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by

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CHAPTER 36

WAVES GENERATED BY A PISTON-TYPE WAVEMAKER

by

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ABSTRACT

When a wavemaker generates a finite number of waves, it has been found that one of the first and one of the last waves in such a burst is considerably larger than the average. A mathematical model, based on the linearized governing equations, is used for the particular problem of the waves generated by a sinusoidally moving piston-type wavemaker starting from rest. Theoretical results for the magnitude of the large wave relative to the average agree fairly well with experiments; however, the actual wave height is smaller in the experiments than predicted by theory. It is shown, by extending the classical wavemaker theory to second order, that finite amplitude effects do not offer an explanation. However, pistons rarely fit the tank dimensions exactly, and an approximate evaluation indicates that the discrepancy between predicted and observed wave heights can be attributed to the effects of leakage around the piston.

1. INTRODUCTION

One of the major problems encountered, when performing tests in a wave tank, is to account for the influence of reflected waves. Within the framework of linear theory we can deal with this problem (see Ursell, et al., 1960), when the magnitude of the reflected wave is small compared with that of the incident wave. However, in cases where the reflection from the far end of the tank is large, this is no longer possible. To overcome this problem, some coastal engineering tests are performed using the "burst method", in which the wavemaker generates waves only so long as no significant reflection from the far end of the tank has yet reached the wavemaker. After the wavemaker is stopped, time is allowed for the reflections to die out, before a new burst is generated. This procedure essentially eliminates the influence of even large reflections, but as is often the case, eliminating one problem creates another.

Figure 1 shows the surface profile recorded by a fixed gage 45 feet from a wavemaker, which generates a burst of 15 waves. A prominent feature is evident: One of the first and one of the last waves arriving at a particular station is considerably larger than the average. The effects of these large waves on test results have been of concern to engineers at the Coastal Engineering Research Center (CERC) where e.g., rip-rap stability is determined from tests employing the burst method.

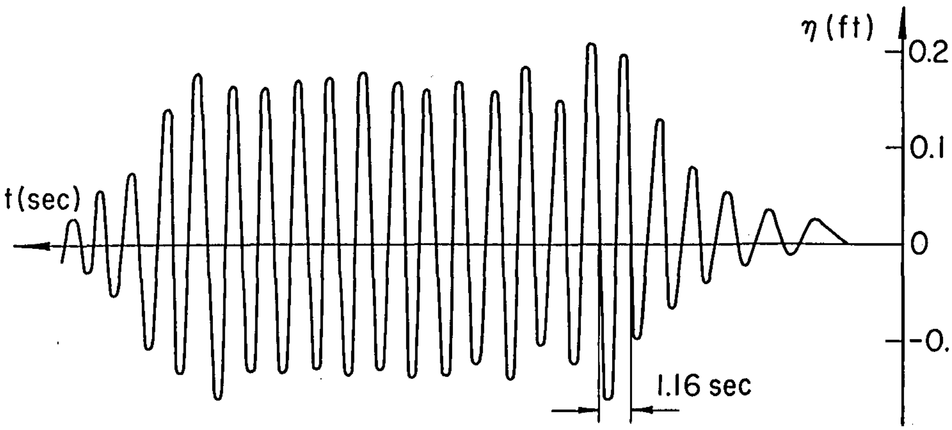

Figure 1: Surface profile recorded 45 feet from a sinusoidal piston-type wavemaker starting from rest and generating 15 waves. (T = 1.16 sec., h = 1.5 ft., Stroke of wavemaker = 0.33 ft.

The study presented in the following was undertaken in an attempt to gain insight into the nature of these large waves and, if possible, to find a way to eliminate them.

The large waves are clearly associated with the transient, i.e., the starting and stopping of the wavemaker, to which the classical wavemaker theory (Havelock, 1929; Biésel and Suquet, 1951; Ursell,et al., 1960) does not apply. However, Kennard (1949) has solved the linearized governing equations based on the assumption of potential flow starting from rest, and we adopt his solution as the theoretical model for the particular problem of a sinusoidally moving piston-type wavemaker starting from rest (Section 2.1).

The question is: How accurately will the linear theory predict the development of laboratory waves, which more often than not are quite nonlinear? From the theory of progressive waves we know that the second order nonlinear Stokes' wave sharpens the crest and flattens the trough when compared with the linear first order solution, but that the wave height remains unchanged. Thus focusing on wave height, rather than amplitude, the linear solution is likely to cover at least slightly nonlinear waves. This is supported by the experimental confirmation of the classical wavemaker theory by Ursell, et al. (1960). A much more serious limitation of the results from a linearized theory stems from the instability of sinusoidal waves, i.e., for relatively short waves (depth/length = $h/L > 0.216$) the Benjamin-Feir side-band instability (Benjamin, 1967), and for moderately long waves ($h/L < 0.09$), the occurrence of secondary crests (Galvin, 1968; Madsen et al., 1970).

With these limitations in mind, the theoretical results for wave heights are tested against experiments for three cases (h/L = 0.24 0.197, 0.132) in Section 2.2, and it is found that the predicted and observed magnitudes of the large waves relative to the average agree reasonably well. However, the theory overestimates the actual wave height. This was also found by Ursell et al. (1960) for waves of fairly large steepness and was attributed to possible nonlinear effects. In Section 3 the classical (linear) wavemaker theory is advanced to second order, and it is found that finite amplitude effects cannot be considered responsible for the difference between observed and predicted wave heights.

