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Finite-Difference Schemes Compared for Wave-Deformation Characteristics in Mathematical Modeling of Two-Dimensional Long-Wave Propagation

by

R. J. Sobey

TECHNICAL MEMORANDUM NO. 32
OCTOBER 1970

U. S. ARMY, CORPS OF ENGINEERS

COASTAL ENGINEERING RESEARCH CENTER

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ABSTRACT

The wave-deformation characteristics of several difference schemes for two-dimensional long-wave propagation are compared by means of the propagation factor introduced by J. J. Leendertse. The schemes compared are those proposed by N. S. Heaps, R. O. Reid and B. R. Bodine, J. J. Leendertse, and M. B. Abbott, respectively. The study also demonstrates the differing behavior of explicit and implicit schemes.

FOREWORD

This report is published because of the useful manner in which several finite-difference schemes for two-dimensional long-wave propagation are compared. The suitability of a particular scheme is based on a comparison of its numerical solution with the analytical solution. Such comparisons provide the coastal engineer with guidance and a method for evaluating and selecting the most appropriate difference scheme for resolving problems concerned with long-wave motion.

Mr. R. J. Sobey of Australia made this study in partial fulfillment of the Individual Study requirements of the 1968-69 International Course in Hydraulic Engineering, Delft Technological University, The Netherlands. The topic for this study was suggested by Ir. J. Siemons of the Delft Hydraulics Laboratory, and his assistance throughout the study was appreciated. The author also acknowledges the general assistance of Dr. M. B. Abbott, Reader, International Course in Hydraulic Engineering, who introduced him to, and stimulated his interest in, numerical modeling in tidal and coastal engineering. Mr. Sobey is now studying for his doctorate at the Imperial College of Science and Technology in London, where he prepared the final draft of this report.

Mr. B. R. Bodine, a hydraulic engineer at CERC, was a fellow-student of Mr. Sobey at Delft Technological University. He was co-author with R. O. Read of one of the difference schemes compared in this study.

At the time of publication, the Director of the Coastal Engineering Research Center was Lieutenant Colonel Edward M. Willis; the Technical Director was Joseph M. Caldwell.

NOTE: Comments on this publication are invited. Discussion will be published in the next issue of the CERC Bulletin.

This report is published under authority of Public Law 166, 79th Congress, approved July 31, 1945, as supplemented by Public Law 172, 88th Congress, approved November 7, 1963.

CONTENTS

ILLUSTRATIONS

LIST OF SYMBOLS

Symbol	Definition	Dimension
arg ( )	argument of complex number	-
b_A, b_L, b_R	computational parameters	-
e	base of Naperian logarithms	-
g	acceleration of gravity	LT^{-2}
g_{ij}	element in i th row and j th column of amplification matrix G	-
h	wave height above M.S.L.	L
h_o	water depth below M.S.L.	L
h^*_m	h-amplitude of m th Fourier component	L
$h^n_{j,k}$	wave height at spatial position ($j.\Delta x$, $k.\Delta y$) and at time $n.\Delta t$	L
i	complex operator	-
j	integer number designating grid position in x-direction	-
k	integer number designating grid position in y-direction	-
m	integer subscript designating a particular Fourier component	-
n	integer number designating time in steps of Δt	-
p_1, p_2	computational parameters	-
s	coordinate dimension in direction of wave propagation and in plane of M.S.L.	L
t	time	T
u	vertically averaged velocity in x-direction	LT^{-1}
u^*_m	u-amplitude of m th Fourier component	LT^{-1}
$u^n_{j,k}$	x-component of vertically averaged velocity at spatial position ($j.\Delta x$, $k.\Delta y$) and at time $n.\Delta t$	LT^{-1}

