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A Method of Analyzing Geothermal Data in Permafrost

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INTRODUCTION

Geothermal measurements in permafrost regions, particularly at deep layers, are of primary importance in studying the permafrost. Dr. G. R. MacCarthy's recent paper (1952) is very valuable, as it is one of the very few presentations of geothermal data on permafrost at deep layers. But, in analyzing data of this kind, great care must be taken in estimating the undisturbed temperature.

MacCarthy writes (p. 591): "In any form of drilling operations the ground temperatures immediately adjacent to the drill hole are greatly disturbed and, on the Arctic Slope, require from many weeks to one or even two years or more to regain thermal equilibrium." It is the geothermal measurements of the undisturbed layers that we need. So, for a certain depth of a drill well, he measured the temperature as a function of time and made a "cooling curve." Then he made an empirical formula from this curve and calculated the undisturbed temperature, or the equilibrium temperature, by extrapolating the curve to $t = \infty$. The extrapolation of an empirical formula is not desirable at any time, and extrapolation to infinity is especially dangerous. Extrapolation to infinity is allowable only when the formula has some theoretical foundation and the parameters are calculated from the observed data in a certain range.

One of the most important problems concerning permafrost is whether the permafrost in a certain region is growing or diminishing at present. There is a possibility of solving this problem if we can make precise geothermal measurements on deep layers in the permafrost. The temperature gradient in permafrost is small, usually of the order of 0.01°C/ft. It is certain that secular variation of the climate affects the geothermal data in permafrost, but the variation is considered as only several hundreths of a degree. Temperature measurements can be accurate to 0.01°C, if we use well calibrated thermistors and exercise proper care in electrical measurement. This accuracy will be quite enough for solving most of the fundamental problems concerning the permafrost. However, not just each measurement, which is always affected by disturbance due to drilling, but the undisturbed value or the equilibrium temperature must be accurate to 0.01°C. Therefore, analysis of the observed data must also be accurate in the order of 0.01°C, or the precise measurements in the field have no meaning. For that purpose, we need a formula with some theoretical background which gives the required value within the error of field measurements.

EMPIRICAL FORMULA OF HYPERBOLIC FORM

MacCarthy gives one example of the cooling curve from a spot 595 feet deep in a certain well. The observed temperatures are plotted against the elapsed time in Figure 1. He employed an empirical formula of hyperbolic form:

$$Y = (7\text{-}x)\ /\ (0.31\,x + 8.06) - 3.34. \quad (1)$$

He writes: "This formula gives such a close approximation to the observed temperatures that it should be possible to estimate with fair accuracy the final equilibrium temperature at this depth." Thus he calculated the equilibrium temperature at 595 feet in this particular well by extrapolating equation (1) to infinity, and obtained −6.57°C as the equilibrium temperature. However, the coincidence of the observed and calculated values does not mean that this formula can be extrapolated to infinity. We can choose another set of parameters which give a curve fitting more closely to the observed values and give a different equilibrium temperature when extrapolated to infinity. One example is an empirical formula of the form:

$$Y = (7 - x) / (0.302\,x + 10.4) - 3.34. \quad (2)$$

The standard deviation from the observed data is 0.039 for this formula, 0.042 for MacCarthy's formula (Table I). The equilibrium temperature calculated from equation (2) is −6.65°C rather than −6.57°C. We cannot say that this difference of 0.08°C is negligible, because so much care was taken in getting an accuracy of 0.01°C in the field measurements. Nor can we conclude that −6.65°C is the correct equilibrium temperature because the standard deviation of equation (2) is less than that of equation (1). We may find another set of parameters which gives a still smaller standard deviation and a different equilibrium temperature. Any effort to get equilibrium temperature from an empirical formula without physical meaning is in vain. Hyperbolic equations like (1) and (2) are very slow in approaching their asymptotic values for large values of x. For example, according to equation (1), more than 50 years are required to reach the value of 0.01°C above the equilibrium value, that is x for $y = -6.56$°C. This point also suggests the inadequateness of this form of empirical formula for obtaining the equilibrium temperature.