An approximate evaluation of the amount of leakage through the gaps between the piston and the tank walls and the influence of this leakage on the height of the generated wave is performed in Section 4. It is found that the discrepancy between observed and predicted wave heights may be attributed to leakage around the piston, which establishes confidence in the wave heights predicted by the linear theoretical model adopted.

Section 5 discusses the possibility of utilizing the large effect of leakage on the height of the generated waves to eliminate the large waves in a burst.

2. LINEAR SOLUTION FOR A WAVEMAKER STARTING FROM REST

2.1 Theory

Assuming irrotational motion the linearized equations governing the motion generated by a wavemaker (see Figure 2) are

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} = 0 \qquad -h \leq y \leq 0 \tag{2.1}$$

$$\phi_y = 0 \qquad y = -h \tag{2.2}$$

$$\eta_t - \phi_y = 0 \qquad y = 0 \tag{2.3}$$

$$\phi_t + g\eta = 0 \qquad y = 0 \tag{2.4}$$

and at the wavemaker, which is characterized by its position, $\xi(y,t)$:

$$\phi_x = \xi_t\,(y,t) = u(y,t) \qquad x = 0 \tag{2.5}$$

where subscripts indicate partial differentiation and g is the acceleration of gravity.

Assuming the motion to start from rest,

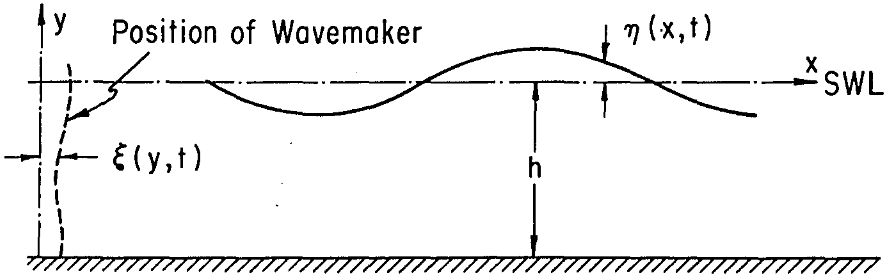

Figure 2: Definition of Symbols

$$\phi(x,y,0) = \phi_t(x,y,0) = \xi(y,0) = \xi_t(y,0) = 0 \tag{2.6}$$

the solution for the surface profile has been obtained by Kennard (1949)

$$\eta(x,t) = -\frac{2}{\pi}\int_0^\infty dk \int_0^t d\tau \int_0^{-h} dy \quad \frac{\cos \sigma(t-\tau)\,\cos kx\, u(y,\tau)\,\cosh k(y+h)}{\cosh kh}, \tag{2.7}$$

where

$$\sigma^2 = gk \tanh kh \tag{2.8}$$

k being the wave number.

For the particular case of a sinusoidal piston-type wavemaker, which runs for a length of time, t', we have

$$u(y,t) = \begin{cases} 0 & t \le 0 \\ U\sin(\omega t + \delta) & 0 < t \le t' \\ 0 & t > t' \end{cases} \tag{2.9}$$

and inserting this in (2.7), we can perform the integration with respect to y and τ to obtain

$$\eta(x,t,\delta) = \frac{2}{\pi}\;\frac{U}{\omega}\int_0^\infty dk \;\frac{\tanh kh}{k}\;\frac{\omega^2}{\omega^2 - \sigma^2}\;\cos kx$$

$$[(\cos\sigma t - \cos\omega t)\cos\delta + (\sin\omega t - \frac{\sigma}{\omega}\sin\sigma t)\sin\delta] \tag{2.10}$$

Realizing that

$$\frac{U}{\omega} = \xi_o$$

is the amplitude of the wavemaker motion and introducing the dimensionless variables (indicated by asterisks)

$$(x^*,y^*) = h^{-1}\ (x,y)$$
$$t^* = \sqrt{\frac{g}{h}}\ t \qquad (2.11)$$
$$k^* = kh$$

the solution can be written

$$\frac{\pi}{2}\ \frac{\eta(x,t,\delta)}{\xi_o} = I_1 \cos\delta + I_2 \sin\delta \qquad (2.12)$$

where

$$I_1 = \int_0^\infty dk^*\ \frac{\tanh k^*}{k^*}\ \frac{\omega^{*2}}{\omega^{*2} - k^*\tanh k^*}\ (\cos\sqrt{k^*\tanh k^*}\,t^* - \cos\omega^* t^*)\ \cos k^* x^* \qquad (2.13)$$

and

$$I_2 = \int_0^\infty dk^*\ \frac{\tanh k^*}{k^*}\ \frac{\omega^{*2}}{\omega^{*2} - k^*\tanh k^*}(\sin\omega^* t^* - \frac{\sqrt{k^*\tanh k^*}}{\omega^*}\ \sin\sqrt{k^*\tanh k^*}\,t^*)\ \cos k^* x^* \qquad (2.14)$$

This solution as it stands is valid only so long as $t < t'$. However, we may add the solution satisfying the boundary condition at $x = 0$:

$$\phi'_x = u'(y,t) = \begin{cases} 0 & t \le t' \\ -U \sin(\omega(t-t') + \delta') & t > t' \end{cases} \qquad (2.15)$$

and due to the linearity of the governing equations the sum of these solutions will satisfy the boundary condition given by (2.9) provided

$$\delta' = \delta + \omega t' - 2m\pi \qquad (2.16)$$

where m is an integer.

