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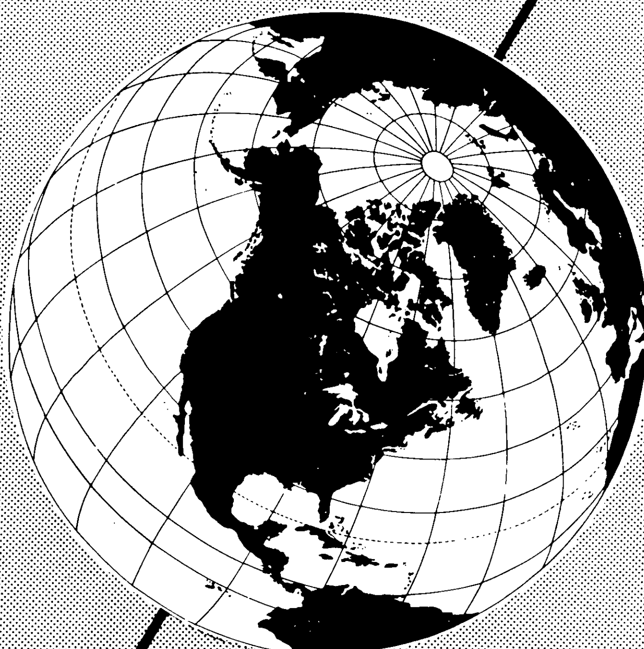

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DENSITY OF SINGLE CRYSTALS OF ICE FROM A TEMPERATE GLACIER

by
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INTRODUCTION

The variability in the measurements of density of ice has been given considerable study, using ice of various ages and from numerous sources, and various techniques of measurement. However, since there is no apparent agreement of the values obtained by the different experiments, it can be assumed that the density of ice is dependent on the type of ice, on the conditions under which it was frozen, on its history, and on its crystal structure. Whether the ice is in the form of a single crystal or is a crystal aggregate may affect the values of densities obtained.

Dorsey (1940) gives a rather thorough compilation of results obtained by various experimenters who give density values varying between 0.908 g/cm^3 to 0.918 g/cm^3 at 0°C. These values are for both natural and artificial ice aggregates.

Ginnings and Corruccini (1947) calculated the density of ice from the calibration of their ice calorimeter, the heat of fusion of ice, and the densities of water and mercury, and arrived at a value of 0.91671 g/ml for ice at 0°C and one atmosphere. This compares favorably with the value of 0.9168±0.0005 g/ml given in the International Critical Tables (1928). No evidence was obtained with the ice calorimeter that the true density of ice is anything but constant.

None of the investigators appears to have attempted accurate determination of the density of single ice crystals. The experimental work reported here was carried out in the refrigerated laboratories of SIPRE at Wilmette, Illinois.

LIST OF SYMBOLS USED

W_s Weight
V_s Volume
D_s Density
s = 1 for empty 10-ml pycnometer
s = 2 for 10-ml pycnometer plus liquid
s = 3 for empty 100-ml pycnometer
s = 4 for 100-ml pycnometer plus liquid
s = 5 for ice in air
s = 6 for submerged ice
s = 7 for 10-ml pycnometer plus mercury.

The primed weight symbols (W'_s) refer to the vacuum-corrected weights

t_1 = temperature at which density of liquid was determined (−4.5°C)
t_2 = temperature at which density of ice specimen was determined
$D_a{}^{t_2}$ = density of air at temperature t_2
D_w = density of balance weights
$D_m{}^{t_1}$ = density of mercury at temperature t_1
$D_s{}^{t_2}$ = density of ice at temperature t_2
$D_L{}^{t_1}$ = density of liquid at temperature t_1
ΔV = change in volume of liquid between t_1 and t_2
V_w = volume of balance weights
α = coefficient of cubical expansion of pyrex glass
k = coefficient of cubical expansion of liquid.

EXPERIMENTAL PROCEDURE

Any list of the physical properties of a substance usually includes its density or specific gravity. If density values obtained for the same substance by different observers or different methods are to be relied upon, they must be both accurate and comparable. Therefore, the previously mentioned disagreement of ice density values obtained by different observers and different methods, assuming both to be reliable, must be due to other factors.

