Analytic solutions for the depth of frost or thaw penetration in homogeneous soil have been developed for various boundary conditions (Aldrich and Paynter, 1953; Jumikis, 1952; Ruckli, 1950). However, for the non-homogeneous or multilayer system which occurs typically below highway and airfield pavements, approximate computation techniques must be used. A number of approaches representing varying degrees of approximation have been suggested but no thorough study of their comparative accuracy and simplicity in use has been undertaken. One reason for this is that no exact solution for the multilayer case is known and thus no basis for comparison exists.

An exact solution is virtually impossible by a direct analytical approach. Consequently, the "exact" solution must be found from a numerical solution of the fundamental thermal equations or from an analog approach using equipment such as an electronic or hydraulic analog computer.

In this report, the computational methods for the stratified case are reviewed and clarified particularly with reference to the modified Berggren equation where the correction factor λ may be misinterpreted. The approximate techniques for solving the multilayer case are presented and compared with sample solutions.

APPROXIMATE TECHNIQUES

Latent heat only, Stefan equation

The simplest approach is based on the Stefan formula, which ignores the volumetric specific heat of the soil and therefore can be expected to give computed depths of frost or thaw penetration which are greater than the actual.

For uniform soil, the Stefan equation may be written:

$$X = \sqrt{\frac{48\ k\ F}{L}} \tag{1}$$

where X is the depth of frost penetration in feet. For stratified soil, two approaches yielding identical results may be used. The so-called "St. Paul" method makes use of the following relationship for the partial freezing index required to freeze any layer "n":

$$F_n = \frac{L_n d_n}{24}\left(\frac{d_1}{k_1} + \frac{d_2}{k_2} + \ldots\ldots\frac{d_n}{2k_n}\right)$$

or:

$$F_n = \frac{U_n}{24}\left(R_1 + R_2 + \ldots\ldots\frac{R_n}{2}\right) \tag{2*}$$

where:

$U_n = L_n d_n$; the latent heat removed when freezing the n^{th} layer of thickness d_n.

$R_n = \frac{d_n}{k_n}$; thermal resistance.

If the freezing index for the locality in question is known, the frost depth can be determined by finding those layers whose sum of partial freezing indexes equals the total for the period. The total freezing index for any period follows from eq 2:

$$F = F_1 + F_2 + \ldots\ldots F_m$$

$$= \frac{U_1}{24}\left(\frac{R_1}{2}\right) + \frac{U_2}{24}\left(R_1 + \frac{R_2}{2}\right) + \ldots\ldots\frac{U_m}{24}\left(R_1 + R_2 + \ldots\ldots\frac{R_m}{2}\right)$$

$$= \sum_{n=1}^{m} \frac{U_n}{24}\left[\frac{R_n}{2} + \sum_{1}^{n-1} R_p\right] \tag{3}$$

*The derivation of eq 2 is given on page 44 of Aldrich and Paynter (1953).

which is eq 2-54 in Aldrich and Paynter (1953) when $U_n = L_n d_n$. It has been shown that the total freezing index may also be written:

$$F = \sum_{n=1}^{m} \frac{R_n}{24} \left[\frac{U_n}{2} + \sum_{p=n+1}^{m} U_p\right]. \tag{4}$$

A procedure equivalent to that given above is to determine an effective "L/k" for the frozen zone and compute the depth of frost penetration from eq 1. The "St. Paul" method is preferred to this procedure because it is more direct. However, this alternate approach is fundamental to understanding the nomograph procedure (Aldrich and Paynter, 1953, p. 49) where the correction factor λ is introduced.