Symbol	Definition	Dimension
v	vertically averaged velocity in y-direction	LT^{-1}
v_m^*	v-amplitude of m th Fourier component	LT^{-1}
$v_{j,k}^n$	y-component of vertically averaged velocity at spatial position $(j.\Delta x,\ k.\Delta y)$ and at time $n.\Delta t$	LT^{-1}
x	horizontal Cartesian coordinate in plane of M.S.L.	L
y	horizontal Cartesian coordinate in plane of M.S.L.	L
$A_1,\ A_2,\ A_A, A_L$	computational parameters	-
G	amplification matrix	-
Im ( )	imaginary part of complex number ( )	-
L	wavelength;	L
	or dimension of length	-
Re ( )	real part of complex number ( )	-
T	complex propagation factor;	-
	or dimension of time;	-
	or superscript indicating the transpose matrix	-
$\overline{U}$	solution vector, $(u,\ v,\ h)^T$	LT^{-1}, LT^{-1}, L
$\overline{U}_m^*$	$\overline{U}$-amplitude of m th Fourier component	LT^{-1}, LT^{-1}, L
$\overline{U}^n$	$\overline{U}$ at time $n.\Delta t$	LT^{-1}, LT^{-1}, L
$\overline{U}_{j,k}^n$	$\overline{U}$ at spatial position $(j.\Delta x,\ k.\Delta y)$ and at time $n.\Delta t$	LT^{-1}, LT^{-1}, L
β	real wave frequency	T^{-1}
β_m	β of m th Fourier component	T^{-1}

Symbol	Definition	Dimension
$\beta_{1,2,3}$	β of steady-state flow, positive characteristic, and negative characteristic, respectively	T^{-1}
β'	complex number such that $Re(\beta')$ is the computed wave frequency and $Im(\beta')$ is an indirect measure of the amplitude deformation	T^{-1}
$\beta'_{1,2,3}$	β' of steady-state flow, positive characteristic, and negative characteristic, respectively	T^{-1}
γ	orientation of wave propagation direction (i.e. s-direction) to positive x-axis (radians)	-
γ_m	γ of m th Fourier component	-
λ	eigenvalues of amplification matrix	-
ν	number of time steps necessary for propagation of real wave over its wavelength	-
σ	wave number	L^{-1}
σ_m	σ of m th Fourier component	L^{-1}
σ_{1m}, σ_{2m}	x- and y-components of σ_m	L^{-1}
σ_1, σ_2	x- and y-components of σ	L^{-1}
$\emptyset$	computational parameter, $= e^{i\beta'\Delta t}$	-
$\emptyset_{1,2,3}$	$\emptyset$ for steady-state flow, positive characteristic, and negative characteristic, respectively	-
Δs	square grid dimension	L
Δt	time step	T
Δx	x-direction grid dimension	L
Δy	y-direction grid dimension	L
$\sum_m (\ )$	summation of ( ) over all m Fourier components	-
$\|(\ )\|$	modulus of number ( )	-

Section I. INTRODUCTION

The finite-difference technique for the solution of partial differential equations has been known for many years. However, in many cases, its wide-spread application has been limited by the computational effort involved. Long-wave propagation in two horizontal spatial dimensions, considered in this study, is such a case.

The introduction in the last decade of large, fast, digital computers has greatly removed this computational limitation, but has served to magnify another difficulty in that there is a wide variety of finite-difference schemes that can be used in each particular case. The suitability of a particular scheme can be measured by considering its order of approximation and its stability, and also by comparison of its numerical solution with the analytical solution, this latter measure being of interest to this study.

Two-dimensional, long-wave propagation has received considerable attention from numerical modelers as the system of equations describes a physical situation of considerable practical interest to coastal engineers and related practitioners. A long-wave can be alternatively described as a nearly horizontal flow, with the implication that vertical accelerations are negligible and that the pressure distribution is hydrostatic. Thus, the long-wave equations can be used to model both storm-surge and tidal-wave propagation.

The comparison of the numerical solution of the difference scheme to the analytical solution of the partial differential equations has a physical interpretation for long-wave propagation in that it will be a measure of the deformation of the computed wave. Leendertse (Reference 1) has used the term *propagation factor* to describe the ratio of the numerical and analytical solutions in the particular case of long-wave propagation. This propagation factor is generally a complex number, thus characterizing the wave deformation in both amplitude and phase. This concept is discussed more fully in Section VI.