The density determinations discussed in this paper were made with the hydrostatic weighing method: the specimen of ice was first weighed in air, and then weighed while suspended in a liquid of known density. The liquid was 2,2,4-trimethylpentane, chosen for its low density and its insolubility in water.

The density of the liquid (saturated with H_2O by contact with ice) was determined by pycnometer. For this, two special 10-milliliter pycnometers were designed, similar to that of Washburn and Smith (1934), with a graduated capillary and fitted with a ground glass cap which could be sealed with grease to prevent any evaporation of the highly volatile liquid.

For the accuracy desired in this work, the effect of the buoyancy in air of the objects on the balance pans was taken into account. The formula for obtaining the true vacuum-corrected weight W' from the weight W needed to balance an object of volume V is given by:

$$W' = W + V_s D_a - \frac{W D_a}{D_w},$$

where $V_s = \frac{W}{D_s}$ is the volume of the solid being weighed, $\frac{W}{D_w}$ is the volume occupied by the balance weights (W) consisting of material having a density D_w, and D_a is the density of air at the time and place of weighing. Since no tables were available giving the density of air for the range of temperature in which the measurements were performed, the standard formula for the density of air

$$D_a{}^t = \frac{0.001293\ (P - 0.0038\ HP^t_{H_2O})}{(1 + 0.00367t)\ 760} \qquad (2)$$

was employed. P is the barometric pressure in millimeters of mercury, H is the relative humidity in percent, $P^t_{H_2O}$ is the vapor pressure of ice at the balance room temperature t°C.

Triple distilled dental mercury was used as the calibrating liquid for the 10-ml pycnometer. The mercury was dripped into the pycnometer with a capillary tube of small enough diameter to allow the displaced air to escape, thus preventing any air from being trapped between the mercury-glass interface. The volume of the 10-ml pycnometer was then calculated from the relation

$$V_1{}^{t_1} = \frac{W'_7 - W'_1}{D_m{}^{t_1}}\ , \qquad (3)$$

where W'_7 and W'_1 are the vacuum-corrected weights of the pycnometer plus mercury and of the empty pycnometer respectively and D_m is the density of mercury at t_1°C. The pycnometers were then filled with 2,2,4-trimethylpentane, the density of which was obtained from

$$D_L{}^{t_1} = \frac{W'_2 - W'_1}{V_1{}^{t_1}}\ , \qquad (4)$$

where W'_2 is the weight of the 10-ml pycnometer plus liquid.

A 100-ml pycnometer of similar design was employed in determining the coefficient of expansion of the liquid. It was necessary to know this in order to determine the density of the liquid at any temperature t_2. The diameter of the uniform capillary of this pycnometer was measured with a filar micrometer mounted on a microscope, and thereby the average volume of each division of the capillary was obtained. Then the 100-ml pycnometer plus liquid was submerged in a constant temperature bath and, varying the temperature of the bath, the change in volume of the liquid in the pycnometer capillary was noted.

The volume, V_3, of the 100-ml pycnometer was obtained by

$$V_3{}^{t_1} = \frac{W'_4 - W'_3}{D_L{}^{t_1}}\ , \qquad (5)$$

where W'_4 and W'_3 are the vacuum-corrected weights of the pycnometer plus liquid and of the empty pycnometer respectively.

The volume of the 100-ml pycnometer V_3 at temperature t_2 was determined by the relation

$$V_3{}^{t_2} = V_3{}^{t_1}\ [1 + \alpha\ (t_2 - t_1)], \qquad (6)$$

where α is the coefficient of cubical expansion of pyrex glass.

$$V_3{}^{t_2} - V_3{}^{t_1} = \alpha\ V_3{}^{t_1}\ (t_2 - t_1)$$

and

$$V_3{}^{t_2} + \Delta V = V_3{}^{t_1}\ [1 + k\ (t_2 - t_1)],$$

where k is the cubical expansion coefficient of the liquid and ΔV is the observed change in volume of the liquid between t_2 and t_1.

$$k' = \frac{\Delta V}{V_1{}^{t_1}\ (t_2 - t_1)}\ ,$$

where k' is the difference between the expansion coefficients of liquid and glass.