FORMULA WITH A THEORETICAL BACKGROUND

The cooling curve tends to drop rapidly in the initial stage and then diminish in an almost exponential form (Fig. 1). The unbroken line in Figure 1 is an exponential curve chosen by trial to fit most of the observed values, except in the initial part of the curve. After about 80 days, all the observed data are very well represented by this exponential curve. This suggests that the cooling curve consists of two parts. The initial stage, in which cooling is very rapid, must be the range where the heating effect of drilling still remains. Another point to be considered is the temperature gradient in the protecting material of the thermistor. As the measurement was started 29 September, the temperature of the thermistor might have been above freezing point when it was inserted into the well. The thermistor itself will take some time to cool, because it is protected with insulating materials. These effects seem to die away in about 80 days, and after that cooling takes place in exponential form. Cooling in the initial stage, which is strongly affected by the method of drilling and the procedure of measurement, should be treated separately from the process of cooling which obeys Lambert's law. The hyperbolic equation tries to include these two modes of cooling in one equation. That is the cause of difficulty. We must exclude the data on the initial stage, which may be affected by the artificial disturbance of the permafrost or the method of installing the thermistors. In any case we need not be concerned with this stage, because we want to know the data for a very large value of t.

After this initial stage, the rate of cooling must be a function of the temperature difference between the place of observation and the undisturbed place, that is, the equilibrium temperature. Expanding the function, we take the first term as the approximate formula,

$$-\frac{dy}{dt} = k\,(y - y_\infty)\,. \quad (3)$$

TABLE I. COMPARISON OF THE TWO EMPIRICAL FORMULAS OF HYPERBOLIC FORM WITH DIFFERENT PARAMETERS AND THE EXPONENTIAL FORMULA

Elapsed time	Observed temp.	Computed temp. by (1)	Difference (1)	Computed temp. (2)	Difference (2)	Computed temp. (5)	Difference (5)
days	°C	°C	°C	°C	°C	°C	°C
7	−3.34	−3.34	0.00	−3.34	0.00	--	--
37	−4.83	−4.88	−0.05	−4.73	+0.10	--	--
53	−5.16	−5.22	−0.06	−5.08	+0.08	--	--
88	−5.54	−5.63	−0.09	−5.53	+0.01	−5.59	−0.05
119	−5.76	−5.83	−0.07	−5.76	0.00	−5.76	0.00
259	−6.18	−6.19	−0.01	−6.18	0.00	−6.19	−0.01
267	−6.20	−6.20	0.00	−6.20	0.00	−6.20	0.00
281	−6.22	−6.22	0.00	−6.22	0.00	−6.22	0.00
295	−6.23	−6.23	0.00	−6.23	0.00	−6.24	−0.01
316	−6.26	−6.25	+0.01	−6.26	0.00	−6.26	0.00
344	−6.29	−6.28	+0.01	−6.29	0.00	−6.29	0.00
Standard deviation			0.042		0.039		
Equilibrium temp.			−6.57°C		−6.65°C		−6.39°C
Difference between (1) and (2)					−0.08°C		
Difference between (1) and (5)							+0.18°C