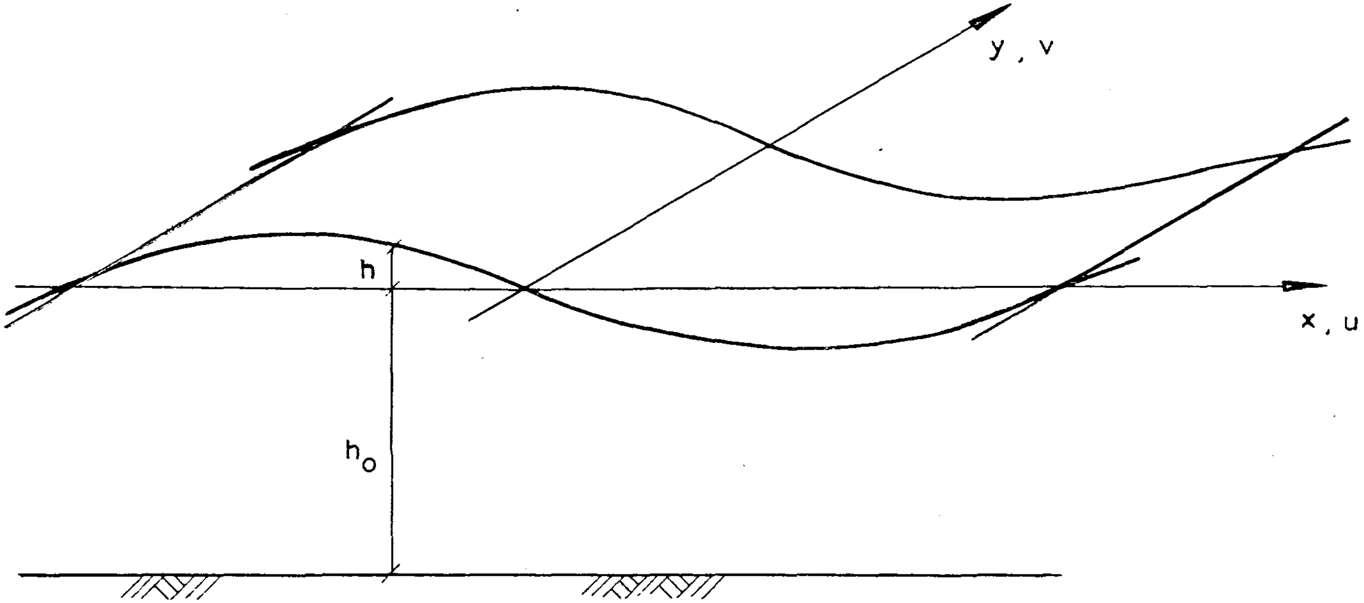
The behavior of this propagation factor, made suitably dimensionless, will provide a useful comparison of the wave deformation characteristics of different finite-difference schemes. A detailed comparison in this manner of four finite-difference schemes that have been proposed for two-dimensional long-wave propagation is the subject of this study.

The four schemes considered are explicit schemes proposed by

1. N. S. Heaps (Reference 2)
2. R. O. Reid and B. R. Bodine (Reference 3)

and implicit schemes proposed by

3. J. J. Leendertse (Reference 1)
4. M. B. Abbott (Reference 4).



X and Y axes are in plane of Mean Sea Level

h is positive upwards.

Figure 1. Definition Sketch

This latter scheme has not been published, but has been analyzed in detail by the present author - a summary of the relevant aspects are included as an Appendix.

Section II. LONG-WAVE EQUATIONS

This study will be restricted to the linearized, vertically averaged equations for two-dimensional long-wave propagation, neglecting convective accelerations, Coriolis accelerations, friction resistance, and surface stresses. These equations are

$$\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} = 0, \tag{1}$$

$$\frac{\partial v}{\partial t} + g\frac{\partial h}{\partial y} = 0, \tag{2}$$

$$\frac{\partial h}{\partial t} + h_o\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{3}$$

where u and v are the vertically averaged velocity components in the horizontal coordinate directions x and y respectively, h_o is the mean depth, h (positive vertically upwards) is the wave height with respect to Mean Sea Level, and t is the time coordinate (see definition sketch, Figure 1).

Section III. REAL WAVE

The analytical solution of the long-wave equations will be called the *real* or *physical wave*.

