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Stress-Strain Relations in Snow Under Uniaxial Compression

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STRESS-STRAIN RELATIONS IN SNOW UNDER UNIAXIAL COMPRESSION.

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ABSTRACT

This paper contains a summary of work done during the past year on the uniaxial compression of snow. Two types of tests were performed; experiments at constant specific velocity and tests at constant load. The visco-plastic behavior of snow is correlated with density, temperature, stress, and snow type. It is found that rate of deformation can be represented by $v = f_1(\epsilon) \cdot f_2(K) \cdot f_3(\sigma) \cdot f_4(t)$, where $f_1(\epsilon)$ and $f_2(K)$ are functions of void ratio and absolute temperature, respectively, and are similar for constant velocity and constant load tests. $f_3(\sigma)$ is a function of stress which is linear for low stress but increases to an approximately cubic relationship for higher stresses. $f_4(t)$ is a function of time which is only important for creep tests. The effect of snow type is small, being less than the relatively large experimental errors encountered.

TESTS AT CONSTANT SPECIFIC VELOCITY

Compression. Stress-time curves were obtained for cylindrical specimens (dia 55 mm, length approx 100 mm) of various snows, both natural and sifted, under uniaxial compression at constant specific velocity. The general characteristics of such curves (Fig. 1) are a short initial slow increase of stress (region I) followed by a sharply increasing stress (II) and finally an approach to an approximately constant stress (III).

The initial region of slow stress increase is probably due to cylinder-end surface irregularities. If a snow sample is compressed slightly and then allowed to relax, this initial region is less prominent on a subsequent test. Therefore, this effect will be neglected in the following discussion.

A significant part of the stress-time curve is an increasing stress at a continuously decreasing rate. Here the stress induced by compression is relieved by a relaxation mechanism. The stress rate decreases until these two processes are equal. The stress rate does not drop to zero, owing to increasing density and perhaps also to strain hardening of the snow. There is, however, a fairly well defined region where the rate decreases abruptly (Fig. 2) and an equilibrium stress is attained.

If the snow is compressed in a series of increasing velocities, a

Figure 1. Constant velocity curve

curve as in Figure 2 is obtained. An equilibrium stress, σ_E , is reached for each specific velocity, $\underline{v}$. These are related by:

$$\underline{v} = k \sigma_E^r \quad (1)$$

k , which will be called the coefficient of plasticity under uniaxial compression, is a function of density and temperature. The velocities used in these experiments range from 10^{-5} sec^{-1} to 10^{-3} sec^{-1} . Values of r vary between 2 and 4 but a large majority fall between 3.0 and 3.2. In the region of investigation ($.3 < \gamma < .6$), r does not vary with density; k , however, varies greatly with density and, within the experimental error, can be represented as proportional to the void ratio, $\epsilon = \frac{\gamma_{ice} - \gamma}{\gamma}$, to a power, p :

$$k = A \epsilon^p, \quad (2)$$

where γ is the density. For densities between .44 and .61, p is about 4.

The temperature dependence of k was determined only in creep tests. It is assumed that this dependence is similar for creep and press tests (see equation 8).

Limited experiments indicate no marked dependence of r or k on snow type. More detailed work would probably show that k , but not r , depends somewhat on snow type. The basis for this prediction is that while various snow types and microcrystalline ice have the same r , the k for snows, when extrapolated using Equation 2, does not agree with the k for microcrystalline ice.

Relaxation. If the press is stopped after equilibrium pressure is reached, relaxation of stress occurs as in Figures 2 and 3.

* This is not the only possible representation.

Assuming elastic and plastic flow processes, the total specific velocity of deformation, $\underline{v}$, is the sum of an elastic velocity, $\underline{v}_E$, and a plastic velocity, $\underline{v}_p$:

$$\underline{v} = \underline{v}_E + \underline{v}_p. \quad (3)$$

Since the elastic strain is proportional to the stress, the specific velocity is

$$\underline{v}_E = -\frac{1}{z} \frac{dz}{dt} = \frac{1}{M} \frac{d\sigma}{dt}, \quad (4)$$

where M is the elastic modulus. Using Equation 1 for $\underline{v}_p$ we get:

$$\underline{v} = \frac{1}{M} \frac{d\sigma}{dt} + k\sigma^r. \quad (5)$$

Since $\underline{v} = 0$, the slope of the relaxation curve must be:

$$\frac{d\sigma}{dt} = -Mk\sigma^r. \quad (6)$$

Figure 2 is an actual experimental curve traced from a recorder graph. The density is assumed constant throughout. Data from this curve, plotted in Figure 4, show good agreement with Equations 1 and 6. The value of r is 3.4 for the step compression curve and 3.1 for the relaxation curve. These two values are considered equal since the results of many experiments do not indicate consistent differences.

