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Elastic Plates with Simply Supported Straight Boundaries, Resting on a Liquid Foundation

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PREFACE

This is one in a series of reports of work performed on USA SIPRE Project 22.2-3, subtask a, Bearing capacity of floating ice sheets. This paper covers a part of investigations by Dr. Kerr, New York University Institute of Mathematics, in the summer of 1958. The work was done for USA SIPRE under contract DA-11-190-ENG-34 with the University of Denver.

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This report has been reviewed and approved for publication by the Office of the Chief of Engineers.

Henry J. Manger

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Acting Director

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SUMMARY

The deflection expression of an infinite plate subjected to a concentrated force is used with the "method of images" to obtain solutions for 6 plates with simply supported edges. The semi-infinite plate, the wedge-shaped plate, and its special case, the rectangular corner plate, are solved in closed form; and the infinite strip, the semi-infinite strip, and the rectangular plate are solved as rapidly convergent series. Behavior under a concentrated force is studied in more detail for the semi-infinite plate and the rectangular corner plate. Relationships for obtaining bending moments, shear forces and reaction distributions as well as derivatives of the kei-function with respect to r and θ are given in the appendices.

ELASTIC PLATES WITH SIMPLY SUPPORTED STRAIGHT BOUNDARIES, RESTING ON A LIQUID FOUNDATION

by

Arnold D. Kerr

INTRODUCTION AND STATEMENT OF THE PROBLEM

The theory of plates on an elastic foundation was formulated by Winkler (1867), who assumed the deflection of the plate at any point proportional to the intensity of the reaction of the elastic foundation at that point.

$$p(x, y) = k\,w(x, y) \tag{1}$$

Because of this assumption, the problem corresponds physically to a plate resting on a liquid base. Eq 1 represents the buoyant pressure at (x, y) where k is the specific weight of the "liquid" under consideration. The deflection surface of the plate is then governed by the differential equation*

$$\Delta\Delta w + \frac{k}{D} w = \frac{q}{D} \tag{2}$$

where Δ is the Laplace operator, D the flexural rigidity of the plate, and q the intensity of the transverse load (per unit area). In the case of concentrated forces, $q = 0$ and eq 2 reduces to

$$\Delta\Delta w + \frac{k}{D} w = 0. \tag{3}$$

As early as 1884 H. Hertz published a paper solving the case of an infinitely extended floating elastic plate subjected to a concentrated load normal to its plane. In the early nineteen twenties the use of concrete plates in foundations and as paving elements for airport runways and roadways increased the practical importance of the theory of beams and plates on elastic foundations. In the following years a number of publications appeared on this subject.** Soon, however, it became obvious that the Winkler assumption does not quite represent the condition of plates resting on a soil foundation, and research activities concentrated on problems of plates resting on an elastic continuum, which are mathematically more difficult. During the Second World War considerable activity took place in northern regions and the problem of transporting and stationing heavy equipment on frozen lakes became of great importance. It was found that ice subjected to loads of short durations responds elastically. Therefore, assuming the ice fields homogeneous and isotropic, eq 2 can be used to determine the deflections and hence the state of stress in the ice plates caused by short-time loads. To facilitate the use of the derived solutions, it was necessary to represent them in terms of well established, easily accessible, tabulated functions. In 1950 M. Wyman derived solutions for the case of an infinite plate subjected to (a) a concentrated force and (b) a uniform circular load, in terms of modified Bessel functions.

Recently R. K. Livesley (1953) presented formal solutions for the case of a semi-infinite plate and an infinite quadrant simply supported along their edges, in terms of double Fourier transforms. In our paper it is shown that, using directly the deflection expression of an infinite plate subjected to a concentrated force in connection with the "method of images", solutions for the following plates with simply supported edges† will be obtained:

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* For derivation see any book on the subject, e.g., Timoshenko (1940), p. 249.

** See for example Happel, 1920, Hayashi, 1921, Westergaard, 1923, Schleicher, 1926, Shechter, 1936, Shapiro, 1942, 1943, and Hetényi, 1946.

† Experience has shown that under certain circumstances ice plates on rivers or lakes can be considered as simply supported along shore lines.

(1) Semi-infinite plate
(2) Wedge-shaped plate
(3) Rectangular corner plate as special case of (2)
(4) Infinite strip
(5) Semi-infinite strip
(6) Rectangular plate.

(1), (2), and (3) are solved in closed form and (4), (5), and (6) as rapidly convergent infinite series.

The behavior of the plate subjected to a concentrated force is studied in more detail for cases (1) and (3).

SOLUTION OF THE BOUNDARY VALUE PROBLEMS FOR SIMPLY SUPPORTED BOUNDARIES

For the plate of infinite extent resting on a liquid foundation and subjected at the origin of the coordinate system ($r = 0$) to a concentrated force $\underline{P}$, eq 3 reduces, because of axial symmetry, to an ordinary differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) w + \frac{k}{D} w = 0. \qquad (4)$$

With $\sqrt[4]{\frac{k}{D}} = \frac{1}{\ell} = \lambda$, the general solution of eq 4 (see Wyman, 1950, p. 295) is

$$w = A_1 \operatorname{ber}(r/\ell) + A_2 \operatorname{bei}(r/\ell) + A_3 \operatorname{ker}(r/\ell) + A_4 \operatorname{kei}(r/\ell) \qquad (5)$$

where ber, bei, ker, kei are modified Bessel functions. Assuming: (a) $w = 0$ for $r \to \infty$, (b) finite deflections at $r = 0$, and (c) that the total restoring force of the water must be equal to the concentrated load $\underline{P}$, eq 5 (see Wyman, p. 296) reduces to

$$w = -\frac{P}{2\pi k \ell^2} \operatorname{kei}(r/\ell) = -\frac{P\lambda^2}{2\pi k} \operatorname{kei}(\lambda r) \qquad (6)$$

where $\underline{r}$ is the distance from $\underline{P}$ to the investigated point of the plate. This is the deflection surface of an infinite plate resting on a liquid foundation subjected at $r = 0$ to a concentrated force $\underline{P}$. For the sake of brevity, it will be referred to in the future as "fundamental deflection function" or simply as "fundamental deflection".