Assume a Fourier series of solution of equations (1) to (3) of form

$$\overline{U} = \sum_m \overline{U}_m^* e^{i(\beta_m t + \sigma_m s)}$$

where $\overline{U} = (u, v, h)^T$, the analytical solution,

$\overline{U}_m^* = (u_m^*, v_m^*, h_m^*)^T$, the amplitude of the m th component,

β_m = real wave frequency of m th component,

σ_m = wave number of m th component,

and s = coordinate dimension of the direction of wave propagation.

The superscript T implies the transpose vector.

Now assume that this wave propagation direction is oriented at angle γ_m to positive x-axis and define the wave number components

$$\sigma_{1m} = \sigma_m \cos \gamma_m$$

$$\sigma_{2m} = \sigma_m \sin \gamma_m .$$

Also the system is linear so that superposition is valid; consequently only one component of the Fourier series need be considered, namely

$$\overline{U} = \overline{U}^* e^{i(\beta t + \sigma_1 x + \sigma_2 y)}. \tag{4}$$

Substitution of equation (4) into equations (1) to (3) yields the following matrix equation

$$\begin{bmatrix} i\beta & 0 & i\sigma_1 gh_o \\ 0 & i\beta & i\sigma_2 gh_o \\ i\sigma_1 & i\sigma_2 & i\beta \end{bmatrix} \begin{bmatrix} u^* \\ v^* \\ h^* \end{bmatrix} = 0.$$

Equating the determinant of the coefficient matrix to zero yields the cubic equation

$$(i\beta)^3 + \sigma^2 gh_o i\beta = 0,$$

which solves to

$$\left.\begin{aligned} \beta_1 &= 0 \\ \beta_{2,3} &= \pm\sigma \sqrt{gh_o} \end{aligned}\right\} \tag{5}$$

Back substitution into equation (4) then describes the real wave.

Section IV. DIFFERENCE SCHEMES

For the purpose of the finite-difference method, the solution vector is made discrete to particular grid points on the solution field. In this context, $\overline{U}^n_{j,k}$ represents the finite-difference solution at spatial position $(j.\Delta x,\ k.\Delta y)$ and at time $n.\Delta t$.

1. Heaps

N. S. Heaps (Reference 2) uses the spatial-solution field which is sketched in Figure 2. His explicit-difference equations are written in spherical coordinates, the equivalent Cartesian forms being

$$\frac{u^{n+1}_{j,k} - u^{n}_{j,k}}{\Delta t} + g\,\frac{h^{n}_{j+1,\,k+1} - h^{n}_{j-1,k+1} + h^{n}_{j+1,\,k-1} - h^{n}_{j-1,\,k-1}}{4.\Delta s} = 0, \tag{6}$$

$$\frac{v^{n+1}_{j,k} - v^{n}_{j,k}}{\Delta t} + g\,\frac{h^{n}_{j+1,\,k+1} - h^{n}_{j+1,\,k-1} + h^{n}_{j-1,\,k+1} - h^{n}_{j-1,\,k-1}}{4.\Delta s} = 0, \tag{7}$$

$$\frac{h^{n+1}_{j,k} - h^{n}_{j,k}}{\Delta t} + h_o\,\frac{u^{n+1}_{j+1,\,k+1} - u^{n+1}_{j-1,k+1} + u^{n+1}_{j+1,k-1} - u^{n+1}_{j-1,\,k-1}}{4.\Delta s}$$

$$+ h_o\,\frac{v^{n+1}_{j+1,\,k+1} - v^{n+1}_{j+1,\,k-1} + v^{n+1}_{j-1,\,k+1} - v^{n+1}_{j-1,\,k-1}}{4.\Delta s} = 0. \tag{8}$$

Further, Heaps shows that all eigenvalues of the amplification matrix are within the unit circle on the imaginary plane when

$$gh_o\frac{\Delta t^2}{\Delta s^2}\,(\sin^2\sigma_1 \cdot \Delta s\,\cos^2\sigma_2.\Delta s \;+\; \cos^2\sigma_1.\Delta s\,\sin^2\sigma_2.\Delta s) \;<\; 4, \tag{9}$$

which he reduces to a stability condition of

$$gh_o\,\frac{\Delta t^2}{\Delta s^2} \;<\; 4. \tag{10}$$

2. <u>Reid and Bodine</u>

R. O. Reid and B. R. Bodine (Reference 3) use the spatial solution field sketched in Figure 3. Their explicit-difference equations are

$$\frac{u^{n+1}_{j,k} - u^{n}_{j,k}}{\Delta t} + g\,\frac{h^{n}_{j+1,k} \quad h^{n}_{j-1,k}}{2.\Delta s} = 0, \tag{11}$$

$$\frac{v^{n+1}_{j,k} - v^{n}_{j,k}}{\Delta t} + g\,\frac{h^{n}_{j,\,k+1} - h^{n}_{j,k-1}}{2.\Delta s} = 0, \tag{12}$$