Equation 1 for uniform soil may be rewritten:

$$\frac{L}{k} = \frac{48}{X^2} F.$$

F from eq 4 for the multilayer case may be substituted to obtain:

$$\frac{L}{k_{eff}} = \frac{2}{X^2} \left[\sum_{n=1}^{m} R_n \left(\frac{U_n}{2} + \sum_{p=n+1}^{m} U_p \right) \right]$$

or:

$$\frac{L}{k_{eff}} = \frac{2}{X^2} \left[\frac{d_1}{k_1} \left(\frac{L_1 d_1}{2} + L_2 d_2 + \ldots\ldots + L_m d_m \right) + \frac{d_2}{k_2} \left(\frac{L_2 d_2}{2} + L_3 d_3 + \ldots\ldots + L_m d_m \right) + \ldots\ldots\ldots\ldots + \frac{d_m}{k_m} \left(\frac{L_m d_m}{2} \right) \right] \tag{5}$$

which is the same equation as that given in Step 5, page 49, Aldrich and Paynter, 1953. Thus, to estimate the frost depth X, determine L/k_{eff} from eq 5, then compute X from eq 1. If the computed value of X differs appreciably from that originally estimated, the computation may be repeated.

The procedure outlined above is identical to that suggested for use with the frost penetration nomograph (Aldrich and Paynter, 1953) except that here the correction coefficient λ has been omitted, which is consistent with an approach that ignores volumetric specific heat.

Latent heat plus volumetric specific heat of frozen soil

As a second somewhat less approximate approach, the heat removed from each frozen layer as it is cooled from the mean annual temperature to 32F may be added to the latent heat, $L_n d_n$. More specifically, it is assumed that the total heat removed from the n^{th} layer is now:

$$U_n = L_n d_n + C_n v_o d_n. \quad (6)$$

Thus, U_n from eq 6 may be used in eq 2-5 to give predicted depths of frost penetration somewhat closer to the actual. However, the heat given off while cooling the unfrozen soil and that released by the frozen soil as it cools from 32F are still neglected. Therefore, the predicted depths are still greater than the actual.

Adaptation of the modified Berggren equation

The soundest approach to frost penetration predictions for multilayered systems is based on the modified Berggren equation. This can be written, with little sacrifice in accuracy, as the depth of frost penetration computed from the Stefan equation (Aldrich and Paynter, 1953) times a correction coefficient λ which accounts primarily for the volumetric specific heat effects.

In the development of the modified Berggren formula the surface temperature was assumed to change suddenly from v_o degrees above freezing to v_s degrees below freezing where it remained constant. This case is referred to as a step-change in surface temperature. The relationship among the dimensionless parameters α, μ, and λ (Fig. 1) is strictly applicable only to this case and:

$$\alpha = \frac{v_o}{v_s} \text{ and } \mu = \frac{C}{L} v_s.$$

In reality a pavement surface experiences daily as well as seasonal temperature fluctuations. Indeed, a sinusoidal variation in surface temperature over an annual period more nearly represents the true case. A correction coefficient λ' will be defined which applies to the seasonal depth (maximum depth) of frost penetration for this case. As before, λ' is a function of α and μ which are expanded to become:

$$\alpha = \frac{v_o}{v_s} = \frac{v_o t}{F} \qquad \mu = \frac{C}{L} v_s = \frac{CF}{Lt}$$

where v_o and v_s are defined as shown in Figure 2. The correction coefficient λ' may be estimated from Table I which compares results of computer solutions for this case with the analytic solution applicable to the step-change in surface temperature.

Frost penetration for four cases is considered as follows:

a. For homogeneous soil assuming a step-change in surface temperature from v_o degrees above freezing to v_s degrees below freezing, λ is given by the curves in Figure 1 and the depth of frost penetration at any time t is:

$$X = \lambda \sqrt{\frac{48\, k\, v_s\, t}{L}}.$$

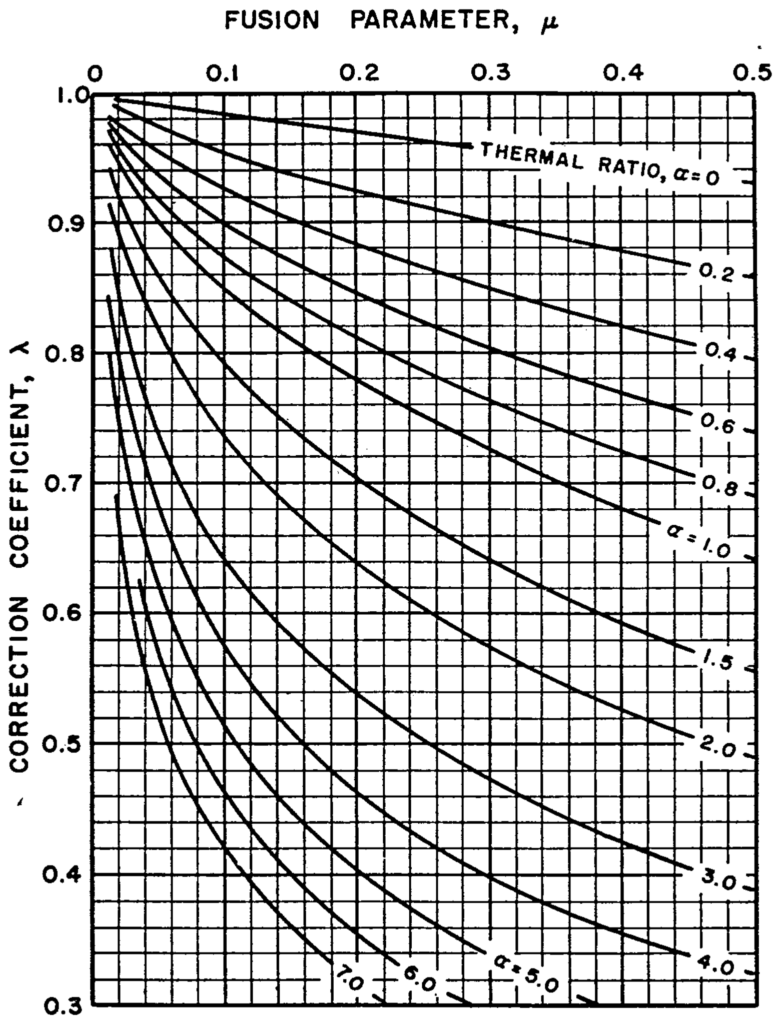


Figure 1. Correction coefficient in the modified Berggren equation.

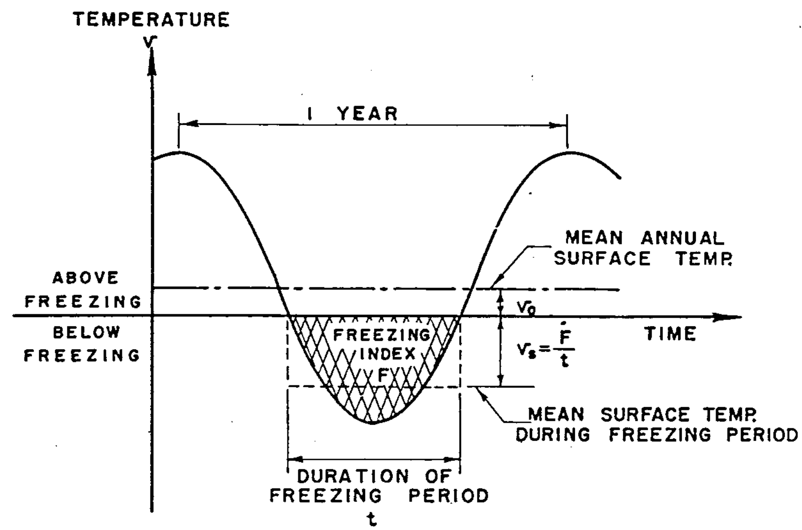


Figure 2. Assumed sinusoidal annual variation in surface temperature.

b. For homogeneous soil assuming a surface temperature varying sinusoidally during the year, λ' may be estimated from Table I and the maximum depth of frost penetration for the freezing season is:

$$X = \lambda' \sqrt{\frac{48\,k\,F}{L}}.$$

It should be noted that λ' is defined only for the entire freezing season. Therefore, this approach must not be used with a partial freezing index to determine frost depth vs time curves.

c. For stratified soil assuming a step-change in surface temperature from v_o degrees above freezing to v_s degrees below freezing, the following procedure for determining a frost depth vs time curve is suggested:

Step 1. Compute weighted values of volumetric specific heat and latent heat within the depth of frost penetration X for which the time is desired:

$$\left. \begin{aligned} C_{wt} &= \frac{C_1 d_1 + C_2 d_2 + \ldots\ldots + C_n d_n}{X} \\ L_{wt} &= \frac{L_1 d_1 + L_2 d_2 + \ldots\ldots + L_n d_n}{X}. \end{aligned} \right\} \quad (7)$$

Step 2. Determine α and μ from:

$$\alpha = \frac{v_o}{v_s} \qquad \mu = \frac{C_{wt}}{L_{wt}} v_s.$$

Step 3. Enter Figure 1 to obtain λ.