Through expansion of $\underline{\operatorname{kei}}(\lambda r)$ it can be shown that eq 6 contains the characteristic singular term $\frac{P}{8\pi D} r^2 \ln r$ of a concentrated force acting laterally on a plate. $r^2 \ln r$ is the only term in eq 6 whose second and third derivatives approach infinity as $r \to 0$.

As eq 3 is linear and homogeneous the fundamental deflection can be used in connection with the "method of images" to construct solutions for various boundary conditions (for plates in bending first suggested by Nadai (1921)). In this method positive and negative concentrated forces $\underline{P}$, which act parallel to the direction of the deflection $\underline{w}$ are arranged on the plate in such a way as to form simply supported boundaries. The deflection of the particular plate is then the sum of the fundamental deflections of the forces involved. Hence, plates with boundaries that can be created by a finite number of forces $\underline{P}$ will have solutions in closed form, whereas deflections of plates with boundaries whose formation will involve an infinite number of forces $\underline{P}$ will be represented by an infinite series of fundamental deflections. For illustration, we start with the simple case of a semi-infinite plate.

Semi-infinite plate

The deflection surface w. Two concentrated forces of equal magnitude $\underline{P}$ but opposite signs arranged on the plate along a straight line (Fig. 1) generate two deflection surfaces, each of them corresponding to that of a semi-infinite plate simply supported along its boundary and subjected to a concentrated force $\underline{P}$ at an arbitrary point ($\underline{r}_0$, 0).

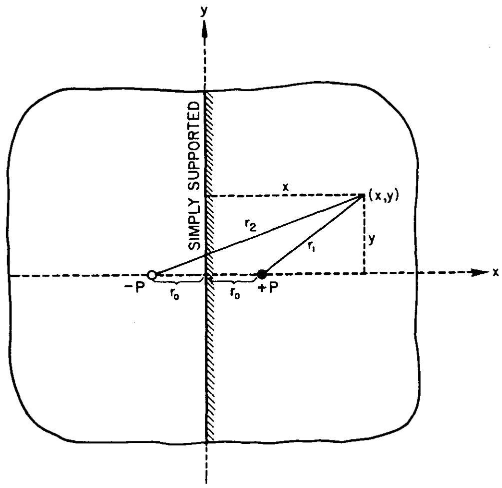

Figure 1.

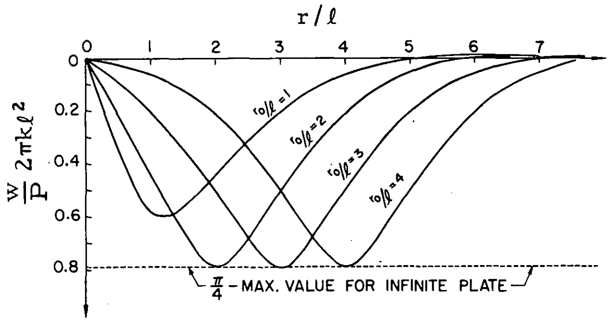

Figure 2.

Summing the deflections caused individually by the two forces P at a point (x, y) we obtain the deflection expression for the semi-infinite plate

$$w = -\frac{P\lambda^2}{2\pi k}\left[\mathrm{kei}(\lambda r_1) - \mathrm{kei}(\lambda r_2)\right] \tag{7}$$

where

$$\left.\begin{aligned} r_1 &= \sqrt{(x - r_0)^2 + y^2} \\ r_2 &= \sqrt{(x + r_0)^2 + y^2}\ . \end{aligned}\right\} \tag{8}$$

It can be seen (Fig. 1) that

$$w = 0 \tag{9}$$

is satisfied along the edge, since for this line $r_1 = r_2$. Therefore, in eq 7 the kei-terms cancel out each other.

As the boundary is parallel to the y-axis and w = 0 along this line, it follows also that $\frac{\partial^2 w}{\partial y^2} = 0$ along the edge. Hence, the second boundary condition for the simply supported edge can be stated

$$\Delta w = 0. \tag{10}$$

It can easily be seen from eq 14 that this condition is satisfied.

To show the influence of the distance of P with respect to the edge line on the deflections, eq 7 was rewritten

$$w = -\frac{P}{2\pi k \ell^2}\left\{\mathrm{kei}\left[\frac{r_0}{\ell}\left(\frac{x}{r_0} - 1\right)\right] - \mathrm{kei}\left[\frac{r_0}{\ell}\left(\frac{x}{r_0} + 1\right)\right]\right\} \tag{11}$$

and then evaluated for different ratios of r_0/ℓ (Fig. 2). For $r_0 = 2\ell$ the maximum deflection is already equal to that of an infinite plate.

Bending moments, shear forces, and reaction distributions can now be easily obtained as higher derivatives of the deflection expression using the relationships as given in Appendix A.

Bending moments M_x and M_y. Derivatives of the kei-function, with respect to x and y were formed and are given in Appendix B.

To get the bending moments we substitute eq 7 into A1 and A2 and obtain

$$M_x = \frac{P}{2\pi}\left[\frac{(x-r_0)^2+\mu y^2}{r_1^2}\ker(\lambda r_1) + (1-\mu)\frac{y^2-(x-r_0)^2}{\lambda r_1^3}\mathrm{kei}'(\lambda r_1) - \frac{(x+r_0)^2+\mu y^2}{r_2^2}\ker(\lambda r_2) - (1-\mu)\frac{y^2-(x+r_0)^2}{\lambda r_2^3}\mathrm{kei}'(\lambda r_2)\right] \quad (12)$$

$$M_y = \frac{P}{2\pi}\left[\frac{\mu(x-r_0)^2+y^2}{r_1^2}\ker(\lambda r_1) - (1-\mu)\frac{y^2-(x-r_0)^2}{\lambda r_1^3}\mathrm{kei}'(\lambda r_1) - \frac{\mu(x+r_0)^2+y^2}{r_2^2}\ker(\lambda r_2) + (1-\mu)\frac{y^2-(x+r_0)^2}{\lambda r_2^3}\mathrm{kei}'(\lambda r_2)\right] \quad (13)$$

and

$$-D\,\Delta w = \frac{M_x+M_y}{1+\mu} = \frac{P}{2\pi}\left[\ker(\lambda r_1) - \ker(\lambda r_2)\right]. \quad (14)$$

