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Plastic Deformation of Floating Ice Plates Subjected to Static Loads

by Arnold D. Kerr

U. S. ARMY SNOW ICE AND PERMAFROST
RESEARCH ESTABLISHMENT
Corps of Engineers
Wilmette, Illinois

PREFACE

This is one in a series of reports of work performed on USA SIPRE Project 22.2.3, subtask a, Bearing capacity of floating ice sheets. This paper covers a part of investigations by Dr. Kerr, New York University Institute of Mathematics, in the summer of 1958. The work was done for USA SIPRE under contract DA-11-190-ENG-34 with the University of Denver.

The author wishes to thank Dr. H. Bader, Chief Scientist, and Dr. A. Assure, project leader, for review of the paper before publication.

Work on this project was under the supervision of Dr. H. Bader, then acting branch chief, Snow and Ice Applied Research Branch.

This report has been reviewed and approved for publication by the Office of the Chief of Engineers.

Walter H. Parsons, Jr.
Colonel, Corps of Engineers
Director

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Unclassified Report

The problem is analyzed mathematically for decreasing and increasing rates of deflection. The analysis is based on the assumptions that for decreasing rates of deflection the floating ice plate will deform under lateral load without failure until the weight of the displaced water is equal to that of the load, and that for increasing rates, deflection increases until the ice plate collapses under and near the load. It is also suggested that the total deflection at a certain time is the result of the elastic deflection surface and the plastic deflection due to shear only, the shear forces obeying Newton's law of viscosity. Deflection equations for plastic deflection due to shear are derived for an
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SUMMARY

The problem is analyzed mathematically for decreasing and increasing rates of deflection. The analysis is based on the assumptions that for decreasing rates of deflection the floating ice plate will deform under lateral load without failure until the weight of the displaced water is equal to that of the load, and that for increasing rates, deflection increases until the ice plate collapses under and near the load. It is also suggested that the total deflection at a certain time is the result of the elastic deflection surface and the plastic deflection due to shear only, the shear forces obeying Newton's law of viscosity. Deflection equations for plastic deflection due to shear are derived for an infinite plate subjected to a line load, an infinite plate subjected to a concentrated force (axially symmetrical flexure), and an infinite plate subjected to uniform circular load. Equations for elastic deflection to be added to the plastic deflection due to shear are suggested. According to the statements and assumptions made and the results obtained, the total system of an ice plate resting on a liquid base can be considered as a Kelvin body for the case of decreasing rates of deflection.

PLASTIC DEFORMATION OF FLOATING ICE PLATES SUBJECTED TO STATIC LOADS

by

Arnold D. Kerr

INTRODUCTION AND STATEMENT OF THE PROBLEM

As early as 1884, H. Hertz published a paper solving the case of an infinitely extended elastic plate resting on a liquid foundation, subjected to a concentrated load normal to its plane. In following years, this theory was extended in connection with pavement problems for airports and highways.

In an extensive theoretical and experimental paper on floating ice plates, S. A. Bernshtein (1929) applied and further developed this elastic theory. However, his experiments as well as several other observations (South Manchurian Railway Co., 1941; Kobeko et al., 1946; and SIPRE experiments) proved that, for loads of long duration, deformations due to plastic flow play the predominant role.