$$k = \alpha + k'.$$

Therefore the expansion coefficient of the liquid was determined by:

$$k = \alpha + \frac{\Delta V}{V_3{}^{t_1}\ (t_2 - t_1)} \qquad (7)$$

With the density of the liquid known at one temperature $D_L{}^{t_1}$, and assuming the coefficient of expansion to be constant through the small range of temperature used, the density of the liquid at any other temperature $D_L{}^{t_2}$ in this range was calculated from

$$D_L{}^{t_2} = D_L{}^{t_1}\ [1 + k\ (t_2 - t_1)]. \qquad (8)$$

The ice specimens used were large, clear, pure single crystals from the Mendenhall Glacier near Juneau, Alaska, which is a temperate glacier (i.e., the temperature of the ice at a depth of more than approximately 40 feet is always the melting point at the corresponding pressure). The ice crystals were machined into cylinders approximately 4 cm in diameter and of approximately 60 g weight. About ½ cm of fine steel wire was frozen into the cylinder so that the ice could be suspended from one arm of the balance into a beaker. Immediately after the weighing in air, the beaker was filled with liquid, and, when the liquid and ice were at the same temperature, the weighing in liquid was made. Temperatures were measured by suspending a copper-constantan thermocouple into the liquid. The accuracy of these measurements was determined to be ±0.02°C.

The density of the specimen of ice at t°C, $D_s^{t_2}$, was calculated from the relation

$$D_s^{t_2} = \frac{W'_5 \, D_L^{t_2}}{W'_5 - W'_6}, \quad (9)$$

where $D_L^{t_2}$ refers to the density of the liquid and W'_5 and W'_6 are the true weights of the ice in a vacuum and in the liquid respectively.

RESULTS AND DISCUSSION

For determining the density of 2,2,4-trimethylpentane, two 10-ml pycnometers were used. Two volumes of these bottles are given in Table I. Table II gives the values of the density of the liquid determined at −4.5°C. The two independent determinations yielded the same value to the fifth decimal place, and this value was used in calculating the volume of the 100-ml pycnometer given in Table III.

Table IV gives the measurements made in determining the coefficient of expansion of 2,2,4-trimethylpentane. The value of k obtained was used to determine the density of the liquid at t_2, that is, at any other temperature within the range.

Table V gives the measurements made for the determination of the density of the single ice crystal. Column 1 gives the number arbitrarily assigned to each single ice crystal. The letters a, b, c, etc. refer to different samples taken from one single crystal. Column 2 gives the volume of each specimen, and Column 4 the computed densities at t_2°C, given in column 3. Column 5 refers to values in Column 4 corrected to −3.5°C, using an approximate coefficient of cubical expansion of 1.6×10^{-4}/°C. This value is very nearly that of 1.58×10^{-4} given by Dorsey (1940). Column 6 lists the mean of the corrected values of each individual sample, and Column 7 the deviation from the mean. How individual specimens deviate from pure, single ice crystals is noted in Column 8. Specimens 21a and 21b were cut from a block of clear commercial ice with no air bubbles, to give a comparison between single ice crystals and another type of ice.

Examination of the data shows that the maximum deviation from the mean of several determinations is ±0.00001. An error calculation was performed. This error is an estimated error; that is, it was computed on the basis of the estimated error of each individual measurement, such as the weighings and the density determination of the liquid. The estimated error was calculated using the general formula for computing the error of a function where

$$N = f(u_1, u_2, u_3 \ldots \ldots \ldots u_n)$$

and

$$\Delta N = \frac{\partial N}{\partial u_1} \Delta u_1 + \frac{\partial N}{\partial u_2} \Delta u_2 + \frac{\partial N}{\partial u_3} \Delta u_3 + \cdots\cdots\cdots + \frac{\partial N}{\partial u_n} \Delta u_n.$$

$\Delta u_1, \Delta u_2, \Delta u_3 \ldots\ldots \Delta u_n$ are the estimated errors involved in each measurement. A thorough discourse on this error computation is given in the Appendix.

The calculated estimated error of the density determinations of the ice specimens is $\pm 2.1 \times 10^{-5}$ g/cm³. The maximum deviation from the mean of several determinations falls well within the limits of this error.