For bending moments along the x-axis $(y=0)$ we get the equations

$$[M_x]_{y=0} = \frac{P}{2\pi}\left\{\ker[\lambda|x-r_0|] - \frac{(1-\mu)}{\lambda|x-r_0|}\mathrm{kei}'[\lambda|x-r_0|] - \ker[\lambda|x+r_0|] + \frac{(1-\mu)}{\lambda(x+a)}\mathrm{kei}'[\lambda(x+r_0)]\right\} \quad (15)$$

$$[M_y]_{y=0} = \frac{P}{2\pi}\left\{\mu\ker[\lambda|x-r_0|] + \frac{(1-\mu)}{\lambda|x-r_0|}\mathrm{kei}'[\lambda|x-r_0|] - \mu\ker[\lambda(x+r_0)] - \frac{(1-\mu)}{\lambda(x+r_0)}\mathrm{kei}'[\lambda(x+r_0)]\right\}. \quad (16)$$

The numerical evaluation of eq 15 and 16 is shown in Figures 3 and 4. The infinite moments at the points of application of the concentrated force P, are a consequence of the assumptions of the thin plate theory. These moment values will be finite once it is assumed that the force is distributed over a finite area, which is usually the case in practice.

Wedge-shaped Plate

Solutions of plates of this type can be obtained by arranging an even number of forces P on a circle, around the tip of the plate (Fig. 5). The solution for a plate with an

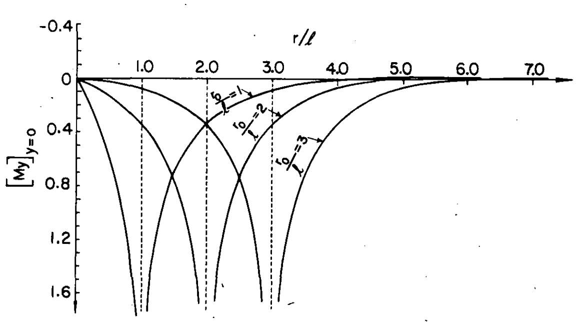

Figure 3.

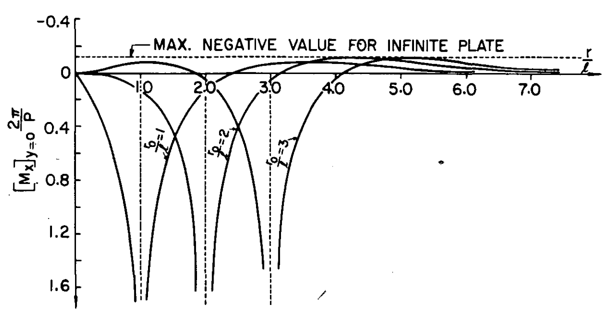

Figure 4.

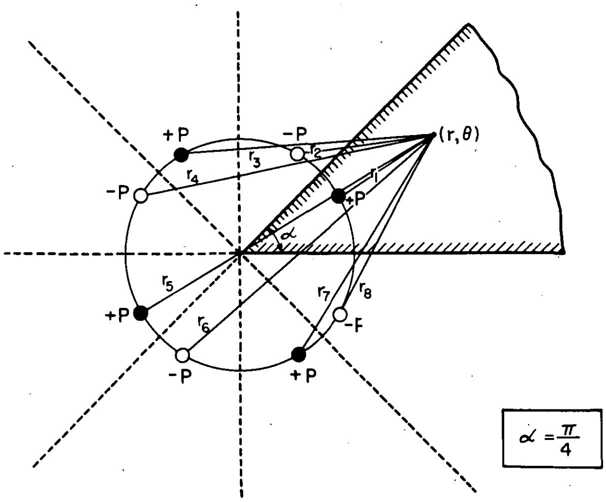

Figure 5.

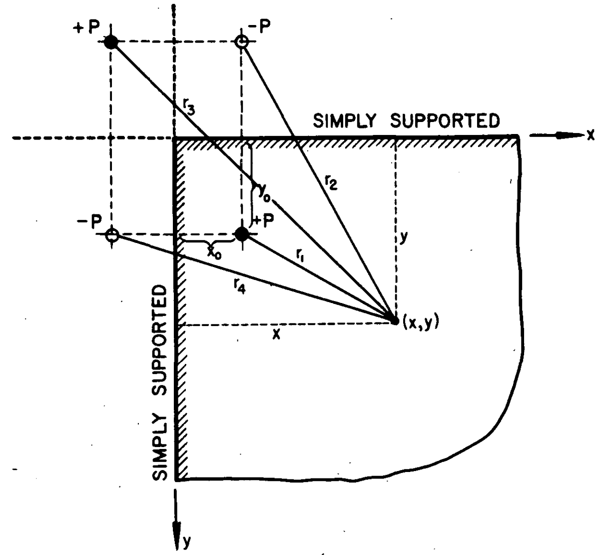

Figure 6.

opening angle $\alpha = \frac{\pi}{m}$ (m = 1, 2, ...), subjected to a force $\underline{P}$ at an arbitrary point inside the plate is

$$w = -\frac{P\lambda^2}{2\pi k} \sum_{n=1}^{2m} (-1)^{n+1} \operatorname{kei}(\lambda r_n) . \qquad (17)$$

It should be noted that m = 1 corresponds to the case of the semi-infinite plate and indeed eq 17 for m = 1 is identical with eq 7.

In the following another important problem, namely the wedge-shaped plate with $\alpha = \frac{\pi}{2}$ will be investigated in detail.

Rectangular corner plate

The deflection surface w. The deflection of the corner plate due to a concentrated force $\underline{P}$ at an arbitrary point inside the plate is, according to eq 17

$$w = -\frac{P\lambda^2}{2\pi k}\left[\operatorname{kei}(\lambda r_1) - \operatorname{kei}(\lambda r_2) + \operatorname{kei}(\lambda r_3) - \operatorname{kei}(\lambda r_4)\right] \qquad (18)$$

where

$$\left.\begin{aligned} r_1 &= \sqrt{(x-x_0)^2 + (y-y_0)^2} \\ r_2 &= \sqrt{(x-x_0)^2 + (y+y_0)^2} \\ r_3 &= \sqrt{(x+x_0)^2 + (y+y_0)^2} \\ r_4 &= \sqrt{(x+x_0)^2 + (y-y_0)^2} \ . \end{aligned}\right\} \qquad (19)$$

It can be seen from Figure 6 that w = 0 is satisfied along the supports, since $r_1 = r_4$ and $r_2 = r_3$, for the edge along the $\underline{y}$-axis, and $r_1 = r_2$ and $r_3 = r_4$ for the edge along the $\underline{x}$-axis. That $\Delta w = 0$ is satisfied along the supports follows from eq 28 by a similar argument (Fig. 7).