Analytical treatments of plastic deformation of ice plates are very scarce. From the papers known to the author, it was found that this problem was approached basically in two different ways. The first approach utilizes the solutions for bending of floating elastic plates and tries to fit the experimental data (a) by modifying the modulus of elasticity (Bregman and Proskuriakova, 1943, p. 53) or (b) by multiplication of the "elastic" results by a time factor (e.g., $1 + at^b$ where t is time and a and b are constants to be determined from experimental data (South Manchurian Railway Co., 1941, eq 177)). However, since the deflection surface due to elastic bending is different in nature from that observed due to creep, this approach does not seem justified. The second approach is founded on Zubov's hypothesis, based on observations on ice fields deformed by transverse loads. It states that for ice plates, especially under comparatively high temperatures, Kirchhoffs' hypothesis (elastic bending) should be replaced by a deformation model of vertical elements, interconnected elastically and plastically and deformed by shear only (Lagutin, 1954, p. 49). In order to verify Zubov's assumption N. V. Zvolinskii carried out a number of theoretical investigations. He extended the theory of bending of thin elastic plates (ibid., p. 19 - 35), including the approximate effect of shear, and then applied it to the study of the deformation of a floating thin plate. Zvolinskii found that for the usual values of moduli of elasticity for ice, the effect of shear on the deformation of the plate is insignificant (ibid., p. 20). He proceeded then to treat the plastic state, assuming that the deformations are entirely due to shearing action and that shear forces follow Newton's law of viscosity. In this case, these assumptions led to the preservation of the linear character of the problem, in spite of the presence of finite deformations. The equation governing plastic deformation as derived by Zvolinskii is a relatively simple, linear partial differential equation of the third order for the vertical deflection w. The solution obtained is rather involved and quoting Zvolinskii (ibid., p. 21): "In this formula the result is not self-evident, and analyzing it does not help us to visualize the picture of the phenomenon." He suggests numerical evaluation of the deflection for different values of time t.

Zvolinskii used as an initial elastic deflection (at $t = 0$) the deflection caused by shear only. Experiments have shown, however, that shortly after load application the deflection surface coincides very closely with the elastic deflection surface due to bending (e.g., Bernshtein, 1929, Fig. 18). Besides, since the elastic deflections are relatively small, it seems that the effect of replacing the conditions of initial deformation by the condition of $w = 0$ for $t = 0$, is negligible in comparison to the introduced error of assuming shear as the only force responsible for plastic deformation.

Observations of the effect of static loads on the deformation of floating ice fields showed that in some cases the rate of deflection decreased after application of the load, and the plate came to equilibrium after a certain time interval (e.g., Bernshtein, p. 47, also quoted in Zubov, 1942, p. 146) whereas in other cases the rate of deflection increased until the plate collapsed. There are indications that temperature plays an important role here. It is also expected that the load and its distribution, as well as the ice thickness, will considerably influence the rates of deflection. However, experimental data which could help to clarify these points are not known to the author.

In the following, an attempt is made to formulate and solve the problem for decreasing and increasing rates of deflection. For decreasing rates it is assumed that the floating ice plate will deform under lateral load without failure until the weight of the displaced water is equal to the weight of the load. Plate deflection will then stop. For increasing rates it is assumed that the deflection increases until the ice plate collapses under and near the load.

It is assumed that the total deflection at a certain time t is

$$w = w_e + w_{p\,(shear)} \tag{1}$$

where w_e is an elastic deflection surface (**see Conclusions**) and $w_{p\,(shear)}$ is the plastic deflection due to shear only ($t > 0$). The shear forces obey Newton's law of viscosity.

Deflection equations for $w_{p\,(shear)}$ are derived for an infinite plate for the following load systems:

1. Line load (cylindrical flexure)
2. Concentrated force (axially symmetrical flexure)
3. Uniform circular load.

Equations for elastic deflection, to be added to $w_{p\,(shear)}$, are suggested.

INFINITE PLATE SUBJECTED TO A LINE LOAD

Decreasing rates of deflection

Mathematical formulation of the problem. We assume, as usual, the coordinate axes x and y in the middle plane of the plate and the z-axis normal to it. Since the line load Q (per unit length) acts along the y-axis normal to the middle surface, we can restrict the investigation to the consideration of a strip cut from the plate by two planes parallel to the x, z-plane a unit distance apart. Considering the vertical equilibrium of a part of this strip as shown in Figure 1, we obtain

$$Q - 2N - 2 \int_0^x \gamma\, w\, dx = 0. \tag{2}$$

Substituting

$$N = \mu h \frac{\partial^2 w}{\partial x\, \partial t} \qquad \text{(Newtonian flow)} \tag{3}$$

into (2) and differentiating with respect to x and then with respect to t,

$$\frac{\partial^4 w}{\partial x^2\, \partial t^2} + \frac{\gamma}{\mu h} \frac{\partial w}{\partial t} = 0 \tag{4}$$

where γ = unit weight of the liquid foundation

μ = coefficient of viscosity of the ice plate.