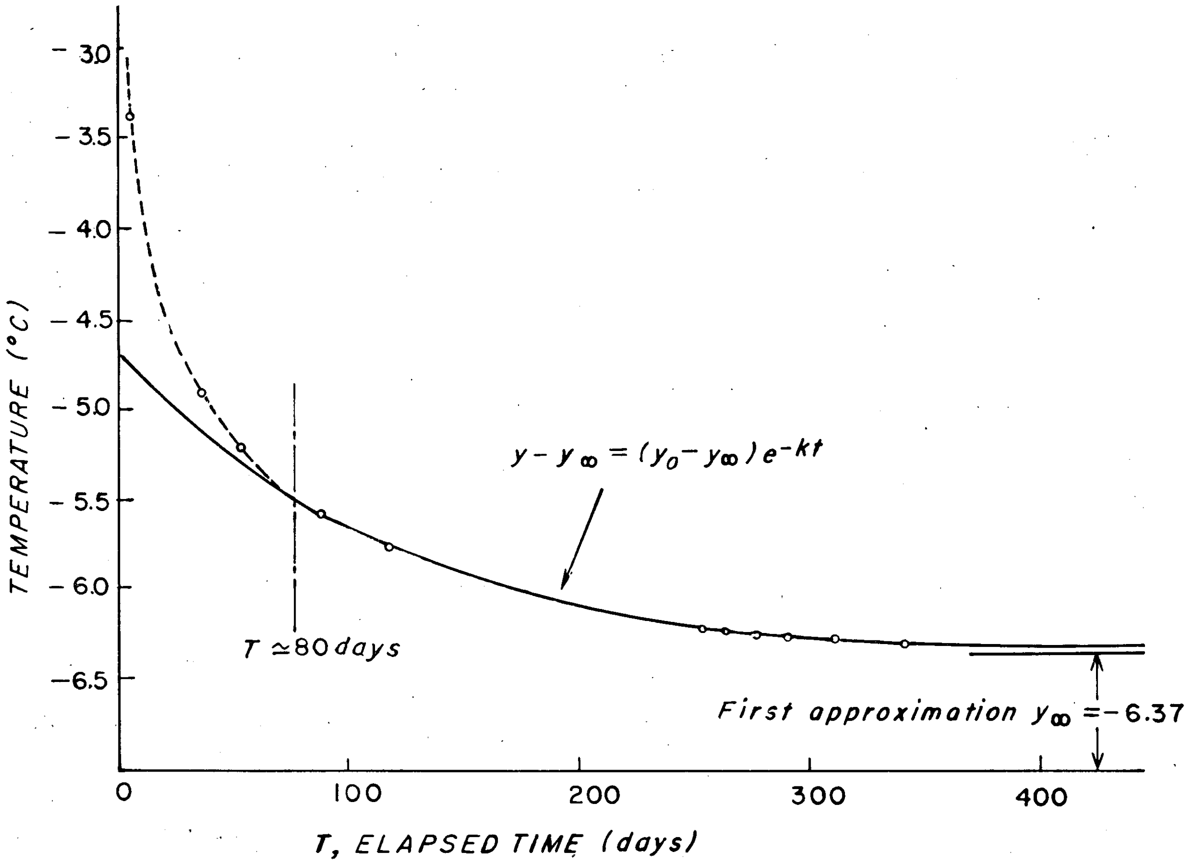

FIGURE 1. COOLING CURVE FOR SOUTH BARROW TEST WELL NO. 3 Depth, 595 feet. Small circles indicate temperatures recorded by MacCarthy (1952).

Integrating (3) we get,

$$(y - y_\infty) = (y_0 - y_\infty)\ e^{-kt}. \tag{4}$$

If we can find y_0 and k from the observed data, we can calculate the equilibrium temperature y_∞. We may extrapolate the formula to infinity in this case because equation (4) has a theoretical background as shown in (3). The most probable values of y_0, k, and y_∞ can be determined quickly from the observed data if an adequate calculating machine or a differential analyzer is available.

The adequateness of the exponential formula (4) is better shown by plotting ln $(y - y_\infty)$ against time (Fig. 2). From equation (4)

$$\ln\ (y - y_\infty) = \ln\ (y_0 - y_\infty) - kt.$$

Curve A in Figure 2 becomes a straight line after $t = 80$ days. For plotting $\ln(y - y_\infty)$ against time, we must assume the value of y_∞. As a first approximation, $-6.37°$C is chosen, by a simple extrapolation of the exponential portion of the observed values in Figure 1.

METHOD OF CALCULATING THE EQUILIBRIUM TEMPERATURE

When an adequate calculating machine is not available, we must determine the parameters by a more laborious procedure. The method of least squares is the best, but it would take several weeks' calculation, as this equation contains an exponential function. Therefore, a method of successive approximations is preferable. First we find the asymptotic value by a graphical extrapolation from the exponential portion of the observed values in Figure 1. The rough approximation thus obtained is $y_\infty = -6.37$. Equation (4) becomes,

$$\ln(y + 6.37) = \ln(y_0 + 6.37) - kt.$$

Plotting $\ln(y + 6.37)$ against t gives us line A in Figure 2. From the slope of line A, we get:

$$k = 0.0086.$$

From $y_\infty = -6.37$ and $k = 0.0086$; we can calculate $(y_0 - y_\infty)$. It is wise to use a point on line A in Figure 2, because this line shows the average of observed data. To minimize the error, a point near the middle of the line should be chosen. So we pick $t = 200$. From this point (P) we read:

$$\ln(y_{200} - y_\infty) = -1.19.$$

$$\ln(y_o + 6.37) = -1.19 + 0.0086 \times 200 = 0.53,$$

$$y_o + 6.37 = 1.71,$$

$$y_o = -4.66.$$

We now have all of the parameters, but the value of y_∞, which is the most important, is still an assumption. In the next approximation, we recalculate the value of y_∞ from the values of y_o and k now obtained. In other words from the equation:

$$y - y_\infty = (-4.66 - y_\infty)\, e^{-0.0086t}$$

we determine y_∞. In this calculation also, it is wise to use the value of $t = 200$ on line A in Figure 2. Then:

$$\ln(y_{200} + 6.37) = -1.19,$$

$$(y_{200} + 6.37) = e^{-1.19} = 0.30,$$

which gives:

$$y_{200} = -6.07,\ e^{-0.0086 \times 200} = 0.179.$$

So we obtain $y_\infty = -6.39$.

Using $y_\infty = -6.39$, instead of -6.37 as in the former calculation, we repeat the same process. The $\ln(y - y_\infty)$ line becomes line B in Figure 2, which gives $k = 0.0081$. Using $y_{200} = -6.07$ and $k = 0.0081$, we get $y_o = -4.77$. From these values, the second approximation of y_∞ comes out as -6.40. Using this value of y_∞, we proceed again, and get $k = 0.0079$ and $y_o = -4.95$. The third approximation gives $y_\infty = -6.39$ again. So we can safely say that the equilibrium temperature is -6.39°C ± 0.01°C. The trend of the successive approximation is seen in Table II.

In the final empirical formula, the mean of the second and third values of k is taken, and $y_\infty = -6.39$, $y_o = -4.77$.

$$y + 6.39 = 1.62\, e^{-0.0080t} \qquad (5)$$

The data computed by equation (5) are given in Table I. The difference from the observed data is 0.00 or 0.01, except at $t = 88$ days. The difference 0.05°C at $t = 88$ days can be attributed to the residual effect of the heat applied to the well in the course of drilling. Thus we can conclude that the equilibrium temperature of this particular well at 595 feet is -6.39°C, and the error will be within 0.01°C. The value of k also has a physical meaning. It is a measure of the thermal conductivity of the frozen soil. The rather large difference in y_o among the successive approximation values in Table 2 does not affect the conclusion much, because y_o has little physical meaning.

TABLE II. TREND OF SUCCESSIVE APPROXIMATION

	y_∞	k	y_o	y_∞
Rough approx.	Graphical extrapolation			$-6.37 = (y_\infty)_o$
First approx.	-6.37	0.0086	-4.66	$-6.39 = (y_\infty)_1$
Second approx.	-6.39	0.0081	-4.77	$-6.40 = (y_\infty)_2$
Third approx.	-6.40	0.0079	-4.95	$-6.39 = (y_\infty)_3$

SUMMARY

The method of successive approximation described in this paper seems to be satisfactory. This method can be condensed as follows:

(1) Use the empirical formula:
$$(y - y_\infty) = (y_o - y_\infty)\, e^{-kt},$$
or $\ln(y - y_\infty) = \ln(y_o - y_\infty) - kt$

(2) Plot the observed data against time, and find the rough value of y by a simple graphical extrapolation of the apparently exponential part of the curve. This gives the rough approximation, $(y_\infty)_o$.

(3) Using $(y_\infty)_o$, plot $\ln y - (y_\infty)_o$ against time. The curve will become a straight line A, after the initial stage. From the slope of the straight segment calculate k. This is k_1. When the curve does not become a straight line, this method is not applicable, but such a case will be very rare.

(4) Choose a certain time t', which corresponds to a point near the middle of the straight line part of A. Read the value of $\ln y_{t'} - (y_\infty)_o$ from line A, and calculate $y_{t'}$. This gives $(y_{t'})_1$.

(6) Using $(y_o)_1$, k_1, t' and $(y_{t'})_1$, calculate y_o. This gives $(y_o)_1$.

(6) Using $(y_o)_1$, k_1, t' and $(y_{t'})_1$, calculate y_∞. This gives $(y_\infty)_1$.

(7) If $(y_\infty)_o - (y_\infty)_1$ is within the accuracy required, either $(y_\infty)_o$ or $(y_\infty)_1$ is the equilibrium temperature. Usually they show the difference of several hundredths of a degree.

(8) Starting with $(y_\infty)_1$, repeat the process described in (3) - (6). This gives k_2, $(y_o)_2$, and $(y_\infty)_3$.