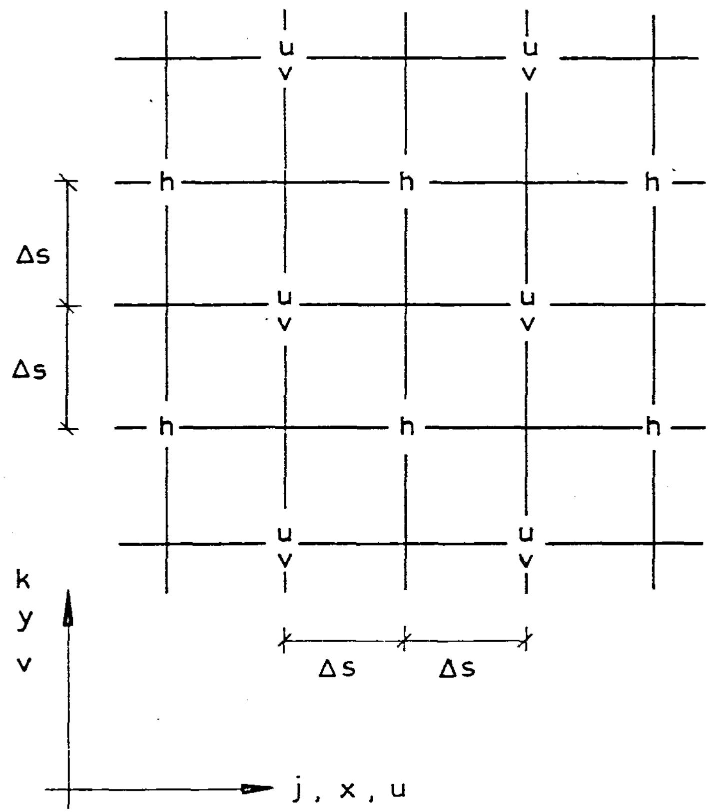


Figure 2. Spatial-solution field of Heaps.

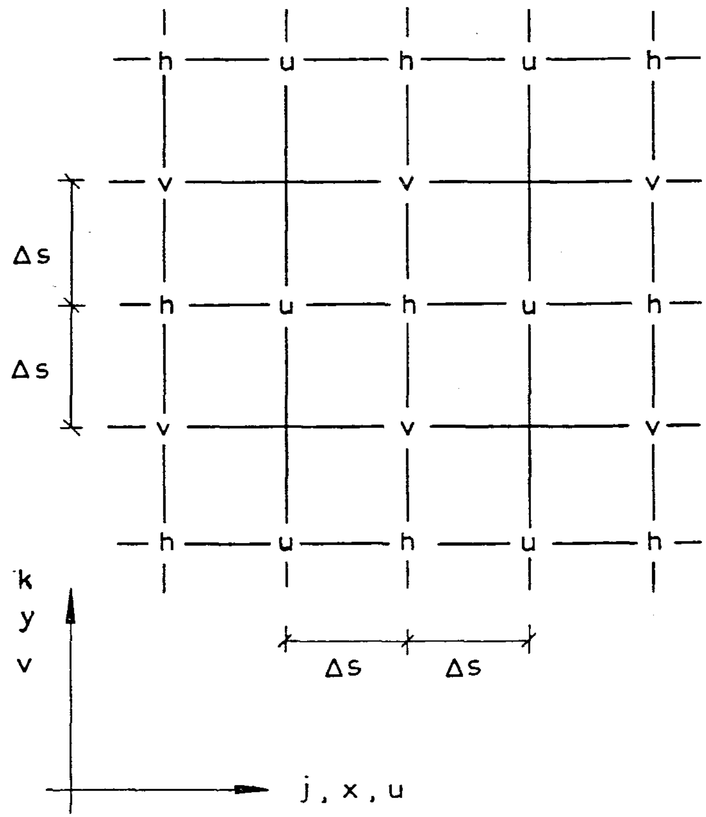


Figure 3. Spatial-solution field of Reid and Bodine; Leendertse and Abbott.