This is the differential equation that governs the deflection due to plastic flow of the floating ice plate subjected to a line load at $x = 0$ (in the sense of the assumptions made).

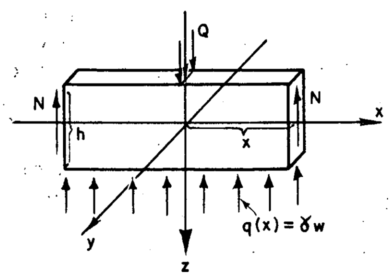


Figure 1.

Solution of the boundary value problem. Assuming the solution in product form

$$w = X(x)\ T(t) \tag{5}$$

and substituting it into eq 4, we obtain two ordinary differential equations whose solutions form the solution of eq 4.

$$w(x, t) = (B_1 + B_2 e^{-k^2 t})(A_1 e^{+\lambda x} + A_2 e^{-\lambda x}) \tag{6}$$

where

$$\lambda = \frac{1}{k}\sqrt{\frac{\gamma}{\mu h}} = \frac{\omega}{h^{1/2} k}; \qquad \omega = \sqrt{\frac{\gamma}{\mu}}$$

and k is an arbitrary real constant. Since for $x \to \infty$ the deflection has to be zero, it follows that $A_1 = 0$. Assuming $w = 0$ for $t = 0$ (see p. 1) it follows that $B_2 = -B_1$. Then, with $B_1 A_2 = A$, eq 6 reduces to

$$w(x, t) = A(1 - e^{-k^2 t})\, e^{-\lambda x} \qquad x \geq 0. \tag{7}$$

For $t \to \infty$ and $x = 0$ it follows from eq 7 that

$$w_0 = A. \tag{8}$$

Hence the deflection surface due to plastic flow of the ice field subjected to a line load can be represented as

$$w(x, t) = w_0 (1 - e^{-k^2 t})\, e^{-(\omega / h^{1/2} k) x} \qquad x \geq 0 \tag{9}$$

where w_0 is the deflection ordinate under the line load, after plate deflection due to creep ceases.

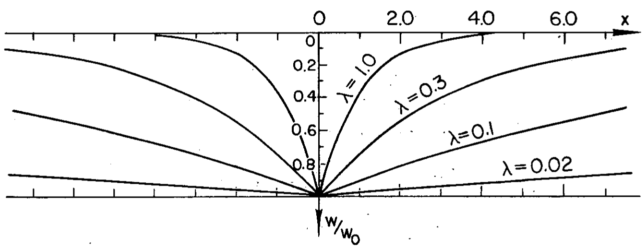
For $t \to \infty$ eq 9 reduces to

$$w(x) = w_0 e^{-\lambda x} \qquad x \geq 0 \tag{10}$$

which agrees with the equation given by Zubov (1942, p. 24 and 1945, p. 148). Zubov based his derivation on geometrical properties of deformed ice plates whose deformation ceased a certain time after load application.

The values k and ω in eq 9 can be determined as functions of temperature, salinity, and other influences on ice properties, comparing the deflection curves obtained by theory and corresponding experiments. In this way Zubov (1942), using eq 10 found as average value $\lambda = 0.1$ (1/meter). (His experiments showed that $0.004 \leq \lambda \leq 0.19$).

Graphs of w/w_0 according to eq 10 are shown in Figure 2. The sensitivity of the curves for relatively small variations of λ in the vicinity of $\lambda = 0.1$ is noteworthy. The discontinuity in slope along the line of application of the "concentrated" line load (Fig. 2) is a consequence of the assumptions made here. This strong discontinuity will disappear once the line load is replaced by one of finite width, which is usually the case in the field (see p. 6 ff).