Thus in short we may write the solution as

$$\eta(x,t) = \begin{cases} \eta(x,t,\delta) & 0 < t \leq t' \\ \eta(x,t,\delta) - \eta(x,t-t',\delta') & t > t' \end{cases} \qquad (2.17)$$

where the right-hand sides are calculated from (2.12).

The integrals I_1 and I_2 have removable singularities and are evaluated by numerical integration using the trapezoidal rule with a stepsize equal to 0.05 the period of the integrand and the upper limit of integration equal to 16. The results were tested for accuracy by varying the stepsize and the upper limit of integration. As a further check the numerical solution was found to approach the classical solution to the wavemaker problem (Biésel and Suquet, 1951) as t became large.

Taking x = constant in (2.12) we can compute the surface elevation at a particular station along the tank as a function of time, which corresponds to the surface profile recorded by a fixed gage. If we define the wave height as the difference in surface elevation between a trough and the preceding crest, we may describe the development taking place some distance from the wavemaker in terms of the sequence of wave heights as the waves arrive at this station. For a station 30 times the depth from the wavemaker, the computed variation in wave heights relative to the wave height in the final periodic state, H/H_p, is shown in Figure 3 for three depth-to-length ratios.

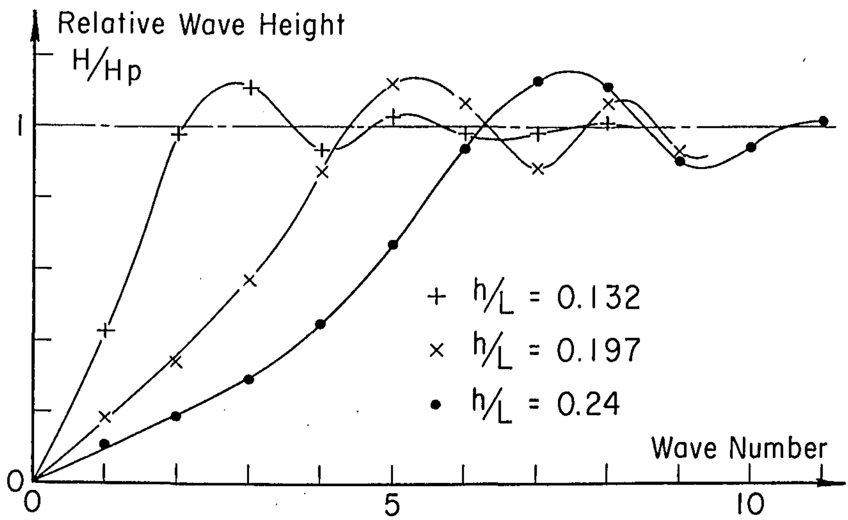

Figure 3: Computed variation in relative wave height of waves generated by a piston-type wavemaker starting from rest, as they reach a station 30 x depth from the generator.

It shows that after the disturbance arrives, the wave height increases, overshoots but finally attains the constant value corresponding to the final periodic state. This type of behavior, which predicts a large first wave, is analogous to the response of a slightly damped mechanical system to an exciting force.

2.2 Comparison with Experiments

The large waves in a burst had previously been studied experimentally in CERC's 72-foot tank by John Ahrens, Hydraulic Engineer at CERC, and records of the surface profile obtained 16 and 45 feet from the wavemaker serve for a detailed comparison with the result computed from (2.12).

The wavemaker, piston type, is electronically controlled and was set to start from its mean position going backwards, i.e., corresponding to $\delta = -\pi/2$ in (2.9). The wavemaker was stopped manually, and if it was stopped off its mean position it would abruptly take this position. Since it is very difficult to avoid this final impulse, which will influence the last part of the burst, the camparison is only carried out for the first waves in a burst. The depth, h, was 1.5 feet and for each period the experiment was repeated three times, giving practically identical results for the part of the burst used for the comparison.

The experiments show second order effects (wave steepness = H/L = 0.03 - 0.06) in that crest amplitudes are larger than trough amplitudes; however, as was indicated previously and in view of the results obtained in Section 3, this effect is essentially eliminated by comparing wave heights rather than amplitudes. The comparison is presented in Table 1 as the variation in wave heights as the waves arrive at the particular stations, and it is obvious from the results that the actual wave heights differ considerably. However, if we compare the variation in wave heights relative to the wave height in the final periodic state, which for the experiments is taken as the average wave height in a burst excluding the large waves, the agreement between computed and experimental wave heights in this sense seems good. The larger discrepancy for the shortest wave, $h/L = 0.24$, may be attributed to the Benjamin-Feir side-band instability.

3. APPROXIMATE SECOND ORDER WAVEMAKER THEORY

The application of a linear theory as a predictor of the motion generated by a wavemaker starting from rest was found to be relatively successful. The large difference between the actual computed and experimental wave heights noted in Table 1 is naturally of minor importance once a particular wavemaker has been calibrated. However, if the observed discrepancy can be explained as other than the inadequacy of the linear solution, an understanding of the responsible mechanism will not only help us in designing more efficient wavemakers and maybe eliminate the need for calibrations, but it will give us confidence in the results obtained from the linearized governing equations, also for problems different from the one treated here. We therefore proceed to investigate if the large difference in wave heights can be attributed to finite amplitude effects as Ursell et al. (1960) suggested.