(9) Proceed in this manner until $(y_\infty)_{n-1} - (y_\infty)_n$ comes within the required accuracy. Either $(y_\infty)_{n-1}$ or $(y_\infty)_n$ is the equilibrium temperature.

The above procedure looks very tedious, but usually it requires only a few hours calculation, once the method is established, and the observed data cannot be utilized otherwise.

There is another method for finding the

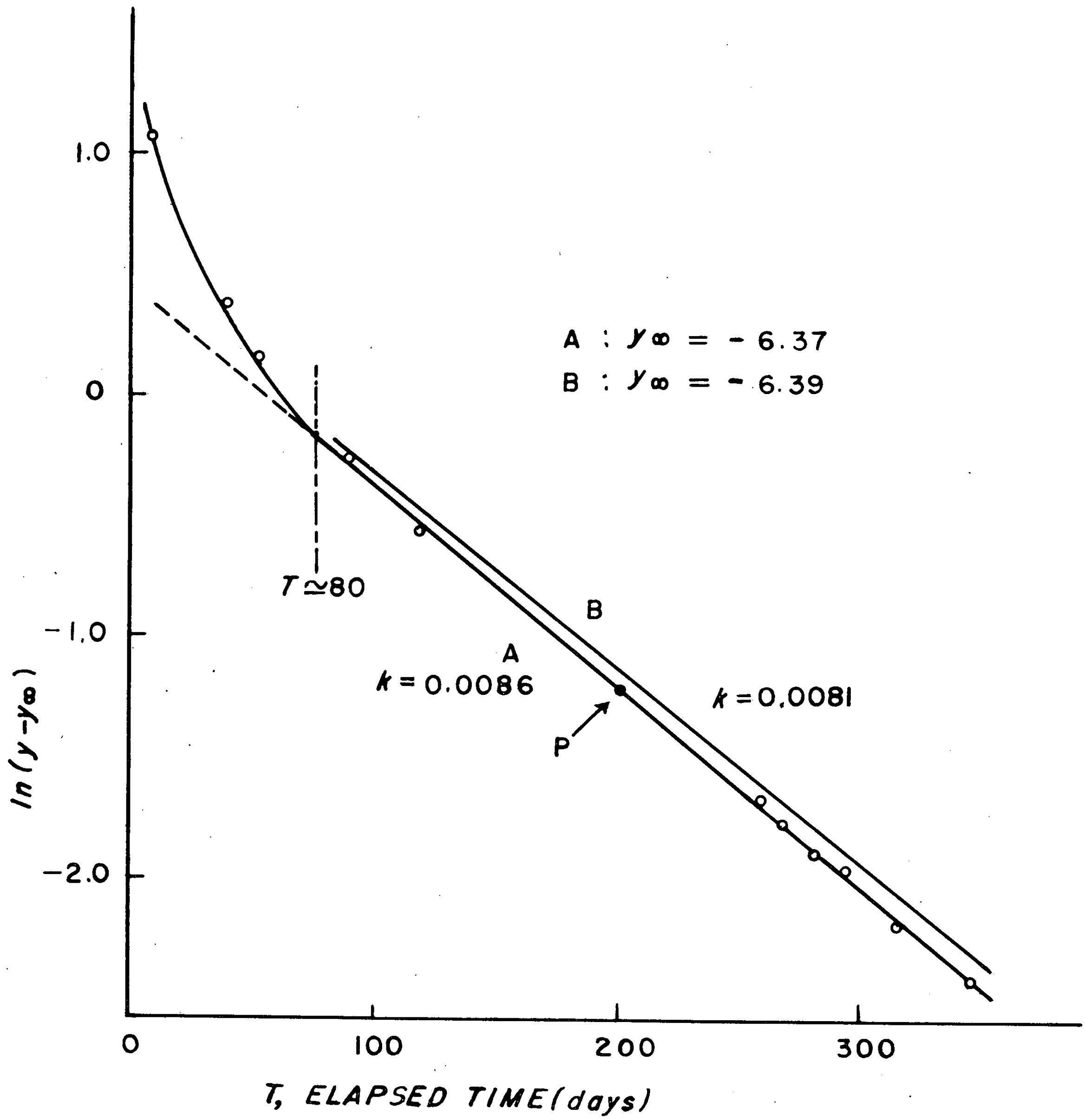

FIGURE 2. CURVE FOR COMPUTING k

values of y_∞, k, and y_0. Choose t_1, t_2, t_3 so that $t_2 - t_1 = t_3 - t_2$, and read the corresponding values of y; that is, y_1, y_2, and y_3.