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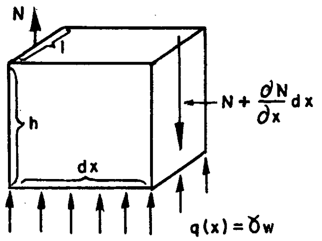

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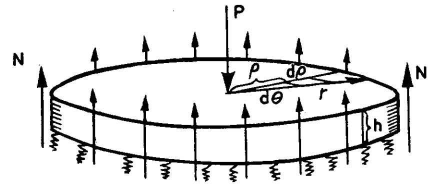

Figure 4.

Increasing rates of deflection

Mathematical formulation of the problem. Considering the vertical equilibrium of a strip element of a floating ice plate as shown in Figure 3, we obtain

$$\frac{\partial N}{\partial x} - q(x) = 0. \tag{11}$$

Substituting eq 3

$$N = \mu h \frac{\partial^2 w}{\partial x\, \partial t} \qquad \text{(Newtonian flow)}$$

and $q = \gamma w$ into eq 11 and differentiating it with respect to t,

$$\frac{\partial^4 w}{\partial x^2\, \partial t^2} - \frac{\gamma}{\mu h} \frac{\partial w}{\partial t} = 0. \tag{12}$$

This is the differential equation for the case of increasing rates of deflection.

The formulation of eq 4 is based on an equilibrium between the load and the buoyant forces, but this is not implied in the formulation of eq 12, in accordance with the assumption that deflection increases until the ice plate collapses.

Solution of the boundary value problem. Proceeding as in eq 5 - 7, with

$$\lambda_1 = \frac{1}{k_1}\sqrt{\frac{\gamma}{\mu h}} = \frac{\omega}{h^{1/2} k_1}; \qquad \omega = \sqrt{\frac{\gamma}{\mu}}$$

the solution of eq 12 is

$$w(x, t) = (D_1 + D_2 e^{k_1^2 t})(C_1 e^{+\lambda_1 x} + C_2 e^{-\lambda_1 x}) \tag{13}$$

which reduces to

$$w(x, t) = C(e^{k_1^2 t} - 1)\, e^{-(\omega/h^{1/2} k_1)x} \qquad x \geq 0. \tag{14}$$

INFINITE PLATE SUBJECTED TO A CONCENTRATED FORCE

Decreasing rates of deflection

Mathematical formulation of the problem. Considering the vertical equilibrium of a centrally loaded region, separated from the rest of the plate by a cylindrical surface of radius r (Fig. 4), and summing all forces in the vertical direction, we obtain

$$P - 2\pi r N - \gamma \int_0^{2\pi} \int_0^{r} w \rho \, d\rho \, d\theta = 0 \tag{15}$$

or

$$P - 2\pi r N - \gamma\, 2\pi \int_0^r w\rho \, d\rho = 0. \qquad (16)$$

Substituting into eq 16

$$N = \mu h \frac{\partial^2 w}{\partial r\, \partial t} \qquad \text{(Newtonian flow)} \qquad (17)$$

and differentiating with respect to r and then with respect to t we obtain

$$r^2 \frac{\partial^4 w}{\partial r^2\, \partial t^2} + r \frac{\partial^3 w}{\partial r\, \partial t^2} + \frac{\gamma}{\mu h} r^2 \frac{\partial w}{\partial t} = 0 \qquad (18)$$

where γ and μ have the same meaning as in eq 4, p. 2.

This is the differential equation that governs the deflection due to viscous flow of the floating ice plate subjected to a concentrated force at $r = 0$.

Solution of the boundary value problem. Assuming the solution in product form

$$w = R(r)\; T(t) \qquad (19)$$

and proceeding as before, with

$$\lambda = \frac{1}{k}\sqrt{\frac{\gamma}{\mu h}} = \frac{\omega}{h^{1/2} k}$$

we obtain

$$w = (B_1 + B_2 e^{-k^2 t})[A_1 I_0(\lambda r) + A_2 K_0(\lambda r)] \qquad (20)$$

where $I_0(\lambda r)$ and $K_0(\lambda r)$ are modified Bessel functions and k is an arbitrary real constant. Since for $r \to \infty$ deflection must be zero, $A_1 = 0$. Assuming $w = 0$ for $t = 0$, $B_2 = -B_1$. Then, with $B_1 A_2 = B$, eq 20 reduces to

$$w(r, t) = B(1 - e^{-k^2 t})\, K_0(\lambda r). \qquad (21)$$

For $t \to \infty$,

$$w \doteq BK_0(\lambda r). \qquad (22)$$

The graphical representation of eq 22 for different values of λ is shown in Figure 5.