$$\frac{h_{j,k}^{n+1} - h_{j,k}^{n}}{\Delta t} + h_o\left(\frac{u_{j+1,k}^{n+1} - u_{j-1,k}^{n+1}}{2.\Delta s} + \frac{v_{j,k+1}^{n+1} - v_{j,k-1}^{n+1}}{2.\Delta s}\right) = 0, \tag{13}$$

In addition, it can be shown that all eigenvalues of the amplification matrix lie on the unit circle in the imaginary plan when

$$gh_o\frac{\Delta t^2}{\Delta s^2}\ (\sin^2\sigma_1.\Delta s + \sin^2\sigma_2.\Delta s) \le 4. \tag{14}$$

Reid and Bodine have stated the stability requirements as

$$gh_o\frac{\Delta t^2}{\Delta s^2} < 2. \tag{15}$$

3. Leendertse

J. J. Leendertse (Reference 1) uses the spatial grid sketched in Figure 3. His implicit-difference equations, which constitute a so-called "leap-frog" operation, are written as

$$\frac{u_{j,k}^{n+\frac{1}{2}} - u_{j,k}^{n}}{\frac{1}{2}.\Delta t} + g\,\frac{h_{j+1,k}^{n+\frac{1}{2}} - h_{j-1,k}^{n+\frac{1}{2}}}{2.\Delta s} = 0, \tag{16}$$

$$\frac{v_{j,k}^{n+\frac{1}{2}} - v_{j,k}^{n}}{\frac{1}{2}.\Delta t} + g\,\frac{h_{j,k+1}^{n} - h_{j,k-1}^{n}}{2.\Delta s} = 0, \tag{17}$$

$$\frac{h_{j,k}^{n+\frac{1}{2}} - h_{j,k}^{n}}{\frac{1}{2}.\Delta t} + h_o\left(\frac{u_{j+1,k}^{n+\frac{1}{2}} - u_{j-1,k}^{n+\frac{1}{2}}}{2.\Delta s} + \frac{v_{j,k+1}^{n} - v_{j,k-1}^{n}}{2.\Delta s}\right) = 0, \tag{18}$$

$$\frac{u_{j,k}^{n+1} - u_{j,k}^{n+\frac{1}{2}}}{\frac{1}{2}.\Delta t} + g\,\frac{h_{j+1,k}^{n+\frac{1}{2}} - h_{j-1,k}^{n+\frac{1}{2}}}{2.\Delta s} = 0, \tag{19}$$

$$\frac{v_{j,k}^{n+1} - v_{j,k}^{n+\frac{1}{2}}}{\frac{1}{2}.\Delta t} + g\,\frac{h_{j,k+1}^{n+1} - h_{j,k-1}^{n+1}}{2.\Delta s} = 0, \tag{20}$$

$$\frac{h_{j,k}^{n+1} - h_{j,k}^{n+\frac{1}{2}}}{\frac{1}{2}.\Delta t} + h_o\left(\frac{u_{j+1,k}^{n+\frac{1}{2}} - u_{j-1,k}^{n+\frac{1}{2}}}{2.\Delta s} + \frac{v_{j,k+1}^{n+1} - v_{j,k-1}^{n+1}}{2.\Delta s}\right) = 0. \tag{21}$$

Further, Leendertse shows that all eigenvalues of the amplification matrix lie on the unit circle in the imaginary plane, this result being unconditional as the scheme is implicit.