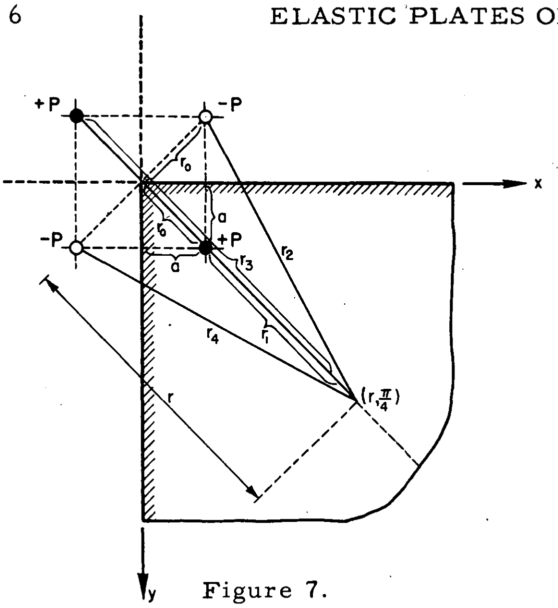

Figure 7.

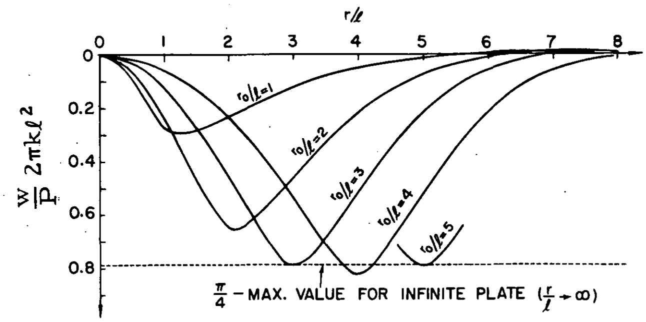

Figure 8.

With

$$\left.\begin{aligned} r_1 &= |r - r_0| \\ r_2 &= \sqrt{r^2 + r_0^2} \\ r_3 &= r + r_0 \\ r_4 &= \sqrt{r^2 + r_0^2} \end{aligned}\right\} \quad (20)$$

eq 18 reduces to the expression for deflections along the bisector radius when P acts at a point along this line.

$$w = -\frac{P}{2\pi k \ell^2}\left\{\text{kei}\left[\frac{r_0}{\ell}\left|\frac{r}{r_0} - 1\right|\right] - 2\,\text{kei}\left[\frac{r_0}{\ell}\sqrt{\left(\frac{r}{r_0}\right)^2 + 1}\right] + \text{kei}\left[\frac{r_0}{\ell}\left(\frac{r}{r_0} + 1\right)\right]\right\} \quad (21)$$

Eq 21 was evaluated numerically for different ratios r_0/ℓ (Fig. 8). For $r_0 = 3\ell$ maximum deflection is approximately equal to that of an infinite plate.

Corner reaction R. Because of the assumption made in deriving eq A9 and A10, namely the replacement of M_{xy} along the simply supported edge by a couple of vertical forces, a negative concentrated corner reaction R appears in the analytical solution in addition to the reactions distributed along the boundaries.

$$R = 2[M_{xy}]_{\substack{x=0\\y=0}} = 2D(1-\mu)\left[\frac{\partial^2 w}{\partial x\,\partial y}\right]_{\substack{x=0\\y=0}} . \quad (22)$$

Substituting eq 18 into 22 we obtain

$$R = -\frac{(1-\mu)P}{\pi}\left\{\frac{4x_0 y_0}{x_0^2 + y_0^2}\left[\text{ker}(\lambda\sqrt{x_0^2 + y_0^2}) - \frac{2}{\lambda\sqrt{x_0^2 + y_0^2}}\,\text{kei}'(\lambda\sqrt{x_0^2 + y_0^2})\right]\right\} . \quad (23)$$

This is the corner reaction caused by P acting at any arbitrary point (x_0, y_0) inside the plate. For $x_0 = y_0 = a$ and noting that $r_0^2 = x_0^2 + y_0^2$ eq 23 reduces to

$$R_{\substack{x_0=a\\y_0=a}} = -\frac{2(1-\mu)P}{\pi}\left[\text{ker}(\lambda r_0) - \frac{2}{\lambda r_0}\,\text{kei}'(\lambda r_0)\right] . \quad (24)$$

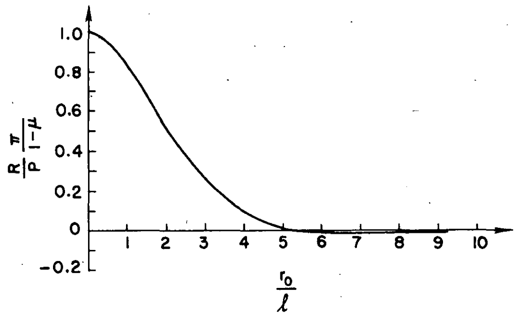

Figure 9.

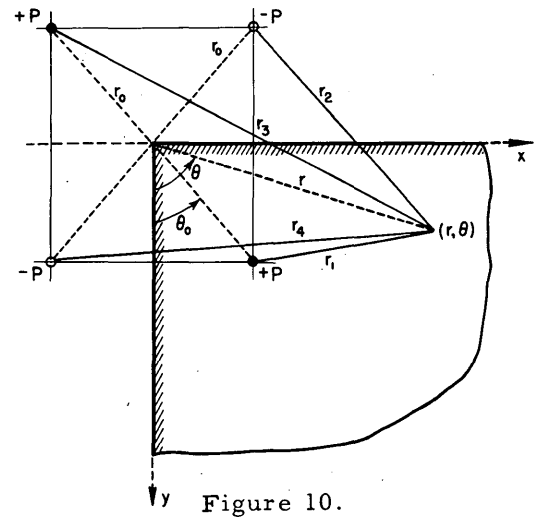

Figure 10.