Step 4. The time t required to freeze to depth X is now computed from:

$$t = \frac{F}{v_s \lambda^2}$$

where F is given by eq 3 or 4. Note that this time is equal to the time computed from the Stefan equation divided by λ^2.

d. For stratified soil subjected to a step-change in surface temperature over a complete freezing season for a given locality, the maximum depth of frost penetration may be determined using the procedure exactly as suggested in Aldrich and Paynter (1953, p. 47-49). This procedure uses:

$$X = \lambda \sqrt{\frac{48\,F}{\left(\frac{L}{k}\right)_{eff}}}$$

where λ is a semi-empirical value given in Aldrich and Paynter (1953, Fig. 9A). The problem involving a stratified soil subjected to a sinusoidal surface temperature is discussed in Aldrich and Nordal (1957).

Again, this procedure must not be used with partial freezing indexes for obtaining frost depth vs time curves for the season.

Table I. Effect of sinusoidally varying surface temperature on correction coefficient.

Thermal ratio α	Fusion parameter μ	Correction coefficient λ	λ'
0	0.191	0.97	0.998
1.025	0.195	0.78	0.724
2.040	0.196	0.64	0.595
0	0.382	0.94	0.998
0.519	0.386	0.80	0.804

λ : Given in Figure 1, modified Berggren equation for step-change in surface temperature.

λ': Determined from results of studies on IBM-701 computer for surface temperature varying sinusoidally.

SAMPLE SOLUTIONS

Computational procedures for a step-change in surface temperature are given in the following sample solutions. These will also serve to indicate, in a quantitative sense for the cases used, the error involved in ignoring volumetric specific heat.

Two extreme cases of stratified soil have been selected. Data for the solutions are given in Table II and the results of the computations are shown on Figures 3 and 4. Case 1 is selected for a sample computation and determination of the time required to freeze the soil to a depth of 0.3 ft.

Latent heat only, Stefan equation, "St. Paul" approach

The surface freezing index required to freeze the soil to X = 0.3 ft may be determined from eq 3:

$$F = \frac{U_1}{24}\left(\frac{R_1}{2}\right) + \frac{U_2}{24}\left(R_1 + \frac{R_2}{2}\right)$$

$$= \frac{L_A d_A}{24}\left(\frac{d_A}{2k_A}\right) + \frac{L_B d'_B}{24}\left(\frac{d_A}{k_A} + \frac{d'_B}{2k_B}\right)$$

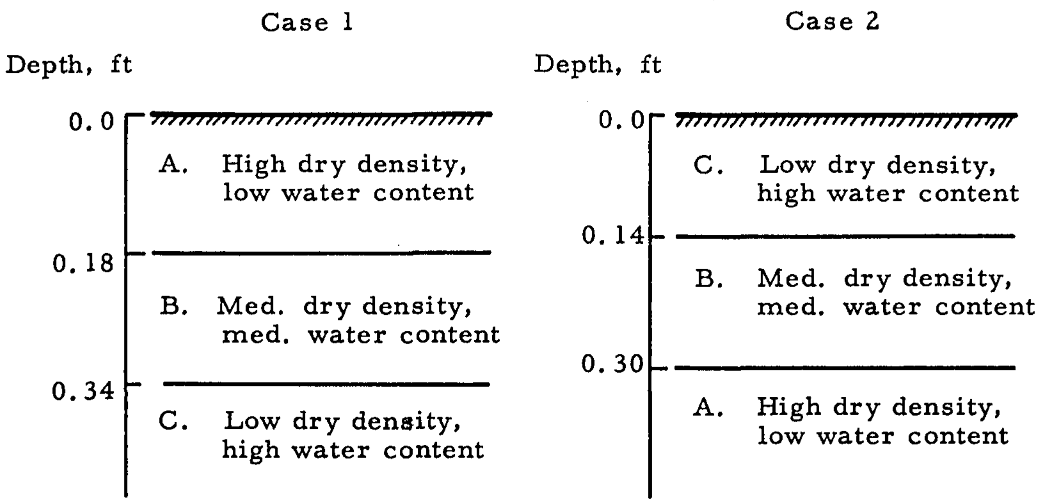
$$= \frac{(800)(0.18)}{24}\left(\frac{0.18}{2 \times 1.70}\right) + \frac{(1750)(0.12)}{24}\left(\frac{0.18}{1.70} + \frac{0.12}{2 \times 1.45}\right)$$