4. Abbott

M. B. Abbott (Reference 4) also proposes the spatial grid sketched in Figure 3. His implicit-difference equations, which also constitute a "leap-frog" operation, are written as

$$\frac{u^{n+1}_{j,k} - u^{n}_{j,k}}{\Delta t} + g\,\frac{h^{n+\frac{1}{2}}_{j+1,k} - h^{n+\frac{1}{2}}_{j-1,k}}{2.\Delta s} = 0, \tag{22}$$

$$\frac{h^{n+\frac{1}{2}}_{j,k} - h^{n}_{j,k}}{\frac{1}{2}.\Delta t} + h_o\Big(\tfrac{1}{2}\,\frac{u^{n+1}_{j+1,k} - u^{n+1}_{j-1,k}}{2.\Delta s} + \tfrac{1}{2}\,\frac{u^{n}_{j+1,k} - u^{n}_{j-1,k}}{2.\Delta s}$$

$$+ \frac{v^{n}_{j,k+1} - v^{n}_{j,k-1}}{2.\Delta s}\Big) = 0, \tag{23}$$

$$\frac{v^{n+1}_{j,k} - v^{n}_{j,k}}{\Delta t} + g\,\Big(\tfrac{1}{2}\,\frac{h^{n+1}_{j,k+1} - h^{n+1}_{j,k-1}}{2.\Delta s} + \tfrac{1}{2}\,\frac{h^{n}_{j,k+1} - h^{n}_{j,k-1}}{2.\Delta s}\Big) = 0, \tag{24}$$

$$\frac{h^{n+1}_{j,k} - h^{n+\frac{1}{2}}_{j,k}}{\frac{1}{2}\,\Delta t} + h_o\Big(\tfrac{1}{2}\,\frac{u^{n+1}_{j+1,k} - u^{n+1}_{j-1,k}}{2.\Delta s} + \tfrac{1}{2}\,\frac{u^{n}_{j+1,k} - u^{n}_{j-1,k}}{2.\Delta s}$$

$$+ \frac{v^{n+1}_{j,k+1} - v^{n+1}_{j,k-1}}{2.\Delta s}\Big) = 0. \tag{25}$$

Further, it can be shown (Reference 4) that all eigenvalues of the amplification matrix lie on or within the unit circle in the imaginary plane. Again, this is an unconditional result as the scheme is implicit.

Section V. COMPUTED WAVES

The numerical solution of the partial differential equations (i.e. the solution of the finite-difference equations) will be called the *computed wave*.

For purposes of comparison it will be assumed that a square finite-difference grid is used in each case, i.e.

$$\Delta x = \Delta y = \Delta s.$$

Now, for each of the four sets of difference equations described in Section IV, a Fourier series solution (for linear operators) will be assumed of form

$$\overline{U} = \overline{U}^* e^{i(\beta' n.\Delta t + \sigma_1 j.\Delta s + \sigma_2 k.\Delta s)} \tag{26}$$

where β' is a complex number of dimension ($time^{-1}$) and such that $Re(\beta')$ is the computed wave frequency and $Im(\beta')$ an indirect measure of the amplitude deformation (see Section VI, equation (52)). The remaining symbols are as previously defined.

Further, put

$$\emptyset = e^{i\beta'\Delta t} \tag{27}$$

for computational convenience.

1. Heaps

Substituting equation (26) into equations (6) to (8) and putting

$$A_1 = \sin(\sigma_1.\Delta s + \sigma_2.\Delta s)$$

$$\text{and} \quad A_2 = \sin(\sigma_1.\Delta s - \sigma_2.\Delta s)$$

yields the matrix equation,

$$\begin{bmatrix} \emptyset - 1 & 0 & i\frac{g.\Delta t}{2.\Delta s}(A_1 + A_2) \\ 0 & \emptyset - 1 & i\frac{g.\Delta t}{2.\Delta s}(A_1 - A_2) \\ i\frac{h_o\Delta t}{2.\Delta s}(A_1 + A_2) & i\frac{h_o\Delta t}{2.\Delta s}(A_1 - A_2) & \emptyset - 1 \end{bmatrix} . \begin{bmatrix} u^* \\ v^* \\ h^* \end{bmatrix} = 0, \tag{28}$$

The solution of which gives the following roots

$$\emptyset_1 = 1$$

$$\text{i.e.} \quad i\beta_1'\Delta t = 0 \quad \text{or} \quad \beta_1' = 0, \tag{29}$$

and

$$\emptyset_{2,3} = 1 \pm i\sqrt{\frac{\Delta t^2}{2.\Delta s^2} gh_o(A_1{}^2 + A_2{}^2)}.$$