Samples 20c and 20d, containing the Tyndall figure (hexagonal prismatic voids), were cut up and the approximate volume of the prisms obtained by measuring the length of the sides of the hexagonal face and the height of the prism under a microscope. Since the hexagonal faces were irregular, the mean of the lengths was used and approximate volumes obtained by the formula for the volume of a hexagonal prism, Volume = 2.598 hs^2, where s is the length of a side and h is the height. This approximation is good because of the small volumes involved. Table VI is a tabulation of the measurements and calculations. Corrected densities were obtained by assuming the Tyndall voids to be filled by ice of the same density as a pure specimen (#20a). The new density value for 20c is well within the estimated error. However, the value for 20d falls outside. This may be due to the very slight fogging that occurs in this sample, or to other, unknown factors.

The conclusions obtained by the examination of all these data are:

(1) Density is slightly but measurably variable from one crystal to another.

(2) Any imperfections, such as appear in samples 20c, 20d, and 20e, are immediately evident in the density determinations, and correction for these imperfections gives a corrected density essentially the same as that of clear samples with no apparent imperfections.

(3) Clear crystal aggregates can have a lower density than pure single ice crystals.

TABLE I. VOLUME OF 10-ML PYCNOMETERS

	Pycnometer #1	Pycnometer #2
Temperature (t_1°C)	−4.5	−4.5
Vacuum-corrected weight of empty pycnometer (g)	12.7853	12.8867
Vacuum-corrected weight of pycnometer plus mercury	137.8514	142.2168
Density of mercury at t°C	13.6063	13.6063
Volume of pycnometer (cm³)	9.1950	9.5102

TABLE II. DENSITY OF 2,2,4-TRIMETHYLPENTANE

	Pycnometer #1	Pycnometer #2
Temperature (t_1°C)	−4.5	−4.5
Vacuum-corrected weight of empty pycnometer (g)	12.7853	12.8867
Vacuum-corrected weight of pycnometer plus liquid (g)	19.3213	19.6332
Density of liquid (g/cm³)	0.71176	0.71176

TABLE III. VOLUME OF 100-ML PYCNOMETER

	Trial 1	Trial 2
Temperature (t°C)	−4.8	−4.5
Vacuum-corrected weight of empty pycnometer (g)	42.0613	42.0613
Vacuum-corrected weight of pycnometer plus liquid (g)	116.0141	116.0131
Volume of pycnometer (cm³)	104.5821	104.5818

Volume of each division of pycnometer capillary is 0.01257 cm³.

TABLE IV.
COEFFICIENT OF EXPANSION OF 2,2,4-TRIMETHYLPENTANE

Temperature, °C	Divisions
+1.20	¾
−0.60	17
−3.20	40
−4.60	52
−4.95	55½
−5.85	63

Expansion equals 8.83 divisions/°C.
Coefficient of expansion, k equals 1.07×10^{-3}/°C.