Eq 21 and 22 contain a logarithmic singularity at $r = 0$, a consequence of the assumptions. The deflection will be finite however, once the load is assumed distributed over a finite area (see p. 6ff).

Increasing rates of deflection

Mathematical formulation of the problem. Considering the vertical equilibrium of an element of a floating ice plate as shown in Figure 6, we obtain

$$r \frac{\partial N}{\partial r} + N - q(r) r = 0. \qquad (23)$$

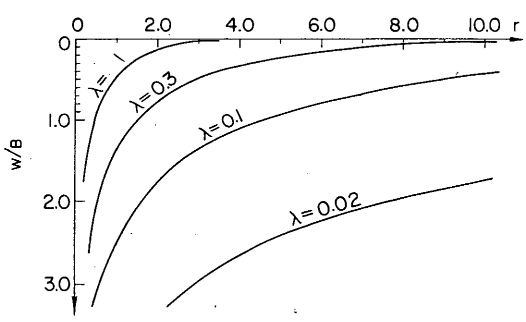

Figure 5.

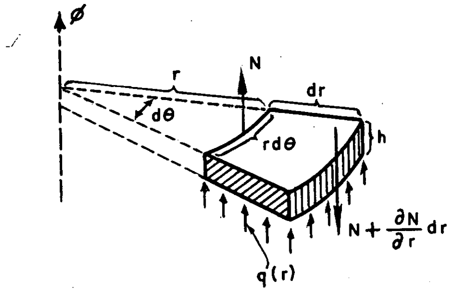

Figure 6.

Substituting eq 17

$$N = \mu h \frac{\partial^2 w}{\partial r\, \partial t} \quad \text{(Newtonian flow)}$$

and $q = \gamma w$ into eq 23, and differentiating with respect to t,

$$r^2 \frac{\partial^4 w}{\partial r^2\, \partial t^2} + r \frac{\partial^3 w}{\partial r\, \partial t^2} - \frac{\gamma}{\mu h} r^2 \frac{\partial w}{\partial t} = 0. \qquad (23)$$

This is the differential equation for the case of increasing rates of deflection. (Equilibrium between the load and the buoyant forces is implied in eq 18 but not in eq 24.)

Solution of the boundary value problem. Proceeding as in eq 19 - 20 we obtain the solution of eq 24 as

$$w(r,\, t) = (D_1 + D_2 e^{k_1^2 t})\, [C_1 I_0(\lambda_1 r) + C_2 K_0(\lambda_1 r)] \qquad (25)$$

which, by an argument similar to eq 21, reduces to

$$w(r,\, t) = D(e^{k_1^2 t} - 1)\, K_0(\lambda_1 r) \qquad (26)$$

where D is a constant.

Under the assumptions made here, the time functions

$$(1 - e^{-k^2 t}) \quad \text{and} \quad (e^{k_1^2 t} - 1)$$

for decreasing and increasing deflection do not depend on the type of loading. The time functions for different values of k^2 are shown in Figures 7 and 8.