Based on a Lagrangian formulation Fontanet (1961) derived the complete second order solution to the wavemaker problem. His solution, however, is extremely difficult to evaluate, and for this reason we outline an approach using the more familiar Eulerian description. Due to the nonlinear instabilities mentioned previously (Benjamin-Feir instability for short waves, secondary waves for long waves), we need

Wave No.	16 feet from Wavemaker: H feet Exp.	H feet Comp.	H/H_p Exp.	H/H_p Comp.	45 feet from Wavemaker: H feet Exp.	H feet Comp.	H/H_p Exp.	H/H_p Comp.	
1	0.305	0.370	0.85	0.9	0.155	0.183	0.43	0.44	h/L = 0.132
2	0.368	0.412	1.02	1.0	0.305	0.405	0.85	0.98	
3	0.387	0.424	1.08	1.03	0.387	0.456	1.08	1.11	
4	0.345	0.408	0.96	0.99	0.354	0.387	0.98	0.94	
5	0.357	0.418	0.99	1.02	0.360	0.424	1.00	1.03	
6	0.362	0.396	1.01	0.96	0.361	0.404	1.00	0.98	
1	0.142	0.181	0.42	0.44	0.061	0.068	0.19	0.16	h/L = 0.197
2	0.331	0.393	0.99	0.95	0.112	0.138	0.34	0.34	
3	0.352	0.457	1.05	1.11	0.194	0.236	0.59	0.57	
4	0.341	0.384	1.02	0.93	0.296	0.363	0.91	0.88	
5	0.334	0.408	1.00	0.99	0.369	0.468	1.13	1.13	
6	0.343	0.412	1.02	1.00	0.315	0.442	0.96	1.07	
7	0.330	0.391	0.98	0.95	0.318	0.366	0.97	0.89	
8	0.333	0.419	0.99	1.02	0.337	0.444	1.03	1.08	
9	0.339	0.416	1.01	1.01	0.327	0.384	1.00	0.93	
1	0.110	0.130	0.30	0.28	0.045	0.049	0.15	0.11	h/L = 0.24
2	0.254	0.287	0.71	0.62	0.068	0.085	0.20	0.18	
3	0.420	0.479	1.17	1.03	0.117	0.136	0.35	0.29	
4	0.366	0.512	1.02	1.10	0.170	0.209	0.51	0.45	
5	0.369	0.437	1.02	0.94	0.275	0.318	0.82	0.68	
6	0.368	0.459	1.02	0.99	0.403	0.441	1.21	0.95	
7	0.359	0.465	0.99	1.00	0.372	0.524	1.12	1.13	
8	0.364	0.472	1.01	1.02	0.300	0.519	0.90	1.12	
9	0.359	0.466	0.99	1.00	0.363	0.427	1.09	0.92	
10					0.310	0.446	0.93	0.96	
11					0.347	0.474	1.04	1.02	

Table 1. Comparison Between Computed and Experimental Wave Heights
(Depth = 1.5 feet)
(Underlined values correspond to the largest wave)

only consider intermediate waves, i.e., $0.21 \leq h/L \leq 0.09$. For large t the solution to the first order (linear) equations given in Section 2 is the classical wave-maker theory (see e.g., Biésel and Suquet, 1951). For a piston-type wavemaker,

$$u(y,t) = \xi_t = U \sin\omega t \; ; \; \xi = -\xi_o \cos\omega t \tag{3.1}$$

and denoting the first order solution by a superscript, $^{(1)}$, the classical solution reads:

$$\phi^{(1)} = C_o \cosh k_o(y+h) \cos(k_o x-\omega t) - \sin\omega t \sum_{n=1} C_n \cos k_n(y+h)\, e^{-k_n x} \tag{3.2}$$

$$\eta^{(1)} = \eta_o \sin(k_o x-\omega t) - \cos\omega t \sum_{n=1} \eta_n\, e^{-k_n x} \tag{3.3}$$

and

$$p^{+(1)} = p^{(1)} + \rho g y = \text{pressure due to the wave} =$$

$$- \rho g[\eta_o \frac{\cosh k_o(y+h)}{\cosh k_o h} \sin(k_o x-\omega t) + \cos\omega t \sum_{n=1} \eta_n e^{-k_n x} \frac{\cos k_n(y+h)}{\cos k_n h}] \tag{3.4}$$

where

$$C_o = \frac{2\, U \sinh k_o h}{k_o h + \sinh k_o h \cosh k_o h} \frac{1}{k_o} \qquad g k_o \tanh k_o h = \omega^2 \tag{3.5}$$

$$C_n = \frac{2\, U \sin k_n h}{k_n h + \sin k_n h \cos k_n h} \frac{1}{k_n} \quad (n \geq 1) \qquad g k_n \tan k_n h = -\omega^2 \quad (n-\tfrac{1}{2})\pi < k_n h < n\pi \tag{3.6}$$

$$\eta_o = \frac{\tanh k_o h}{n_1} \xi_o \qquad n_1 = \tfrac{1}{2}[1 + \frac{2k_o h}{\sinh 2k_o h}] \tag{3.7}$$

$$\eta_n = \frac{2 \sin^2 k_n h}{k_n h + \sin k_n h \cos k_n h} \xi_o \tag{3.8}$$