$$= 0.32 + 1.29 = 1.61 \text{ degree-days.}$$

Since the surface temperature is 22F, v_s = 32 - 22 = 10F. Thus: $t = F/v_s$ = 1.61/10 = 0.161 days = 3.86 hours. The plotted point is shown in Figure 3.

Table II. Soil profile, thermal properties and boundary conditions for sample problems; Case 1 and Case 2.

Soil Profiles:



Thermal Properties:

Layer	k Btu/ft hr °F	C Btu/cu ft °F	L Btu/cu ft
A. High dry density, low water content	1.70	28.0	800
B. Med. dry density, med. water content	1.45	29.5	1750
C. Low dry density, high water content	1.10	32.0	3000

Boundary Conditions:

1. Soil initially at 42F throughout (v_o = 10 F)
2. Surface temperature dropped suddenly to 22 F. (v_s = 10 F)

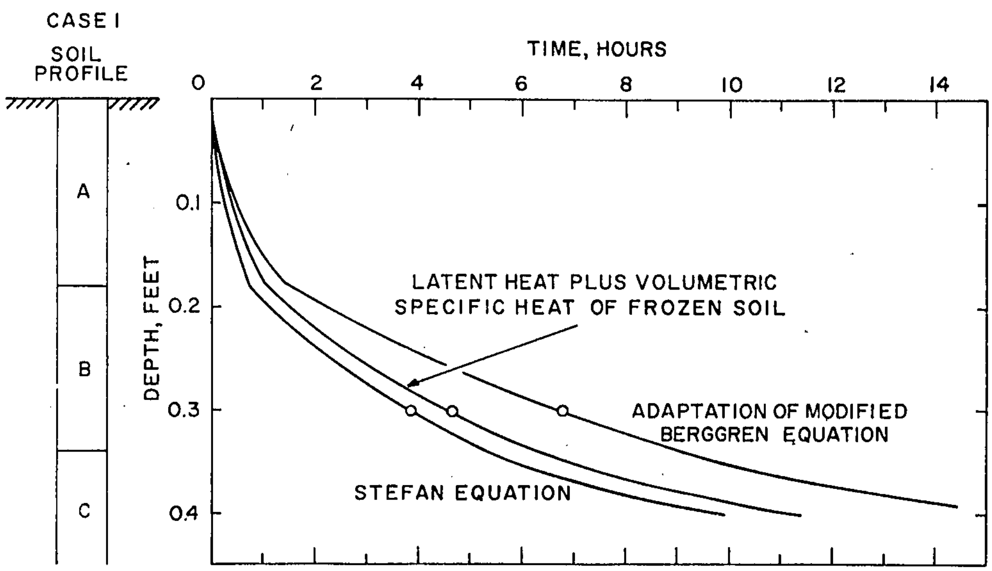


Figure 3. Prediction of frost penetration by various analytical techniques; Case 1 solution. Soil initially at 42F; surface temperature suddenly changed to 22F.