The graphical representation of eq 24 is shown in Figure 9. Note that the corner reaction is negligible for $r_0 > 5\ell$.

Bending moments M_r and M_t. The magnitudes of r_n ($n = 1, 2, 3, 4$) expressed in polar coordinates are (Fig. 10):

$$\left.\begin{aligned} r_1^2 &= r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0) \\ r_2^2 &= r^2 + r_0^2 + 2rr_0\cos(\theta+\theta_0) \\ r_3^2 &= r^2 + r_0^2 + 2rr_0\cos(\theta-\theta_0) \\ r_4^2 &= r^2 + r_0^2 - 2rr_0\cos(\theta+\theta_0) \end{aligned}\right\} . \tag{25}$$

Derivatives of the kei-function with respect to r and θ were found and are given in Appendix B. Substituting eq 18 into A3 and A4 gives the bending moments of M_r and M_t for the rectangular corner plate, subjected to a concentrated force P at an arbitrary point (r_0, θ_0) inside the plate.

$$\begin{aligned} M_r = \frac{P}{2\pi}\Bigg\{ & \frac{[r - r_0\cos(\theta-\theta_0)]^2 + \mu r_0^2\sin^2(\theta-\theta_0)}{r_1^2}\,\mathrm{ker}(\lambda r_1) \\ & - \frac{[r + r_0\cos(\theta+\theta_0)]^2 + \mu r_0^2\sin^2(\theta+\theta_0)}{r_2^2}\,\mathrm{ker}(\lambda r_2) \\ & + \frac{[r + r_0\cos(\theta-\theta_0)]^2 + \mu r_0^2\sin^2(\theta-\theta_0)}{r_3^2}\,\mathrm{ker}(\lambda r_3) \\ & - \frac{[r - r_0\cos(\theta+\theta_0)]^2 + \mu r_0^2\sin^2(\theta+\theta_0)}{r_4^2}\,\mathrm{ker}(\lambda r_4) \\ & + \frac{(1-\mu)}{\lambda}\Bigg[\frac{r_0^2\sin^2(\theta-\theta_0) - [r - r_0\cos(\theta-\theta_0)]^2}{r_1^3}\,\mathrm{kei}'(\lambda r_1) \\ & - \frac{r_0^2\sin^2(\theta+\theta_0) - [r + r_0\cos(\theta+\theta_0)]^2}{r_2^3}\,\mathrm{kei}'(\lambda r_2) \\ & + \frac{r_0^2\sin^2(\theta-\theta_0) - [r + r_0\cos(\theta-\theta_0)]^2}{r_3^3}\,\mathrm{kei}'(\lambda r_3) \\ & - \frac{r_0^2\sin^2(\theta+\theta_0) - [r - r_0\cos(\theta+\theta_0)]^2}{r_4^3}\,\mathrm{kei}'(\lambda r_4)\Bigg]\Bigg\} . \end{aligned} \tag{26}$$

$$
\begin{aligned}
M_t = \frac{P}{2\pi} \Bigg\{ & \frac{r_0^2 \sin^2(\theta - \theta_0) + \mu[r - r_0 \cos(\theta - \theta_0)]^2}{r_1^2} \operatorname{ker}(\lambda r_1) \\
& - \frac{r_0^2 \sin^2(\theta + \theta_0) + \mu[r + r_0 \cos(\theta + \theta_0)]^2}{r_2^2} \operatorname{ker}(\lambda r_2) \\
& + \frac{r_0^2 \sin^2(\theta - \theta_0) + \mu[r + r_0 \cos(\theta - \theta_0)]^2}{r_3^2} \operatorname{ker}(\lambda r_3) \\
& - \frac{r_0^2 \sin^2(\theta + \theta_0) + \mu[r - r_0 \cos(\theta + \theta_0)]^2}{r_4^2} \operatorname{ker}(\lambda r_4) \\
- \frac{(1-\mu)}{\lambda} \Bigg[& \frac{r_0^2 \sin^2(\theta - \theta_0) - [r - r_0 \cos(\theta - \theta_0)]^2}{r_1^3} \operatorname{kei}'(\lambda r_1) \\
& - \frac{r_0^2 \sin^2(\theta + \theta_0) - [r + r_0 \cos(\theta + \theta_0)]^2}{r_2^3} \operatorname{kei}'(\lambda r_2) \\
& + \frac{r_0^2 \sin^2(\theta - \theta_0) - [r + r_0 \cos(\theta - \theta_0)]^2}{r_3^3} \operatorname{kei}'(\lambda r_3) \\
& - \frac{r_0^2 \sin^2(\theta + \theta_0) - [r - r_0 \cos(\theta + \theta_0)]^2}{r_4^3} \operatorname{kei}'(\lambda r_4) \Bigg] \Bigg\}
\end{aligned}
\tag{27}
$$

and

$$
-D \Delta w = \frac{M_r + M_t}{(1+\mu)} = \frac{P}{2\pi} \left\{ \operatorname{ker}(\lambda r_1) - \operatorname{ker}(\lambda r_2) + \operatorname{ker}(\lambda r_3) - \operatorname{ker}(\lambda r_4) \right\} \tag{28}
$$

The numerical calculation of the bending moments is facilitated by the fact that the second halves of eq 26 and 27 are identical but of opposite sign, and the first halves are similar in structure.