The first terms in (3.2), (3.3) and (3.4) represent a progressive wave, whereas the summations (inertia terms) express the correction necessary to account for the wavemaker motion not being exactly that of the particles in a progressive wave with the given period and depth. For long progressive waves, we know that the horizontal particle velocity is practically independent of y, and this suggests that the necessary correction, i.e., the sum of the inertia terms, is small when a piston-type wavemaker generates long waves. Similarly, large corrections are needed when short waves are generated by a piston-type wavemaker. This was clearly demonstrated in calculations by Biésel and Suquet (1951). We also note, (3.2) combined with (3.6), that the exponential behavior of the inertia terms means that their importance becomes negligible within a small distance, of the order 3h, from the wavemaker.

In advancing the theory to second order we proceed as outlined by Stoker (1957) by assuming a perturbation solution and expanding the boundary conditions at the free surface and at the wavemaker around y = 0 and x = 0, respectively. Denoting the second order solution by superscript $^{(2)}$ we have the governing equations,

$$\nabla^2\phi^{(2)} = \phi^{(2)}_{xx} + \phi^{(2)}_{yy} = 0 \qquad -h \leqq y \leqq 0 \tag{3.9}$$

$$\phi^{(2)}_{y} = 0 \qquad y = -h \tag{3.10}$$

$$\phi^{(2)}_{y} + \frac{1}{g}\phi^{(2)}_{tt} = -\frac{1}{g}\left\{(\phi^{(1)}_{ty}\,\eta^{(1)})_t + \phi^{(1)}_{xt}\,\phi^{(1)}_{x} + \phi^{(1)}_{yt}\,\phi^{(1)}_{y}\right\} - \phi^{(1)}_{yy}\,\eta^{(1)} + \phi^{(1)}_{x}\,\eta^{(1)}_{x} \qquad y = 0 \tag{3.11}$$

where (3.11) is obtained by combining the dynamic and kinematic conditions to obtain a condition in $\phi^{(2)}$ only. The second order profile, $\eta^{(2)}$, is given by

$$\eta^{(2)} = -\frac{1}{g}\left\{\phi^{(2)}_{t} + \phi^{(1)}_{ty}\,\eta^{(1)} + \frac{1}{2}[(\phi^{(1)}_{x})^2 + (\phi^{(1)}_{y})^2]\right\}_{y=0} \tag{3.12}$$

and the boundary condition at the wavemaker

$$\phi^{(2)}_{x} = -\phi^{(1)}_{xx}\,\xi \qquad x = 0 \tag{3.13}$$

where we have used that $\xi = O(\eta)$ except for very long waves (see (3.7)).

The troublesome part of the determination of $\phi^{(2)}$ from (3.9), (3.10), (3.11) and (3.13) is the inhomogeneous equation (3.11). However, under the assumption of $t\to\infty$, $\phi^{(1)}$ and $\eta^{(1)}$ are as given by (3.2) and (3.3) and it can be shown that at y = 0 the maximum value of the inertia terms is of the order 1/4 of the term associated with the progressive wave so long as $h/L < 1/4$. It, therefore, seems to be a reasonable approximation to neglect the inertia terms when substituting $\phi^{(1)}$ and $\eta^{(1)}$ into (3.11) With this approximation the solution satisfying the equations (3.9) through (3.11) is the well known Stokes' second order solution for progressive waves (see e.g., Ippen, 1966).

$$\phi_P^{(2)} = -\frac{3}{8}\omega\eta_o^2 \frac{\cosh 2k_o(y+h)}{\sinh^4 k_o h} \sin 2(k_o x-\omega t) \quad (3.14)$$

and

$$\eta_P^{(2)} = -\frac{1}{4}k_o\eta_o^2 \frac{\cosh k_o h\ (\cosh 2k_o h+2)}{\sinh^3 k_o h} \cos 2(k_o x-\omega t) \quad (3.15)$$

However, we have not yet included the boundary condition x = 0, (3.13), which reads

$$\phi_x^{(2)} = -\phi_{xx}^{(1)}\ \xi(y,t) =$$

$$[C_o k_o^2 \cosh k_o(y+h) \cos\omega t + \sin\omega t \sum_{n=1} C_n k_n^2 \cos k_n(y+h)]\ \xi(y,t) \quad (3.16)$$

We have previously justified the neglect of the inertia terms in the boundary condition at y = 0; however, this cannot be justified for all y's at x = 0 unless the wave length is restricted further ($h/L < O(0.1)$). Introducing (3.1) in (3.16) and using the relationship among η_o and C_o we obtain

$$\phi_x^{(2)} = -\xi_o\ [\frac{g\eta_o}{\omega}\ k_o^2\ \frac{\cosh k_o(y+h)}{\cosh k_o h} \cos^2\omega t + \sin\omega t \cos\omega t \sum_{n=1} C_n k_n^2 \cos k_n(y+h)]. \quad (3.17)$$