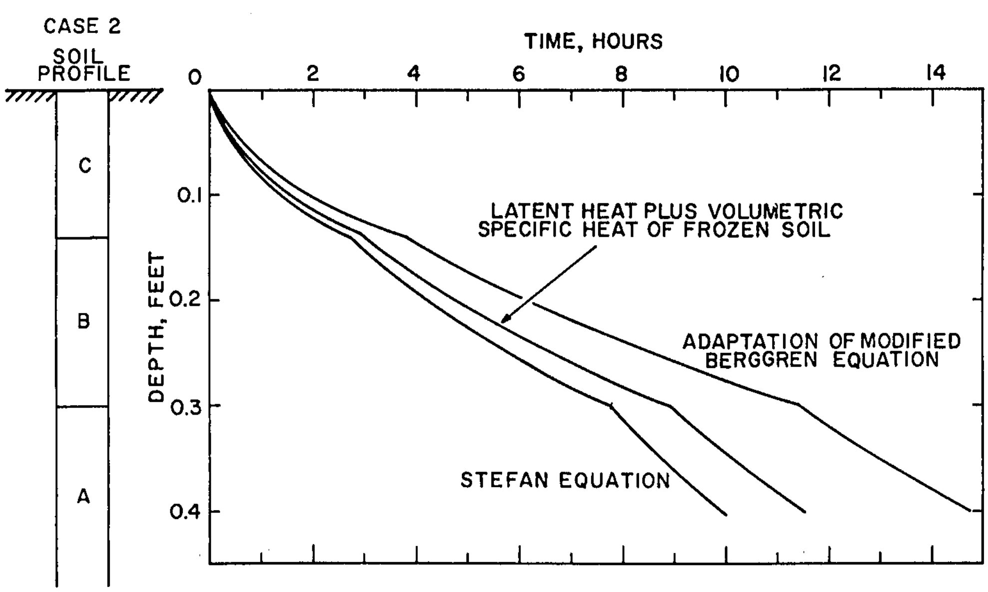


Figure 4. Prediction of frost penetration by various analytical techniques; Case 2 solution. Soil initially at 42F; surface temperature suddenly changed to 22F.

Latent heat plus volumetric specific heat of frozen soil

In this case eq 6 is used:

$$U_n = L_n d_n + C_n v_o d_n$$

in place of:

$$U_n' = L_n d_n.$$

Thus:

$$U_1 = (800)(0.18) + (28.0)(10)(0.18) = 194 \text{ Btu/sq ft}$$

$$U_2 = (1750)(0.12) + (29.5)(10)(0.12) = 245 \text{ Btu/sq ft}$$

and from eq 3:

$$F = 1.94 \text{ degree-days}$$

and:

$$t = 4.65 \text{ hours}$$

in the same manner as before.

Adaptation of modified Berggren equation

A step-change in surface temperature is considered here and, therefore, the procedure suggested previously may be used.

Step 1. $$C_{wt} = \frac{(28.0)(0.18) + (29.5)(0.12)}{0.30} = 28.6 \text{ Btu/cu ft °F}$$

$$L_{wt} = \frac{(800)(0.18) + (1750)(0.12)}{0.30} = 1180 \text{ Btu/cu ft}$$

Step 2. $$\alpha = \frac{v_o}{v_s} = \frac{10}{10} = 1.0$$

$$\mu = \frac{C_{wt}}{L_{wt}} v_s = \frac{28.6}{1180} (10) = 0.24.$$

Step 3. Figure 1 gives $\lambda = 0.755$.

Step 4. From the Stefan "St. Paul" approach:

$$t = 3.86 \text{ hours.}$$

Thus, for this case:

$$t = \frac{3.86}{\lambda^2} = 6.78 \text{ hours.}$$

Results of all computations for the two cases are given in Figures 3 and 4.

At this writing, exact numerical solutions for these two cases are not available. However, both problems were solved on the hydraulic analog computer. In Case 1, the analog depth was about 3% greater than that computed by the modified Berggren equation while results for Case 2 were in even closer agreement.

CONCLUSIONS AND RECOMMENDATIONS

The best approach for determining the depth of frost penetration in non-uniform soil is believed to be by use of an adaptation of the modified Berggren equation. While a computational procedure is recommended, additional study is needed to improve the semi-empirical curves for the correction factor λ (Aldrich and Paynter, 1953; Fig. 9A).

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