For $\underline{M}_r$ and $\underline{M}_t$ along the bisector radius, when $\underline{P}$ acts along this line ($\theta = \theta_0 = \frac{\pi}{4}$), eq 26 and 27 reduce to

$$
\begin{aligned}
[M_r] = \frac{P}{2\pi} \Bigg\{ & \operatorname{ker}[\lambda |r - r_0|] + \operatorname{ker}[\lambda(r + r_0)] - 2\frac{r^2 + \mu r_0^2}{r^2 + r_0^2} \operatorname{ker}[\lambda \sqrt{r^2 + r_0^2}] \\
& -(1-\mu)\Bigg[\frac{1}{\lambda|r - r_0|} \operatorname{kei}'[\lambda|r - r_0|] + \frac{1}{\lambda(r + r_0)} \operatorname{kei}'[\lambda(r + r_0)] \\
& + \frac{2(r_0^2 - r_0^2)}{\lambda(r^2 + r_0^2)^{3/2}} \operatorname{kei}'[\lambda\sqrt{r^2 + r_0^2}]\Bigg]\Bigg\}
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
[M_t] = \frac{P}{2\pi} \Bigg\{ & \mu\Big[\operatorname{ker}[\lambda |r - r_0|] + \operatorname{ker}[\lambda(r + r_0)]\Big] - 2\frac{r_0^2 + \mu r^2}{r^2 + r_0^2} \operatorname{ker}[\lambda \sqrt{r^2 + r_0^2}] \\
& + (1-\mu)\Bigg[\frac{1}{\lambda|r - r_0|} \operatorname{kei}'[\lambda|r - r_0|] + \frac{1}{\lambda(r + r_0)} \operatorname{kei}[\lambda(r + r_0)] \\
& \frac{2(r_0^2 - r^2)}{\lambda(r^2 + r_0^2)^{3/2}} \operatorname{kei}'[\lambda\sqrt{r^2 + r_0^2}]\Bigg]\Bigg\}
\end{aligned}
\tag{30}
$$

Eq 29 and 30 were evaluated for different ratios r_0/ℓ (Figs. 11 and 12). From Figure 12 it can be seen that for $r_0/\ell < 4$ there will be a negative bending moment in the corner area, larger than the value of the negative moment of an infinite plate.

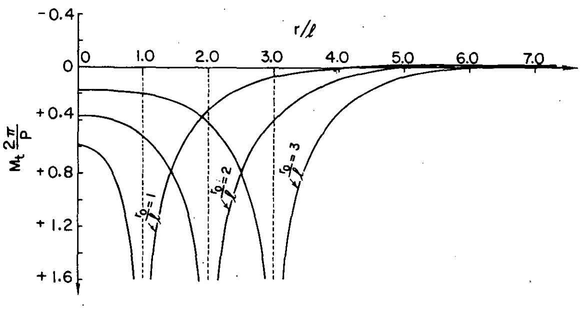

Figure 11.

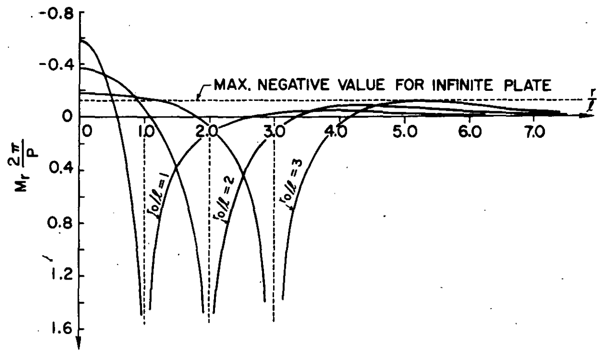

Figure 12.

Infinite strip

An infinite number of concentrated forces P, arranged on the plate along a straight line (Fig. 13), generate deflection surfaces, each of them equivalent to that of an infinite strip simply supported along its boundaries and subjected at an arbitrary point (x_0, y_0) to a concentrated force P. Summing the deflections caused individually by all forces P at a point (x, y), we obtain the deflection surface of the infinite strip.

$$w = -\frac{P\lambda^2}{2\pi k} \sum_{n=-\infty}^{+\infty} [\mathrm{kei}(\lambda r_n) - \mathrm{kei}(\lambda r'_n)] \tag{31}$$

where

$$\left.\begin{aligned} r_n &= \sqrt{(2na + x_0 - x)^2 + y^2} \\ r'_n &= \sqrt{(2na - x_0 - x)^2 + y^2} \end{aligned}\right\} . \tag{32}$$

Semi-infinite strip

A system of forces P (Fig. 14) produces deflection surfaces, each equivalent to that of a semi-infinite strip simply supported along its boundaries and subjected at an arbitrary point inside the plate (x_0, y_0) to a concentrated force P. The deflection is

$$w = -\frac{P\lambda^2}{2\pi k} \sum_{n=-\infty}^{+\infty} [\mathrm{kei}(\lambda r_n) - \mathrm{kei}(\lambda r'_n) + \mathrm{kei}(\lambda \rho'_n) - \mathrm{kei}(\lambda \rho_n)] \tag{33}$$

where

$$\left.\begin{aligned} r_n &= \sqrt{(x - x_0 - 2na)^2 + (y - y_0)^2} \\ r'_n &= \sqrt{(x + x_0 - 2na)^2 + (y - y_0)^2} \\ \rho_n &= \sqrt{(x - x_0 - 2na)^2 + (y + y_0)^2} \\ \rho'_n &= \sqrt{(x + x_0 - 2na)^2 + (y + y_0)^2} \end{aligned}\right\} . \tag{34}$$

Rectangular plate

Extending the force system of the preceding section, we will assume a set of periodically recurring forces in the positive and negative direction of the y-axis as shown in Figure 15. These forces produce a series of deflection surfaces, each of them equivalent

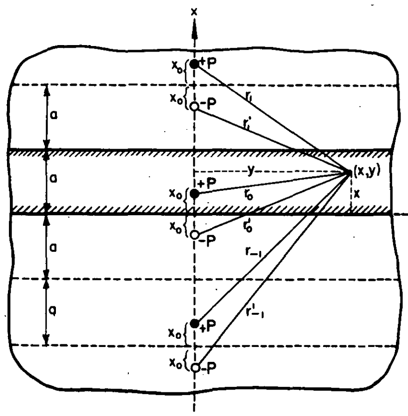

Figure 13.

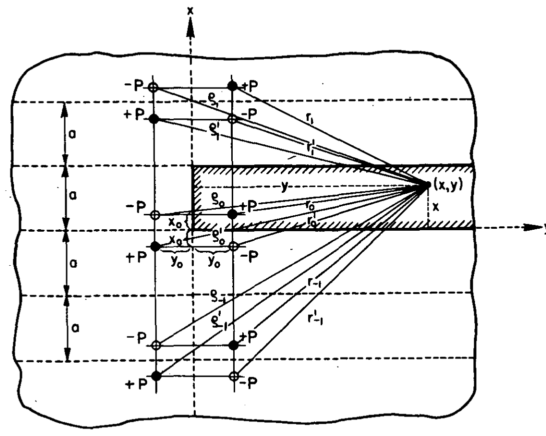

Figure 14.