Clearly the solution (3.14) does not satisfy this boundary condition. Using $\cos^2\omega t = \frac{1}{2}(\cos 2\omega t + 1)$ and taking only the periodic part of (3.17) we see that we have a residual periodic boundary condition at x = 0.

$$(\phi_R^{(2)})_x = \phi_x^{(2)} - (\phi_P^{(2)})_x = -\frac{g\eta_o^2 k_o^2}{2\omega}\left\{\frac{n_1 \cosh k_o(y+h)}{\sinh k_o h} - \frac{3}{2}\frac{\cosh 2k_o(y+h)}{\sinh^3 k_o h \cosh k_o h}\right\}\cos 2\omega t - \sin 2\omega t\ \frac{\xi_o}{2}$$

$$\sum_{n=1} C_n k_n^2 \cos k_n(y+h) = U_R^{(2)}(y) \cos 2\omega t + \sin 2\omega t \sum_{n=1} (V_R^{(2)}(y))_n \quad (3.18)$$

Introducing

$$\phi_R^{(2)} = \phi^{(2)} - \phi_P^{(2)} \tag{3.19}$$

in the governing equations, which are linear in $\phi^{(2)}$ we see that $\phi_R^{(2)}$ is the solution to (3.18) and the homogeneous equations

$$\nabla^2 \phi_R^{(2)} = 0 \qquad -h \le y \le 0 \tag{3.20}$$

$$(\phi_R^{(2)})_y = 0 \qquad y = -h \tag{3.21}$$

and

$$(\phi_R^{(2)})_y + \frac{1}{g} (\phi_R^{(2)})_{tt} = 0 \qquad y = 0 \tag{3.22}$$

i.e., the same governing equations as those for the first order solution, except for the more complicated boundary condition (3.18) at $x = 0$. The solution, however, can be found from the classical solution to the linear problem as a sum of ϕ_R's, which we may combine and the progressive part can be written as

$$\phi_{R,\ \mathrm{Progressive}}^{(2)} = C_o^{(2)} \cosh k_o^{(2)} (y+h) \cos(k_o^{(2)} x - 2\omega t + \psi) \tag{3.23}$$

where $C_o^{(2)}$ and ψ are found by combining the ϕ_R's and

$$4\,\omega^2 = g\, k_o^{(2)} \tanh k_o^{(2)} h \tag{3.24}$$

We can therefore express the velocity potential far from the wavemaker to the second order

$$\phi \simeq \phi^{(1)}_{\mathrm{Progressive}} + \phi_P^{(2)} + \phi^{(2)}_{R,\ \mathrm{Progressive}} \tag{3.25}$$

or in physical terms the periodic waves generated by a wavemaker can be expressed as

(1) A first harmonic linear wave of amplitude η_o.

(2) A second harmonic coupled with the first harmonic to give the second order Stokes' wave corresponding to the linear solution.

(3) A second harmonic free wave of small amplitude.

This description agrees with that of Fontanet (1961), and when combined, we see that the surface elevation at a fixed value of x can be expressed as

$$\eta = \eta_o \left(\sin\omega t + \epsilon_1 \sin(2\omega t + \psi_1)\right) \qquad (3.26)$$

where $\epsilon_1 \ll 1$. This type of surface elevation was shown by Ursell et al., (1960) to give a wave height, $H = 2\eta_o(1+0(\epsilon_1^2))$. Thus for small ϵ_1, the wave height of nonlinear waves, as recorded by a fixed gage, is practically the same as that predicted by a linear theory.

This analysis was carried out assuming the final periodic state to be reached, and suggests that when results for periodic waves obtained from a linear theory are compared with experiments, a comparison of wave heights should essentially eliminate the influence of finite amplitude. From this we conclude that the large difference between computed and experimental wave heights noted in Table 1 can hardly be attributed to nonlinear effects.

4. THE INFLUENCE OF LEAKAGE ON THE HEIGHT OF THE GENERATED WAVES

The results of Ursell et at. (1960) indicated that the discrepancy between measured and predicted wave heights increases from the order 3% to 10% with an increase in wave steepness from 0.03 to 0.045. This does not agree with the experiments at CERC, which show a larger discrepancy (of the order 15%) between theory and observation as well as the opposite trend, i.e., decreasing discrepancy with increasing wave steepness. In the experiments by Ursell et al. the leakage around the wavemaker was reduced by a rubber foam lining between the piston and the walls and bottom of the tank, whereas no such provision was taken in the CERC experiments. This suggests that leakage around the piston may have a large influence upon the height of the generated wave. A series of experiments performed at CERC (Tenney, 1969) serve as further evidence of the influence of leakage on the height of the generated waves. Two holes were drilled through a piston, the area of the holes was approximately 0.29% of the wetted area of the piston. It was found that the difference in wave height between the waves generated with these holes closed and open was of the order 2.8% - (1.6% - 4.0%).

4.1 Waves Generated by an Oscillating Flow Through a Slot

To evaluate the influence of leakage let us start by examining the waves generated by the oscillating flow through a slot which extends over the width of the generator.