TABLE V. DENSITY OF SINGLE ICE CRYSTALS

(1) Crystal Number	(2) Volume of Specimen (cm³)	(3) Temperature t_2°C	(4) Ice Density (g/cm³) at t_2°C	(5) Ice Density (g/cm³) corrected for −3.5°C	(6) Mean	(7) Maximum Deviation from Mean	(8) Remarks
15	63.2	−4.50	0.917272	0.917125			
		−4.65	0.917287	0.917118	0.917121	±4	
17	65.4	−3.97	0.917299	0.917230			
		−3.92	0.917283	0.917221	0.917225	±5	
2	49.0	−3.78	0.917329	0.917288			
		−3.68	0.917302	0.917276	0.917282	±6	
18	62.0	−3.82	0.917296	0.917249			
		−3.78	0.917305	0.917264			
		−3.73	0.917292	0.917258	0.917257	±8	
8	59.8	−3.95	0.917266	0.917200			
		−3.90	0.917269	0.917210			
		−3.80	0.917263	0.917219	0.917210	±10	
10	60.2	−3.85	0.917256	0.917204			
		−4.60	0.917376	0.917215	0.917209	±6	
19a	63.0	−4.45	0.917276	0.917137			
		−4.24	0.917242	0.917133			
		−2.91	0.917051	0.917138			
		−3.45	0.917125	0.917132	0.917135	±3	
19b	63.8	−3.78	0.917193	0.917150			
		−3.68	0.917168	0.917142			
		−3.97	0.917211	0.917142	0.917145	±5	
19c	64.0	−4.30	0.917266	0.917149			
		−4.45	0.917276	0.917137	0.917143	±6	
20a	75.1	−3.87	0.917188	0.917134			
		−3.95	0.917204	0.917138	0.917136	±2	
20b	64.0	−4.00	0.917216	0.917143			
		−4.01	0.917211	0.917136			
		−4.04	0.917216	0.917137	0.917139	±4	
20c	70.0	−3.99	0.917120	0.917048			Contains thin Tyndall figure
		−4.16	0.917143	0.917046			
		−4.17	0.917131	0.917033	0.917042	±9	
20d	70.1	−4.44	0.917046	0.916908			Contains thick Tyndall figure plus slight fog
		−4.38	0.917048	0.916909			
		−4.32	0.917038	0.916918			
		−4.16	0.917006	0.916909	0.916911	±7	
20e	73.7	−3.75	0.917128	0.917091			Contains slight fog
		−3.65	0.917109	0.917087			
		−3.57	0.917083	0.917073	0.917084	±9	
21a	94.3	−4.18	0.917122	0.917022			Crystal aggregate
		−4.10	0.917111	0.917023			
		−4.17	0.917099	0.917014	0.917020	±6	
21b	94.4	−4.17	0.917082	0.916984			Crystal aggregate
		−4.06	0.917059	0.916977			
		−3.99	0.917050	0.916978	0.916980	±4	

APPENDIX

The errors involved in each measurement are estimated errors. Only the absolute values of these errors are considered in the following calculations.

The errors of all direct measurements were considered to be the same as the reading accuracy of the measuring instruments. For example, the accuracy of the balance used in all the weighings was ±0.0001 grams. This value was considered as the estimated error of all weighings. Where values were taken from reliable tables, the accuracy was considered as that of the last decimal place stated. An example of this is the density of mercury, given as 13.6068 with an accuracy of ±0.0001.

The errors involved in the determination of the density of the ice were calculated on the basis of measurements on Crystal #2, which was the smallest specimen used.

The error involved in the determination of the volume of the weights,

$$V_w = \frac{W}{D_w},$$

is given by:

$$\Delta V_w = \frac{D_w \Delta W - W \Delta D_w}{(D_w)^2}.$$

D_w is the density of the weights = 8.40 g/cm³
ΔD_w = 0.01 g/cm³

The weight of the weights:

W_1 = 12.9 g for empty 10-ml pycnometer
W_2 = 19.3 g for 10-ml pycnometer and liquid
W_3 = 42.0 g for empty 100-ml pycnometer
W_4 = 115.9 g for 100-ml pycnometer and liquid
W_5 = 46.1 g for ice in air
W_6 = 10.4 g for submerged ice
W_7 = 137.2 g for 10-ml pycnometer and mercury

$\Delta W_1 = \Delta W_2 = \Delta W_3 = \Delta W_4 = \Delta W_5 = \Delta W_6 = \Delta W_7 = 0.0001$ g.

$\Delta V_{w_1} = 1.8 \times 10^{-3}$ cm³.

$\Delta V_{w_2} = 2.7 \times 10^{-3}$ cm³.

$\Delta V_{w_3} = 5.9 \times 10^{-3}$ cm³.

$\Delta V_{w_4} = 1.6 \times 10^{-2}$ cm³.

$\Delta V_{w_5} = 6.5 \times 10^{-3}$ cm³.

$\Delta V_{w_6} = 1.5 \times 10^{-3}$ cm³.

$\Delta V_{w_7} = 2.2 \times 10^{-2}$ cm³.