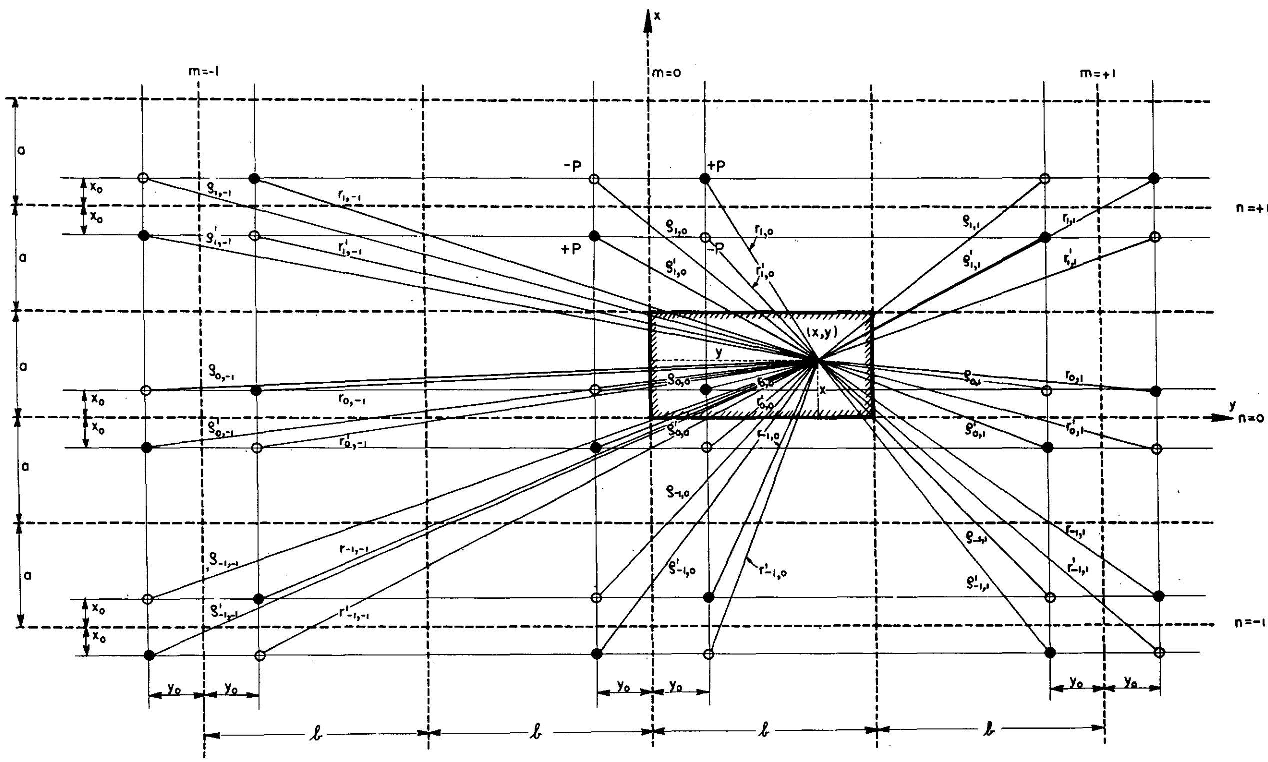

Figure 15.

to that of a rectangular plate simply supported along its boundaries and subjected at an arbitrary point $(\underline{x}_0, \underline{y}_0)$ inside the plate to a concentrated force $\underline{P}$. The deflection surface is

$$w = -\frac{P\lambda^2}{2\pi k} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \left[\mathrm{kei}(\lambda r_{nm}) - \mathrm{kei}(\lambda r'_{nm}) + \mathrm{kei}(\lambda \rho'_{nm}) - \mathrm{kei}(\lambda \rho_{nm}) \right] \tag{35}$$

where

$$\left.\begin{aligned} r_{nm} &= \sqrt{(x - x_0 - 2na)^2 + (y - y_0 - 2mb)^2} \\ r'_{nm} &= \sqrt{(x + x_0 - 2na)^2 + (y - y_0 - 2mb)^2} \\ \rho_{nm} &= \sqrt{(x - x_0 - 2na)^2 + (y + y_0 - 2mb)^2} \\ \rho'_{nm} &= \sqrt{(x + x_0 - 2na)^2 + (y + y_0 - 2mb)^2} \end{aligned}\right\} . \tag{36}$$

It should be noted that for $m = 0$, $n = 0$ the deflection eq 35 reduces to that of a rectangular corner plate (eq 18).

By an argument similar to that for a semi-infinite and a rectangular corner plate, $w = 0$ and $\Delta w = 0$ are satisfied along the edges for an infinite and a semi-infinite strip and for a rectangular plate.

Expressions for bending moments, shear forces, reaction distributions, and (for a semi-infinite strip and a rectangular plate) for corner reactions can be derived by substituting eq 31, 33, or 35 respectively into the relationships given in Appendix A.

The sufficient conditions for validity of term-by-term differentiation of an infinite series of two variables is satisfied in the cases treated here, since eq 31, 33, and 35 as well as their first, second and third derivatives with respect to x and y are continuous functions of x and y in the whole region (except the second and third derivative at $x = y = 0$ – singular term.

For solutions expressed in terms of infinite series, the problem of convergence, as well as of the rate of convergence, is of considerable importance. From the theory of infinite series, it is known that if the terms of a series

$$u_1 - u_2 + u_3 - u_4 + \dots\dots\dots\dots \tag{37}$$

are of alternate sign, it is necessary and sufficient for the convergence of a series that for every value of n

$$\left.\begin{aligned} & u_n \geqq u_{n+1} \\ \text{and} \quad & \lim_{n\to\infty} u_n = 0. \end{aligned}\right\} . \tag{38}$$

The "method of images" in connection with the force systems used here forms an alternating series for an infinite and semi-infinite strip and for a rectangular plate, for each point of the plate. It can easily be proven that the series satisfy the conditions as stated in eq 38. This can also be shown considering the physical picture. Because of the liquid foundation the effects (like w, M, V) are localized and will decrease with increasing distance from P and increasing ratio k/D. Since the terms of increasing n represent the effects of forces P acting at increasing distances from the point under consideration, these terms will decrease rapidly with growing n and will be zero for $n \to \infty$, which proves the conditions in eq 38. In addition, the rapid decrease of the effect ensures rapid convergence of the corresponding infinite series.