For a slot of height Δ a distance Y below the free surface, we have

$$u(y,t) = \begin{cases} 0 & y > -Y + \Delta \\ v_o \sin\omega t & -Y \leq y \leq -Y + \Delta \\ 0 & y < -Y \end{cases} \qquad (4.1)$$

which, with the notation used in Section 3, gives the solution

$$C_o = \frac{4}{2k_o h + \sinh 2k_o h} \frac{1}{k_o} v_o [\sinh k_o(-Y+\Delta+h) - \sinh k_o(-Y+h)] \quad (4.2)$$

For small Δ we may write this as

$$C_o = \frac{4 \sinh k_o h}{2k_o h + \sinh 2 k_o h} \frac{1}{k_o} \left\{ v_o \frac{\Delta k_o \cosh k_o(-Y+h)}{\sinh k_o h} \right\} \quad (4.3)$$

which written in this form by comparison with (3.5) clearly shows the generated progressive wave to be the same as that generated by a piston-type wave generator having a velocity given by

$$U = U' = v_o \frac{k_o \Delta \cosh k_o(-Y+h)}{\sinh k_o h} \quad - h \le y \le 0 \quad (4.4)$$

In particular we see that for a gap between the wavemaker and the bottom of the tank, Y = h,

$$U'_B = v_B \frac{k_o \Delta_B}{\sinh k_o h} = v_B \frac{\Delta_B}{h} \frac{k_o h}{\sinh k_o h} \quad (4.5)$$

with indices introduced for clarity.

Clearly, the influence of leakage between the sidewalls and the piston is not as easy to handle rigorously. This leakage is probably one of the mechanisms responsible for the generation of transverse waves; however, if the width of the tank is small compared with the wave length of the generated waves, it seems physically reasonable that the waves generated by an oscillating flow, $v_s(y) \sin\omega t$, through a vertical slot of width Δ_s may be approximated as the waves generated by a piston-type wavemaker, having the prescribed motion

$$U'_s = v_s \frac{\Delta_s}{b} \quad (4.6)$$

where b is the width of the wave tank, and v_s is the average of $v_s(y)$ over the depth.

Thus in principle, if v_B and v_s are known, we can find, at least with some accuracy, the generated waves, as those generated by an ideal piston-type wavemaker having the prescribed motion

$$u(y,t) = (U'_B + U'_s) \sin\omega t \quad (4.7)$$

However, is the potential theory really appropriate? When the water is forced through the small gaps into the ambient fluid one might expect a considerable energy loss due to turbulence. In a study of the forced heave motion of a rectangular cylinder of large draught, i.e., small distance between bottom and cylinder ($\Delta_B \approx 0.06$ h), Svendsen (1968) found the radiated wave to be accurately predicted by potential theory. Thus, an inviscid theory seems indeed to give reliable results; however, contrary to Svendsen's study, the major problem in our case is determining the velocities.

4.2 Determination of the Leakage Velocity

In order to attack this problem we must first specify the conditions on both sides of the generator blade. If we assume that the region behind the wavemaker is occupied by an absorber beach, which corresponds to the CERC 72-foot tank, a reasonable assumption is that waves are generated in both directions and that these waves are the same, but 180° out of phase.

With this assumption the pressure due to the sinusoidal motion (3.1), of the piston on the front side is given by (3.4) and the pressure due to the wave on the back of the piston is of equal magnitude but opposite sign. If we restrict our analysis to moderately long wave ($h/L < 1/4$) the influence of the terms in (3.4) with exponential behavior in x, the inertia terms, is small compared with that of the term associated with the progressive wave. Thus, we may approximate the pressure difference between the two sides of the piston, Δp, by

$$\Delta p = p^+_{front} - p^+_{back} \approx 2\rho g \eta_o \frac{\cosh k_o(y+h)}{\cosh k_o h} \sin\omega t \qquad (4.8)$$

where η_o is given by (3.7).

This pressure difference will produce a flow through the gaps between the piston and the sides and the bottom of the tank. The velocity, v, of this flow may be estimated from Bernoulli's equation. Neglecting friction and the unsteadiness of the motion we have

$$v^2/2g = \Delta p/\rho g \qquad (4.9)$$

In particular we get for the gap at the bottom, $y = -h$, by introducing (4.8) and defining v to be positive when directed towards the front of the piston, that

$$v(y=-h) = -\text{Sign}\left\{\sin\omega t\right\} 2\sqrt{\frac{g\eta_o}{\cosh k_o h}} \sqrt{\sin\omega t} \qquad (4.10)$$

To comply with the type of boundary condition assumed in (4.1), we can expand the time dependence of (4.10) in a Fourier Series. This has been done by Keulegan (1967), and retaining only the term associated with $\sin\omega t$, we get

$$v(y = -h) \simeq -2.22\sqrt{\frac{g\eta_o}{\cosh k_o h}} \sin\omega t \qquad (4.11)$$

or with the notation used in Section 4.1 we have

$$v_B = -2.22\sqrt{\frac{g\eta_o}{\cosh k_o h}} \qquad (4.12)$$

To find the leakage velocity through the gaps along the sides of the piston, we proceed in a similar manner, and taking the average leakage velocity over the depth, h, as the mean of the velocity at y = 0 and y = h we get

$$v_s \simeq -1.11\sqrt{g\eta_o}\left(1 + \frac{1}{\sqrt{\cosh k_o h}}\right) \qquad (4.13)$$