The errors involved in determining the volume of the solid,

$$V_s = \frac{W_s}{D_s},$$

are given by:

TABLE VI. CORRECTION FOR TYNDALL FIGURES

Crystal #20c

s Length of side (mm)	h Height (mm)	Volume (cm³)	Weight of equivalent volume of ice (g)	New density corrected for −3.5°C (g/cm³)
2.3	0.51	6.4×10^{-3}	0.0059	(1) 0.917132
1.6				(2) 0.917130
2.7				(3) 0.917117
2.4				
1.9				
2.2				
13.1				
Mean = 2.2 mm				Mean density = 0.917126 ±.000009

Crystal #20d (2 Tyndall Figures)

s Length of side (mm) #1	#2	h Height (mm) #1	#2	Volume (cm³) #1	#2	Weight of equivalent volume of ice (g) #1	#2	New density corrected for −3.5°C (g/cm³)
1.9	.98	1.6	0.32	.0106	.00026	.0098	.0002	(1) 0.917050
								(2) 0.917063
2.3	.30							(3) 0.917063
								(4) 0.917046
1.1	1.02							
1.4	.26							
1.4	.50							
1.3	.50							
9.4	3.36							
Mean = 1.6	0.56							Mean density = 0.917057 ± .000006

$$\Delta V_s = \frac{D_s \Delta W_s - W_s \Delta D_s}{(D_s)^2},$$

where W and ΔW are the same as above.

$D_{s_1} = 2.51$ g/cm³ $\Delta D_{s_1} = 0.01$ g/cm³ $\Delta V_{s_1} = 2.0 \times 10^{-2}$ cm³.
$D_{s_2} = 0.95$ g/cm³ $\Delta D_{s_2} = 0.01$ g/cm³ $\Delta V_{s_2} = 2.1 \times 10^{-1}$ cm³.
$D_{s_3} = 2.51$ g/cm³ $\Delta D_{s_3} = 0.01$ g/cm³ $\Delta V_{s_3} = 6.7 \times 10^{-2}$ cm³.
$D_{s_4} = 0.95$ g/cm³ $\Delta D_{s_4} = 0.01$ g/cm³ $\Delta V_{s_4} = 12.8 \times 10^{-1}$ cm³.
$D_{s_5} = 0.917$ g/cm³ $\Delta D_{s_5} = 0.0001$ g/cm³ $\Delta V_{s_5} = 5.4 \times 10^{-3}$ cm³.
$D_{s_7} = 9.63$ g/cm³ $\Delta D_{s_7} = 0.01$ g/cm³ $\Delta V_{s_7} = 1.5 \times 10^{-2}$ cm³.

The errors involved in the vacuum-corrected weights,

$$W' = W + (V_s - V_w)\, D_a{}^t,$$

are given by:

$$\Delta W' = \Delta W + (V_s - V_w)\Delta D_a{}^t + D_a{}^t\,(\Delta V_s - \Delta V_w),$$

where

$D_a{}^t$ is the density of air at time and place of weighing $= 0.00128$ g/cm³.
$\Delta D_a{}^t = 10^{-7}$ g/cm³.

$V_{s_1} = 5.2$ cm³ $V_{w_1} = 1.5$ cm³ $\Delta W'_1 = 1.2 \times 10^{-4}$ g.
$V_{s_2} = 14.3$ cm³ $V_{w_2} = 2.3$ cm³ $\Delta W'_2 = 3.7 \times 10^{-4}$ g.
$V_{s_3} = 16.8$ cm³ $V_{w_3} = 5.1$ cm³ $\Delta W'_3 = 1.8 \times 10^{-4}$ g.
$V_{s_4} = 120.9$ cm³ $V_{w_4} = 13.8$ cm³ $\Delta W'_4 = 1.7 \times 10^{-3}$ g.
$V_{s_5} = 49.2$ cm³ $V_{w_5} = 5.4$ cm³ $\Delta W'_5 = 1 \times 10^{-4}$ g.
$V_{s_6} =$ -- $V_{w_6} = 1.2$ cm³ $\Delta W'_6 = 1 \times 10^{-4}$ g.
$V_{s_7} = 14.3$ cm³ $V_{w_7} = 16.4$ cm³ $\Delta W'_7 = 1 \times 10^{-4}$ g.

The error involved in determination of the 10-ml pycnometer,

$$V_1{}^{t_1} = \frac{W'_7 - W'_1}{D_m{}^{t_1}},$$

is:

$$\Delta V_1{}^{t_1} = \frac{D_m{}^{t_1}\,(\Delta W'_7 - \Delta W'_1) - (W'_7 - W'_1)\,\Delta D_m{}^{t_1}}{(D_m{}^{t_1})^2}$$

$D_m{}^{t_1}$ is the density of mercury $= 13.6$ g/cm³
W'_7 is the vacuum-corrected weight of 10-ml pycnometer + mercury = 137.9 g
W'_1 is the vacuum-corrected weight of empty 10-ml pycnometer = 12.9 g
$\Delta D_m{}^{t_1} = .0001$ g/cm³.