INFINITE PLATE SUBJECTED TO A UNIFORM CIRCULAR LOAD

The results obtained for a concentrated load (eq 21, 26) are used to obtain solutions for the case of a uniform circular load, which is important in practice. Eq 21 and 26 differ by the product of a constant and a time function, which does not affect the intended integration with respect to r. Therefore, for brevity, these products are represented by the expression $B\, f(t)$. Experiments (Volkov, 1940, p. 227) indicate that "as regards the action on ice of a static load, it results in deflections, the magnitude of the deflection

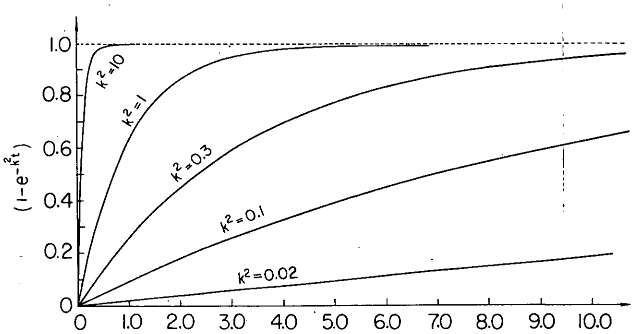

Figure 7.

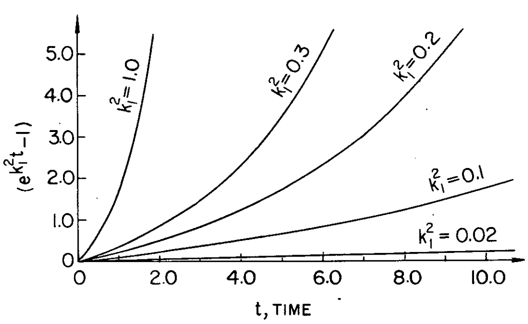

Figure 8

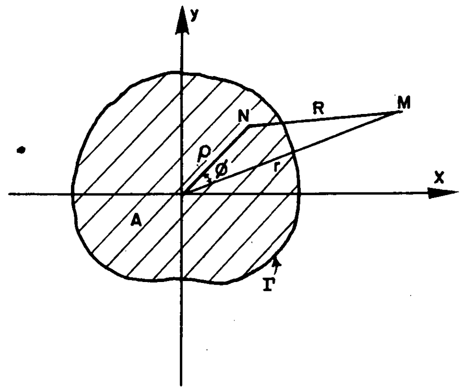

Figure 9.

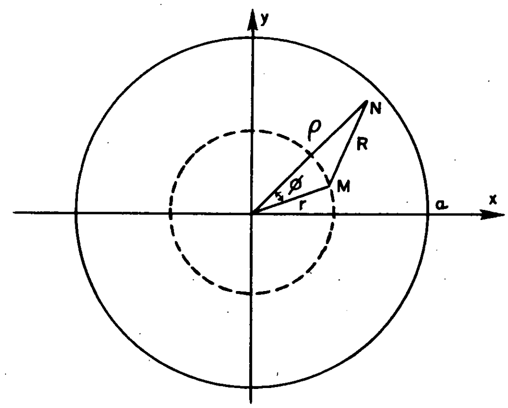

Figure 10.

increasing in proportion to the load." Therefore, we assume

$$B = PG_1 \tag{27}$$

where $\underline{P}$ is the concentrated force and $\underline{G_1}$ is a constant.

We consider a plate loaded in an arbitrary manner over the area $\underline{A}$ inclosed by the curve Γ (Fig. 9). Using eq 21 or 26 and 27, deflection at an arbitrary point $\underline{M}$ due to a load element

$$dP = qdA = q(\rho, \phi)\ \rho d\rho d6 \tag{28}$$

at $\underline{N}$ can be expressed as

$$dw = G_1 f(t)\, K_0(\lambda R)\, q(\rho, \phi)\, \rho d\rho\, d\phi \tag{29}$$

where $q(\rho, \phi)$ is the distributed load at (ρ, ϕ) per unit area. Thus the deflection caused by the total distributed load is

$$w = G_1 f(t) \int\int_A q(\rho, \phi)\ \rho K_0(\lambda R)\ d\rho d\phi \tag{30}$$

with $R^2 = r^2 + \rho^2 - 2r\rho \cos\phi$ Assuming q = const. distributed over the area inclosed by a circle of radius $\underline{a}$ with center at $r = 0$, we obtain

$$w = G_1 q f(t) \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \rho K_0(\lambda R)\ d\rho d\phi \tag{31}$$