4.3 Decrease in Wave Height Due to Leakage Around the Piston

By substituting (4.12) and (4.13) into (4.5) and (4.6) respectively, we get from (4.7) that the leakage around the piston produces waves, whose characteristics are approximately those of the waves generated by an ideal piston-type wavemaker having the prescribed motion:

$$U' = -\sqrt{g\eta_o}\left(2.22\sqrt{\frac{1}{\cosh k_o h}}\frac{\Delta_B}{h}\frac{k_o h}{\sinh k_o h} + 1.11\frac{\Delta_s}{b}\left(1+\sqrt{\frac{1}{\cosh k_o h}}\right)\right) \qquad (4.14)$$

Comparing (4.14) with (3.1) we see that the leakage around the piston will decrease the amplitude of the generated waves by an amount $\Delta\eta_o$, which may be found from

$$\frac{\Delta\eta_o}{\eta_o} = -\left(2.22\sqrt{\frac{1}{\cosh k_o h}}\frac{\Delta_B}{h}\frac{k_o h}{\sinh k_o h} + 1.11\frac{\Delta_s}{b}\left(1+\sqrt{\frac{1}{\cosh k_o h}}\right)\right)\frac{\sqrt{g\eta_o}}{U} \qquad (4.15)$$

where η_o is found from (3.7).

To test the validity of (4.15) some experiments were performed in the CERC 72-foot tank, which has a width, b = 1.5 feet. Measurements gave Δ_B = 0.28 inches and Δ_s = 2x0.1 = 0.2 inches. With water depth h = 1.5 feet, the wave height, H_{exp}, as recorded 16 feet from the generator serve for the comparison presented in Table 2.

h/L	ξ_o (feet)	H_{exp} (feet)	$2\eta_o$ (feet)	Difference Observed %	Difference from (4.15) %
0.132	0.167	0.239	0.274	12.8	17.8
	0.251	0.367	0.411	10.7	14.6
0.197	0.115	0.228	0.272	16.2	15.8
	0.173	0.356	0.410	13.2	12.9
	0.230	0.477	0.545	12.5	11.2
0.240	0.125	0.300	0.352	14.8	12.7
	0.167	0.390	0.471	17.2	11.1
	0.196	0.466	0.553	15.3	10.2

Table 2: Comparison of observed reduction in wave height with that predicted from (4.15)

When considering the number of assumptions made in deriving (4.15), the agreement between predicted and observed reduction in wave height must be considered good. It is therefore concluded that the main part of the discrepancy between observed and predicted wave heights noted in Table 1 can be attributed to the influence of leakage around the piston. As a further check on the reasoning leading to (4.15), the reduction in wave height due to two holes in the piston, corresponding to the experiments by Tenney (1969) mentioned earlier, gave a predicted reduction of 2.1%, which compares favorably with Tenney's 1.6 - 4.0%.

The neglect of friction and unsteadiness made in order to arrive at (4.9) was tested by mounting a fitted 3/4 inch plywood board to the front of the piston blade. This increased the thickness of the blade from 1/4 inch to 1 inch, and since this had no significant effect on the height of the generated wave, this assumption seems justified.

It is interesting to note that the area of the gaps around the piston is of the order 2.7% of the wetted area, and that leakage through these gaps reduces the wave height by about 15%. This large effect of leakage through a small area, which is even more pronounced in the experiments by Tenney, may explain the results of Ursell et al. (1960), whose "seal" around the piston might not have been 100% effective for the large pressure difference between the two sides of the piston associated with the steepest waves. When one considers the large gap, which may exist at the bottom in the case of a flap-type generator hinged at the bottom, the large effect of leakage through this gap may also explain why a group of Neyrpic engineers (1952) observed a discrepancy of 30% between theoretical and measured wave heights of waves generated by a flap-type wavemaker.

5. CONCLUSION

In the preceding sections we have used a theoretical solution, based on the linearized governing equations, to predict the height of the waves generated by a piston-type wavemaker starting from rest. By comparison with experiments it was found that the theoretical model predicted the relative wave heights fairly accurately, and that the difference between predicted and observed wave heights could be attributed to the influence of leakage around the piston rather than to the inadequacy of the linear model. Thus the material presented essentially serves to establish confidence in the theoretical model adopted.

As stated in the Introduction, the problem initiating this study was the large first and last waves in a burst (Figure 1). Since the large last wave can be eliminated by the use of a drop-gate or a similar structure, the preceding sections have concentrated on the development taking place as the wavemaker is started. Computations show that the relative magnitude of the large first wave depends weakly on the position of the wavemaker, as it is started, and that this is smallest, when the wavemaker is started from its extreme positions. This dependence is, however, insignificant at stations far from the wavemaker, and consequently offers no solution to the problem of eliminating the large first wave.

From the analogy between Figure 3 and the response of a mechanical system to an exciting force, it is seen that the large first wave may be eliminated by starting the wavemaker slowly increasing the amplitude of its motion to the desired value. If this is possible and how it is most efficiently achieved can be determined by means of our theoretical model. Most wavemakers being of the type, which does not permit a change in stroke during operation, seems to render this solution impractical. However, this is where it might be possible to utilize the surprisingly large effect of leakage on the height of the generated waves. By controlling e.g. the area of the gap between the bottom and the piston, it would, in principle, be possible to change the height of the waves generated by a wavemaker having a constant stroke. If this effect can be used to eliminate the large first wave remains to be seen. However, this effect does seem to offer the possibility of generating amplitude modulated waves with a conventional wavemaker.

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