$$\frac{y_1 - y_\infty}{y_2 - y_\infty} = e^{k(t_2 - t_1)}$$

$$\frac{y_2 - y_\infty}{y_3 - y_\infty} = e^{k(t_3 - t_2)}$$

$$(y_1 - y_\infty)\,(y_3 - y_\infty) = (y_2 - y_\infty)^2$$

$$y_\infty = \frac{y_1 y_3 - y_2^2}{y_1 - 2y_2 + y_3} \tag{6}$$

y_∞ is thus obtained. Using this value of y_∞, k is calculated by the equation,

$$e^{k(t_2 - t_1)} = \frac{y_1 - y_\infty}{y_2 - y_\infty} \tag{7}$$

After knowing y_∞ and k, the value of y_0 is easily calculated.

From the original graph of Figure 1, three sets of t and y are read.

$$t_1 = 100,\ t_2 = 250,\ t_3 = 400$$

and $\quad y_1 = -5.67,\ y_2 = -6.19,\ y_3 = -6.33.$

From these values y_∞ is calculated as

$$y_\infty = -6.38.$$

This value coincides with the former value of $y_\infty = -6.39$ within the required accuracy. After getting the value of y_∞, k can be easily calculated by equation (7), but the graphical method may be better, because the errors in the measurements are smoothed out when we use the graphical method.

SECULAR VARIATION OF PERMAFROST TEMPERATURE

In the foregoing discussions, y_∞ was calculated under the assumption that it is a constant; that is, the temperature of the permafrost at a certain depth in a certain locality was assumed to be constant for a long period of time, at least in the range of observation. If the observations are continued for several years and a sensible secular variation is existent in that period of observation, the foregoing formula cannot be used to represent data covering the whole period.

When a sensible secular variation does exist, we must consider y_∞ to be a function of time. Let the undisturbed temperature of the permafrost at a certain depth in a certain locality be T.

$$T = F(t')$$

where t' is a long range time measured in units of a year, as distinguished from t, which is a short period of time measured in units of a day or month.

When no secular variation exists,

$$T = y_\infty = \text{constant, always.}$$

If a sensible secular variation is expected,

$$T = y_\infty = \text{constant, for a certain range of } t'.$$

Let the range of t', in which T can be considered as a constant within the experimental error, be $\bar{t'}$, determined by the required accuracy of measurement and the mode of the secular variation. The former can be designated, but the latter is unknown, and can only be determined by measurements over a considerable period of time t'.

Let us consider the simplest case, when the temperature of undisturbed permafrost is decreasing slightly over several years. In this case, the first approximation of $T = F(t')$