<u>Deflections outside the loaded area $(r > \rho)$</u>

For the integration of eq 31 with respect to ϕ, we use eq A5 (see Appendix) and obtain

$$w = 2\pi\ G_1 q f(t)\ K_0(\lambda r) \int_0^a \rho\, I_0(\lambda\rho)\ d\rho. \tag{32}$$

Applying formula 835.3 in Dwight (1947), with $2\pi G_1 = G$, eq 32 reduces to

$$w(r, t) = Gf(t)\ q\frac{a}{\lambda}\ I_1(\lambda a)\ K_0(\lambda r). \tag{33}$$

Deflections inside the loaded area ($r > \rho$)

Eq 31 can be represented as

$$w = G_1 f(t)\, q \Big[\underbrace{\int_0^{2\pi} \int_0^{r} \rho K_0(\lambda R)\, d\rho\, d\phi}_{r > \rho} + \underbrace{\int_0^{2\pi} \int_r^{a} \rho K_0(\lambda R)\, d\rho\, d\phi}_{r < \rho} \Big]. \tag{34}$$

Applying eq A5 to the first integral expression and eq A6 to the second, and setting $2\pi G_1 = G$, we obtain

$$w = Gf(t)\, q \Big[\int_0^{r} \rho K_0(\lambda r)\, I_0(\lambda \rho)\, d\rho + \int_r^{a} \rho K_0(\lambda \rho)\, I_0(\lambda r)\, d\rho \Big] \tag{35}$$

which reduces to

$$w = Gf(t) \frac{q}{\lambda^2} \Big\{ \lambda r \left[K_0(\lambda r)\, I_1(\lambda r) + K_1(\lambda r)\, I_0(\lambda r) \right] - \lambda a I_0(\lambda r)\, K_1(\lambda a) \Big\} \tag{36}$$

(using formula 835.3, ibid.) and with the identity

$$I_0(\lambda r)\, K_1(\lambda r) + K_0(\lambda r)\, I_1(\lambda r) = + \frac{1}{\lambda r} \tag{37}$$

to

$$w(r, t) = Gf(t) \frac{q}{\lambda^2} \left[1 - \lambda a K_1(\lambda a)\, I_0(\lambda r) \right]. \tag{38}$$

Note that in eq 33 and 38, for decreasing rates of deflection,

$$f(t) = (1 - e^{-k^2 t}) \tag{39}$$

and for increasing rates of deflection,

$$f(t) = (e^{k_1^2 t} - 1). \tag{40}$$

The constants G and λ are also different for these two cases.

CONCLUSIONS

It is interesting to observe that the time function obtained in the derivations for decreasing rates of deflection corresponds to that of a Kelvin model (Fig. 11). According to M. Reiner (in Eirich, 1956, p. 47), "the Kelvin body is a solid, i.e., it will come to a state of rest when the spring has taken up the load. Its viscosity is accordingly solid viscosity damping both the appearance and disappearance of the elastic strain." According to the statements and assumptions made and the results obtained, the total system of an ice plate resting on a liquid base can be considered as a Kelvin (also called Voigt) body for the case of decreasing rates of deflection, where the "dashpot" corresponds to the resistance of the ice plate when deformed in its vertical planes by viscous flow and the spring to the liquid foundation. This analogy gives k^2 a physical meaning. In rheology $1/k^2$ is called the "retardation time". k^2 is a measure of the delay caused by the "dashpot" in attaining the final deformation (Fig. 7, 8).

As pointed out in the introduction it was observed from experiments that the deflection surface immediately after load application ($t = 0$) corresponds to that due to elastic bending, whereas after some time ($t \rightarrow \infty$), for decreasing rates of deflection, it corresponds to the deformation surface caused by shear only. In view of this, the following composition of the total deflection w of an ice plate resting on a liquid foundation is suggested for the case of decreasing rates:

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$$w = (w_{e\,(bending)})\, e^{-at} + w_{p\,(shear)} \qquad (35)$$

where $\underline{w}_{e\,(bending)}$ = elastic deflection of a thin plate resting on a liquid foundation (Herz's case),

$\underline{a}$ = constant (a = k or some function of it) and

$\underline{w}_{p\,(shear)}$ = corresponding plastic deflection as obtained in this paper.