$\Delta V_1{}^{t_1} = 6.9 \times 10^{-5}$ cm³.

The error involved in determining the density of the liquid at temperature t_1,

$$D_L{}^{t_1} = \frac{W'_2 - W'_1}{V_1{}^{t_1}},$$

is:

$$\Delta D_L{}^{t_1} = \frac{V_1{}^{t_1}\,(\Delta W'_2 - \Delta W'_1) - (W'_2 - W'_1)\,\Delta V_1{}^{t_1}}{(V_1{}^{t_1})^2}$$

$W'_2 = 19.3$ g $t_1 = -4.5$°C
$W'_1 = 12.9$ g $V_1{}^{t_1} = 9.2$ cm³

$\Delta D_L{}^{t_1} = 2.2 \times 10^{-5}$ g/cm³.

The error involved in determination of the volume of the 100-ml pycnometer,

$$V_3^{t_1} = \frac{W'_4 - W'_3}{D_L^{t_1}},$$

is:

$$\Delta V_3^{t_1} = \frac{D_L^{t_1}(\Delta W'_4 - \Delta W'_3) - (W'_4 - W'_3)\Delta D_L^{t_1}}{(D_L^{t_1})^2}$$

$W'_4 = 116$ g
$W'_3 = 42$ g
$D_L^{t_1} = 0.71$ g/cm³
$\Delta V_3^{t_1} = 1.1 \times 10^{-3}$cm³.

The error involved in determination of the coefficient of expansion of the liquid,

$$k = \alpha + \frac{\Delta V}{V_3^{t_1}(t_2 - t_1)},$$

is:

$$\Delta k = \Delta\alpha + \frac{V_3^{t_1}(t_2 - t_1)\,\Delta[\Delta V] - \Delta V\,[V_3^{t_1}\Delta t_2 + t_2\Delta V_3^{t_1} - V_3^{t_1}\Delta t_1 - t_1\Delta V_3^{t_1}]}{[V_3^{t_1}(t_2 - t_1)]^2}$$

α is the coefficient of cubical expansion of pyrex glass $= 0.033 \times 10^{-4}$

$V_3^{t_1} = 104.1$ g/cm³
$t_2 = -3.5$°C
$\Delta t_2 = \Delta t_1 = .02$°C
$\Delta V = 0.11$ g/cm³
$\Delta[\Delta V] = 0.001$ g/cm³
$\Delta k = 9.6 \times 10^{-6}$.

The error involved in the determination of the density of the liquid at any temperature t_2 within this range,

$$D_L^{t_2} = D_L^{t_1}[1 + k\cdot(t_2 - t_1)],$$

is:

$$\Delta D_L^{t_2} = \Delta D_L^{t_1} + D_L^{t_1}k\Delta t_2 + kt_2\Delta D_L^{t_1} + D_L^{t_1}t_2\Delta k - [D_L^{t_1}k\Delta t_1 + kt_1\Delta D_L^{t_1} + D_L^{t_1}t_1\Delta k]$$

$\Delta D_L^{t_2} = 1.5 \times 10^{-5}$.

The error involved in determining the density of ice,

$$D_s^{t_2} = \frac{W'_5 D_L^{t_2}}{W'_5 - W'_6},$$

is given by:

$$\Delta D_s^{t_2} = \frac{W'_5}{W'_5 - W'_6}\Delta D_L^{t_2} + \frac{D_L^{t_2}}{W'_5 - W'_6}\Delta W'_5 - \frac{W'_5 D_L^{t_2}}{(W'_5 - W'_6)^2}\Delta W'_5 + \frac{W'_5 D_L^{t_2}}{(W'_5 - W'_6)^2}\Delta W'_6$$

$W'_5 = 45.0$ g
$W'_6 = 10.1$ g
$\Delta D_s^{t_2} = \pm\ 2.1 \times 10^{-5}$ g/cm³.

This value is the estimated error of the density determinations discussed in this paper. Larger specimens would have slightly smaller estimated errors.

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