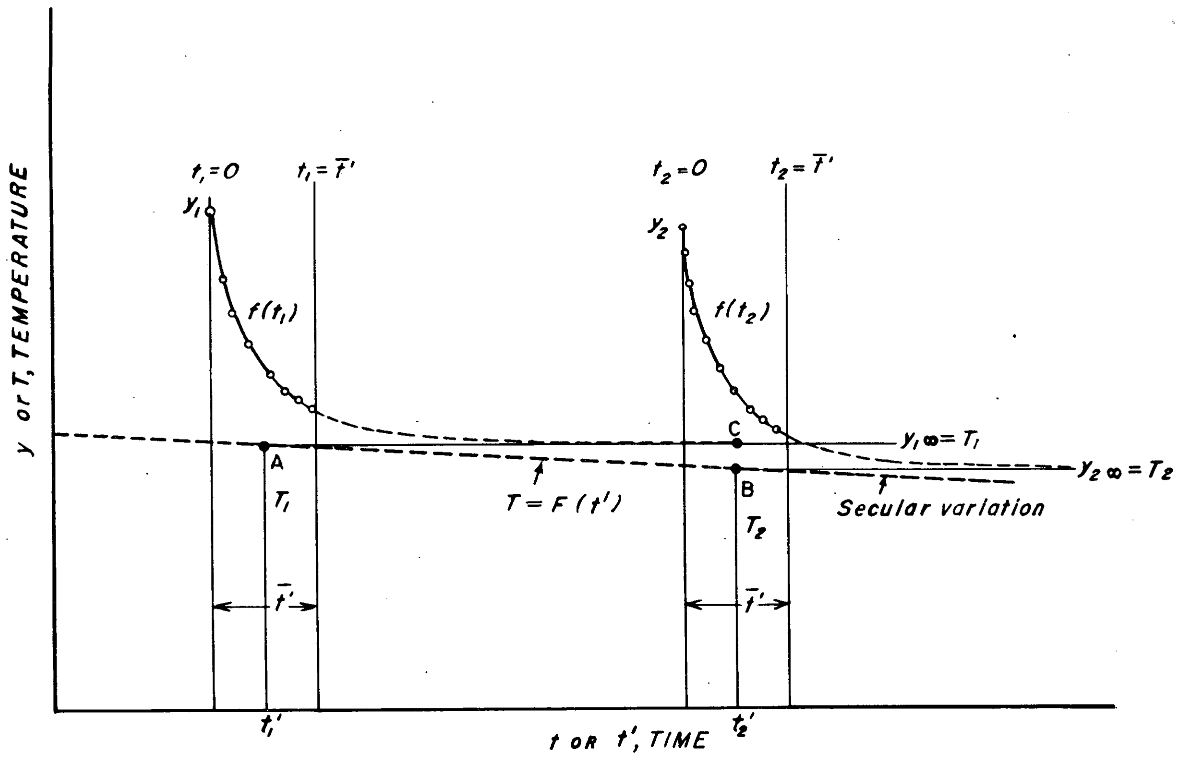

FIGURE 3. PERMAFROST TEMPERATURE WHEN A SECULAR VARIATION EXISTS

will be obtained by two similar measurements at t'_1 and t'_2, as shown in Figure 3. The measured values of $y_1 = f(t_1)$ will give the value of T_1 at t'_1 by extending the extrapolation formula f. The value of T_2 is obtained in a similar manner. The points A and B in Figure 3 thus obtained give the first approximation of the secular change in the temperature of the permafrost. In this calculation, it is better to use the values of y for $0<t<\bar{t}'$.

When it is impossible to drill another well at the same spot after several years, we can derive a formula with the condition that y_∞ is a function of time, so that not only the equilibrium temperature but also its secular variation can be measured in one well by long range measurements extending over several years. However, the geothermal data now available are not sufficiently reliable to justify the application of such a formula with the condition that y_∞ is a function of time. When the long range measurements in a certain well show some discrepancy between the calculated and observed values, we have to be satisfied with presuming that this discrepancy is most likely due to the secular variation of the temperature of the permafrost itself.

One simple example will be given. Referring to Figure 3 again, the measurement started at t'_1 in a certain well is continued for several years and gives the value B at the date t'_2, while the calculated value of the undisturbed temperature is C, which was obtained from the values of y_1 in the range $0<t_1<\bar{t}'$ by using the extrapolation formula f. In such a case, the discrepancy $\overline{BC}$ can be taken as showing the inadequateness of the extrapolation formula f, if f is a mere empirical formula. However, if f has some theoretical background, this discrepancy $\overline{BC}$ is better taken as the secular variation of the permafrost temperature. The author hopes equation (4)

$$y - y_\infty = (y_0 - y_\infty)\ e^{-kt}$$

can be used for giving the secular variation also, if it exists.

The above discussions are founded on the assumption that the observed data are accurate within the required accuracy. Such problems as the accuracy of the thermistor, the leakage in the circuits, etc. are beyond the scope of this discussion. They must be separately checked, and adequate corrections derived. The analysis must be applied to the corrected data.

This formula may be used also as a possible criterion for checking the accuracy of the measurements, especially the constancy of the measuring elements over a long range of time. For example, when the temperature measurements are available for several years from a fixed depth in a selected well, the approximate value of the supposed secular variation can be estimated from the discrepancy between the calculated and the observed data. If this value is reasonable, it is probable that the extrapolation formula and the method of the measurements are both correct, and we can proceed further. If this estimated value of the secular variation is extraordinarily high, there must be some defect either in the extrapolation formula or in the method of measurement, more likely in the latter. If the defect is evident, the next step is to check the accuracy of the measuring equipment.

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