Figure 2. Stress vs. time for a series of velocities
 $T = -10^\circ\text{C}$. $\gamma = .42 \text{ g/cm}^3$.

$$\begin{aligned} v_1 &= 6.34 \times 10^{-5} \text{ sec}^{-1} \\ v_2 &= 12.8 \times 10^{-5} \text{ sec}^{-1} \\ v_3 &= 25.8 \times 10^{-5} \text{ sec}^{-1} \\ v_4 &= 52.1 \times 10^{-5} \text{ sec}^{-1} \end{aligned}$$

Figure 3. Compression and relaxation curve
 $T = -10^\circ\text{C}$. $\gamma = .42 \text{ g/cm}^3$.

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Figure 4. Snow compression and relaxation data, taken from Figure 2
 $T = -10^{\circ}\text{C}$. $\gamma = .42 \text{ g/cm}^3$.

The value of M as calculated from these curves is $1.6 \times 10^9 \text{ dyne/cm}^2$. This is considerably smaller than the averaged Young's modulus for ice ($\bar{E} \approx 10^{11} \text{ dyne/cm}^2$). Further disagreement is found when Equation 5 is applied to the slope of the compression curve (Fig. 3). The initial slopes do not agree with Equation 5. Table I shows the difference in slope between the compression and relaxation curves of Figure 3 for several pressures. According to Equation 5, these differences should be constant.

TABLE I

σ (kg/cm)	$\left(\frac{d\sigma}{dt}\right)_{\text{comp.}} - \left(\frac{d\sigma}{dt}\right)_{\text{relax.}}$ (kg/cm ² -sec)
0.473	0.311
0.947	0.349
1.42	0.734

Equation 5 is probably unsatisfactory for increasing stress because of neglect of the dependence of strain rate on time. The equation holds well for relaxation where the strain rate is zero.

CREEP TESTS

A series of creep tests were performed. In these tests the deformation of snow was measured as a function of time for uniaxial compression at constant load. The stress was produced by dead-weight loads and deformation was measured with a dial gage. A typical series of creep curves is shown in Figure 5. The creep rate decreases continuously with time, having the appearance of asymptotically approaching a linear time de-

pendence. However, the slope of the apparent asymptote decreases with time and, for the longest runs made (~ 15000 min), a straight line is not yet attained. A good representation of these creep curves, then, does not involve an appreciable linear term for times less than many thousands of minutes.

One representation of the creep curve appears satisfactory for a large percentage of the cases:

$$\frac{\Delta z}{z_0} = B \sigma t^s. \quad (7)$$

Here $\frac{\Delta z}{z_0}$ is the relative deformation and B is a parameter depending on temperature and density. σ is the stress, which is considered constant for these experiments, because of the small changes in cross-sectional area. Values of s are usually near 0.8 although extreme values vary between 0.5 and 1.0. Not all creep curves can be described by Equation 7 unless two values of s are used in different time regions.

The linear dependence of deformation on stress, indicated in Equation 7, is good for the range of stress used in the creep tests ($2 < \sigma < 100 \text{ g/cm}^2$). For these low loads, the weight of the upper part of the sample on the lower part contributes appreciably to the deformation. Therefore, the dead-weight load plus one half the weight of the sample is taken as the total load. A typical series of creep curves for identical snows under various loads is shown in Figure 6. In Figure 7, the relative deformation at 1000 min is plotted against the total stress. It should be noted that, beside the good experimental agreement, there is some theoretical justification for using the factor 1/2 for the load contribution due to the weight of the sample. A theoretical development of this point can be found in the appendix.

Creep tests were conducted at -5°C , -10°C , and -20°C . The temperature variation of B can be represented by

$$B = \text{const.} \cdot e^{-\frac{\Delta H}{RK}} \quad (8)$$

where K is the absolute temperature and R is the gas constant. The activation energy, ΔH , for creep is about 14000 cal/mole.

The density dependence of B can be expressed as a function of the void ratio as in Equation 2. Figure 8 shows that the value of p is about 4, in good agreement with the constant

Figure 5. Creep curves
 $\gamma = .42 \text{ g/cm}^3$, $T = -10^\circ\text{C}$.

velocity tests.

Snow type does not appear to have any significant effect on $\bar{B}$ or $\bar{s}$. This means that, if any effect exists, it is masked by the fairly large experimental scatter observed, and that snow type is certainly not a primary factor in these experiments.

CONCLUSIONS

The dependence of uniaxial deformation on stress increases from a linear relationship at low stress to approximately a cubic relationship at high stress. This change occurs roughly in the neighborhood of 1 kg/cm^2 . More detailed study of this relationship is necessary.

The temperature dependence indicated in the creep tests leads to a value for the activation energy of about 14,000 cal/mole. This energy is greater than energies usually associated with diffusion processes and less than those associated with chemical processes. This indicates that the creep process may involve a combination of mechanisms (e.g. intergranular and intragranular slip). Tests on samples of widely differing grain size might shed light on this matter.