For increasing rates, the following composition is suggested:

$$w = w_{e\,(bending)} + w_{p\,(shear)} \, . \qquad (37)$$

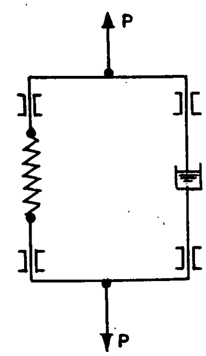

Figure 11.

Floating ice plates are neither homogeneous nor isotropic in either the vertical or the horizontal direction. By introducing simplifications the phenomenon of plastic deformation is mathematically represented here by a differential equation of a relative low order (for the space coordinates, only second order). Hence the results as derived here will need certain modifications. For example it is expected that the constants $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$, and $\underline{G}$ in the deflection equations will in fact also be functions of the plate thickness $\underline{h}$, because the h appearing in the exponent of $\underline{e}$ or in the argument of the Bessel functions affects the shape of the deformed ice plate, but not the magnitude of the deflection (see also eq 27). Thus at this stage of the derivations a sequence of tests is necessary in order to compare results obtained here with actual conditions in the field, to determine the dependence of $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$, and $\underline{G}$ upon $\underline{h}$, and to determine the influence of neglected factors – temperature, salinity, etc. – on the variation of the remaining constants.

In conclusion it should be mentioned that the limiting state between decreasing and increasing rates of deflection forms the necessary safety criterion for the application of loads of long duration on floating ice plates.

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APPENDIX

Some properties of the $K_0(\lambda R)$ - function.

For

$$R^2 = r^2 + \rho^2 - 2r\rho \cos \phi \tag{A1}$$

according to Watson (1945, p. 361, eq 8) with $\nu = 0$

$$K_0(\lambda R) = \sum_{m=-\infty}^{+\infty} K_m(\lambda r)\, I_m(\lambda \rho) \cos m\phi \tag{A2}$$

and because of symmetry of eq A1 in r and ρ

$$K_0(\lambda R) = \sum_{m=-\infty}^{+\infty} K_m(\lambda \rho)\, I_m(\lambda r) \cos m\phi \tag{A3}$$

Multiplying eq A2 and A3 by $d\phi$ then integrating from $\phi = 0$ to $\phi = 2\pi$ and noting that

$$\int_0^{2\pi} \cos m\phi \; d\phi = \begin{cases} 0 & \text{for } m \neq 0 \\ 2\pi & \text{for } m = 0 \end{cases} \tag{A4}$$

we obtain

$$\int_0^{2\pi} K_0(\lambda R)\, d\phi = 2\pi K_0(\lambda r)\, I_0(\lambda\rho) \quad \text{for } r > \rho \tag{A5}$$

and

$$\int_0^{2\pi} K_0(\lambda R)\, d\phi = 2\pi K_0(\lambda \rho)\, I_0(\lambda r) \quad \text{for } r < \rho . \tag{A6}$$

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PLASTIC DEFORMATION OF FLOATING ICE PLATES SUBJECTED TO STATIC LOAD – Arnold D. Kerr

Research Report 57, Sept 1959, 10p-illus-app.
DA Proj 8-66-02-400, SIPRE Proj 22.2-3a.
Unclassified Report

The problem is analyzed mathematically for decreasing and increasing rates of deflection. The analysis is based on the assumptions that for decreasing rates of deflection the floating ice plate will deform under lateral load without failure until the weight of the displaced water is equal to that of the load, and that for increasing rates, deflection increases until the ice plate collapses under and near the load. It is also suggested that the total deflection at a certain time is the result of the elastic deflection surface and the plastic deflection due to shear only, the shear forces obeying Newton's law of viscosity. Deflection equations for plastic deflection due to shear are derived for an

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