Solutions for two cases treated here were already obtained previously by other authors. The infinite strip was solved by Westergaard (1923) and the rectangular plate by Timoshenko (1940) p. 252. Westergaard used Levys' approach and obtained the solution as a simple series. Timoshenko used the Navier approach and obtained the deflection expression as a double sine series. The convergence of series of this type decreases rapidly with an increasing number of differentiations. As the moments, shearing forces and reactions are represented as higher derivatives of the deflection function, their convergence is also decreased. Besides, in the case of a concentrated force (as was already noted by Nadai (1921) for the case of plates in bending) solutions of this kind fail to yield results for the determination of stresses in the neighborhood of the concentrated force (they diverge there).

The convergence of the solutions obtained for infinite and semi-infinite strips and for a rectangular plate are affected relatively little by differentiations. Additionally, they converge rapidly in the neighborhood of P, since the main part of the effect under consideration (M or V) is represented by the first term (with argument λr_0) of the corresponding series.

REMARKS ON THE PROBLEM OF ARBITRARY LOAD DISTRIBUTION

Solutions for plates subjected to concentrated forces are of great importance; in addition to being solutions of specific problems in themselves, they can be used to construct influence fields for deflections, moments, etc. By means of influence fields the corresponding magnitudes, caused by arbitrary distributed loads, can then be determined in a way similar to that for elastic beam systems.

Solutions for an arbitrary load distribution can also be obtained, setting $P = qdA$ in the corresponding solution for a concentrated force, and then integrating over the loaded area A.

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APPENDIX A

Bending moments and shear forces as higher derivatives of ω

Bending moments

a) in Cartesian coordinates

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \quad (A1)$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right) \quad (A2)$$

b) in polar coordinates

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \mu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \quad (A3)$$

$$M_t = -D\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \mu \frac{\partial^2 w}{\partial r^2}\right) \quad (A4)$$

Shear forces

a) in Cartesian coordinates

$$Q_x = -D\frac{\partial}{\partial x}(\Delta w) \quad (A5)$$

$$Q_y = -D\frac{\partial}{\partial y}(\Delta w) \quad (A6)$$

b) in polar coordinates

$$Q_r = -D\frac{\partial}{\partial r}(\Delta w) \quad (A7)$$

$$Q_t = -D\frac{\partial}{r\,\partial \theta}(\Delta w) \quad (A8)$$

Reaction distributions

$$V_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x \partial y^2}\right] \quad (A9)$$

$$V_y = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2 \partial y}\right] \quad (A10)$$

APPENDIX B

a) Derivatives of the kei (λr_n) -function with $r_n = \sqrt{(x+x_0)^2+(y-y_0)^2}$

$$\frac{\partial}{\partial x}\left\{\text{kei}\,(\lambda r_n)\right\} = \frac{\lambda(x+x_0)}{r_n}\,\text{kei}'(\lambda r_n)$$

$$\frac{\partial}{\partial y}\left\{\text{kei}\,(\lambda r_n)\right\} = \frac{\lambda(y-y_0)}{r_n}\,\text{kei}'(\lambda r_n)$$

$$\frac{\partial^2}{\partial x^2}\left\{\text{kei}\,(\lambda r_n)\right\} = \frac{\lambda^2(x+x_0)^2}{r_n^2}\,\text{ker}(\lambda r_n) + \lambda\,\frac{(y-y_0)^2-(x+x_0)^2}{r_n^3}\,\text{kei}'(\lambda r_n)$$

$$\frac{\partial^2}{\partial y^2}\left\{\text{kei}\,(\lambda r_n)\right\} = \frac{\lambda^2(y-y_0)^2}{r_n^2}\,\text{ker}(\lambda r_n) - \lambda\,\frac{(y-y_0)^2-(x+x_0)^2}{r_n^3}\,\text{kei}'(\lambda r_n)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left\{\text{kei}\,(\lambda r_n)\right\} = \lambda^2\,\text{ker}(\lambda r_n)$$

$$\frac{\partial^2}{\partial x\,\partial y}\left\{\text{kei}\,(\lambda r_n)\right\} = \frac{\lambda^2(x+x_0)(y-y_0)}{r_n^2}\left[\text{ker}(\lambda r_n) - \frac{2}{\lambda r_n}\,\text{kei}\,(\lambda r_n)\right]$$

b) Derivatives of the kei (λr_n) -function with $r_n = \sqrt{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)}$

$$\frac{\partial}{\partial r}\left\{\text{kei}\,(\lambda r_n)\right\} = \frac{\lambda[r-r_0\cos(\theta-\theta_0)]}{r_n}\,\text{kei}'(\lambda r_n)$$

$$\frac{\partial}{\partial \theta}\left\{\text{kei}\,(\lambda r_n)\right\} = \frac{\lambda\, rr_0\sin(\theta-\theta_0)}{r_n}\,\text{kei}'(\lambda r_n)$$

$$\frac{\partial^2}{\partial r^2}\left\{\text{kei}\,(\lambda r_n)\right\} = \frac{\lambda^2[r-r_0\cos(\theta-\theta_0)]^2}{r_n^2}\,\text{ker}(\lambda r_n) +$$

$$+\,\lambda\,\frac{r_0^2\sin^2(\theta-\theta_0)-[r-r_0\cos(\theta-\theta_0)]^2}{r_n^3}\,\text{kei}'(\lambda r_n)$$

$$\frac{\partial^2}{\partial \theta^2}\left\{\text{kei}\,(\lambda r_n)\right\} = \frac{\lambda^2 r^2 r_0^2\sin^2(\theta-\theta_0)}{r_n^2}\,\text{ker}(\lambda r_n)$$

$$+\,\lambda\,\frac{(r^2+r_0^2)rr_0\cos(\theta-\theta_0)-2r^2r_0^2}{r_n^3}\,\text{kei}'(\lambda r_n)$$

$$\Delta\left\{\text{kei}\,(\lambda r_n)\right\} = \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left\{\text{kei}\,(\lambda r_n)\right\} = \lambda^2\,\text{ker}(\lambda r_n)$$