The empirical relations discussed in the preceding sections appear to be valid in many cases. Reproducibility is erratic, however, and deviations from these relations do occur. When carefully sifted snows are used, greater reproducibility is obtained than for the natural snows. Nevertheless, deviations of 10 to 20 % are often observed for supposedly identical sifted snows.

Greater reproducibility is necessary if more fundamental relationships are to be found. Alternatively a suitable theory of snow mechanics would provide equations that could be checked by existing data. In either case, relations valid for a wider range of conditions would result.

Plans for future experimental work include study of:

- a) Activation energy as a function of grain size.
- b) Creep behavior of snow-ice and single-crystal ice.
- c) Triaxial mechanics of snow. In particular, correlation between hydrostatic, laterally confined, and uniaxial tests will be attempted.
- d) Creep at stresses between the maximum used in the creep tests and the minimum stresses in the press tests.

Figure 6. Creep at various stresses
 $T = -10^\circ\text{C}$. $\gamma = .42 \text{ g/cm}^3$.

Figure 7. Deformation vs. stress
 for identical snows
 $T = -10^\circ\text{C}$. $\gamma = .42 \text{ g/cm}^3$.

Figure 8. Creep parameter vs. void ratio for various snows. $T = -10^\circ\text{C}$.

APPENDIX

DEFORMATION OF SNOW UNDER CERTAIN IDEALIZED CONDITIONS

Consider a homogeneous layer of snow of initial thickness $\underline{z}_0$ and density γ_0 with an applied uniaxial stress σ . We assume:

$$-\frac{1}{z} \frac{dz}{dt} = k\sigma, \quad (1a)$$

$$k = c\gamma^{-q}, \quad (2a)$$

and
$$\frac{1}{\gamma} \frac{d\gamma}{dt} = -\frac{1}{z} \frac{dz}{dt}. \quad (3a)$$

Assumption (1a) states that the exponent, r , in Equation 1 is unity, (2a) gives a density dependence which differs from that in Equation 2* but allows a straight-forward integration, and (3a) neglects lateral expansion under compression.

Combining (1a), (2a), and (3a) we find:

$$\frac{d\gamma}{\gamma^{1-q}} = c\sigma dt \quad (4a)$$

Integrating, we obtain the density after the load has been applied for a time, $\underline{t}$:

$$\gamma = \gamma_0 (1 + q\gamma_0^{-q} c\sigma t)^{\frac{1}{q}}. \quad (5a)$$

The thickness of the layer is now,

$$z = z_0 (1 + qc\gamma_0^{-q} \sigma t)^{-\frac{1}{q}}. \quad (6a)$$

If we consider the snow as made up of n initially identical layers of total thickness, $\underline{z}_0$, the weight of each layer will add to the stress on lower-lying layers. The stress on the m^{th} layer will be the applied stress plus the weight of the overlying layers:

$$\sigma_m = \sigma + \frac{m}{n} z_0 \gamma_0, \quad 0 \leq m \leq n-1. \quad (7a)$$

The thickness of the snow pack after a time $\underline{t}$ will be the sum of the thicknesses of all $\underline{n}$ layers, or

$$z = \sum_{m=0}^{n-1} \frac{z_0}{n} (1 + qc\gamma_0^{-q} \sigma_m t)^{-\frac{1}{q}}. \quad (8a)$$

As $\underline{n}$ approaches infinity, the summation can be replaced by an integral:

$$z = \int_0^{z_0} (1 + qc\gamma_0^{-q} \sigma_{z'} t)^{-\frac{1}{q}} dz', \quad (9a)$$

* Equations 2 and 2a both represent the experimental data; however, Equation 2 has better boundary conditions and is therefore preferable.

where the stress is now:

$$\sigma z' = \sigma + z' \gamma_0, \quad (10a)$$

z' being the variable distance as measured from the top down. Performing the integration in Equation 9a, we find:

$$z = \frac{1}{(q-1) c \gamma_0^{1-q} t} \left\{ \left[1 + q c \gamma_0^{-q} (\sigma + z_0 \gamma_0) t \right]^{\frac{q-1}{q}} - \left[1 + q c \gamma_0^{-q} \sigma t \right]^{\frac{q-1}{q}} \right\} \quad (11a)$$

This can be expanded for small t and becomes:

$$z = z_0 \left[1 - \left(\frac{z_0 \gamma_0}{2} + \sigma \right) c \gamma_0^{-q} t \right], \quad (12a)$$

which can be rewritten in the form:

$$\frac{\Delta z}{z_0} = \left(\frac{z_0 \gamma_0}{2} + \sigma \right) c \gamma_0^{-q} t. \quad (13a)$$

It can be seen that the effective stress is the applied stress plus one-half the weight of the sample. These results can be considered only as an approximation to real behavior. The simplifying assumptions used in Equations 1a, 2a, 3a, and 12a should account, in part, for the deviation of Equation 13a from the observed time dependence. Probably, neglect of strain-hardening effects also contributes to the lack of good agreement between this theory and experimental data.