The latter roots reduce to

$$\beta'_{2,3}.\Delta t = \pm \tan^{-1}\sqrt{\frac{\Delta t^2}{2.\Delta s^2} gh_o (A_1^2 + A_2^2)}$$

$$- \; i \ln \sqrt{1 + \frac{\Delta t^2}{2.\Delta s^2} gh_o (A_1^2 + A_2^2)} \;. \qquad (30)$$

2. Reid and Bodine

Substituting equation (26) into equations (11) to (13) yields the matrix equation,

$$\begin{bmatrix} \emptyset-1 & 0 & i\, g \frac{\Delta t}{\Delta s} \sin\sigma_1 \Delta s \\ 0 & \emptyset-1 & i\, g \frac{\Delta t}{\Delta s} \sin\sigma_2 \Delta s \\ i\, \emptyset\, h_o \frac{\Delta t}{\Delta s} \sin\sigma_1 \Delta s & i\, \emptyset\, h_o \frac{\Delta t}{\Delta s} \sin\sigma_2 \Delta s & \emptyset-1 \end{bmatrix} . \begin{bmatrix} u^* \\ v^* \\ h^* \end{bmatrix} = 0, \qquad (31)$$

the solution of which gives the following roots

$$\emptyset_1 = 1$$

$$\text{i.e.} \quad i\beta'_1 \Delta t = 0 \quad \text{or} \quad \beta'_1 = 0, \qquad (32)$$

and

$$\emptyset_{2,3} = \frac{-b_R \pm \sqrt{b_R^2 - 4}}{2} \qquad (33)$$

$$\text{where } b_R = p_1^2 + p_2^2 - 2, \qquad (34)$$

$$\left.\begin{aligned} p_1 &= \sqrt{gh_o}\, \frac{\Delta t}{\Delta s} \sin\sigma_1 \Delta s, \\ \text{and} \quad p_2 &= \sqrt{gh_o}\, \frac{\Delta t}{\Delta s} \sin\sigma_2 \Delta s \end{aligned}\right\} \qquad (35)$$

Two cases of equation (33) will be considered:

Case 1 $\quad b_R^2 - 4 > 0.$

In this case, equation (33) reduces to

$$\beta'_{2,3}.\Delta t = -i \ln \left| -\frac{b_R}{2} \pm \sqrt{\left(\frac{b_R}{2}\right)^2 - 1} \right| \tag{36}$$

It is noted here that the condition $b_R^2 - 4 > 0$ can be shown to be equivalent to $p_1^2 + p_2^2 > 4$,

$$\text{i.e.} \quad gh_o \frac{\Delta t^2}{\Delta s^2} (\sin^2\sigma_1\Delta s + \sin^2\sigma_2\Delta s) > 4,$$

which (refer to equation (14)) inplies instability.

<u>Case 2</u> $b_R^2 - 4 \leq 0$.

In this case, equation (33) reduces to

$$\beta'_{2,3}.\Delta t = \pm \sin^{-1} \sqrt{1 - \left(\frac{b_R}{2}\right)^2}. \tag{37}$$

3. <u>Leendertse</u>

It can be shown (Reference 1) that, in this case,

$$\phi_1 = 1$$

$$\text{i.e.} \quad i\beta'_1.\Delta t = 0 \text{ or } \beta'_1 = 0, \tag{38}$$

and

$$\phi_{2,3} = \frac{-b_L \pm \sqrt{b_L^2 - 4}}{2} \tag{39}$$

$$\text{where} \quad b_L = \frac{2\ (A_L - 1)}{A_L + 1}, \tag{40}$$

$$A_L = \frac{p_1^2}{4} + \frac{p_1^2 p_2^2}{16} + \frac{p_2^2}{4}, \tag{41}$$

and p_1, p_2 are defined by equations (35).

Further, it can be shown that $b_L^2 - 4$ is not more than zero. Now, by analogy with Case 2 of Section V, paragraph 2, but noting the different definition of b, equation (39) reduces to

$$\beta'_{2,3}.\Delta t = \pm \sin^{-1} \sqrt{1 - (\frac{b_L}{2})^2} . \qquad (42)$$

4. <u>Abbott</u>

It can be shown (Reference 4) that substitution of equation (26) into the reduced two-level form of equations (22) to (25), represented by equations (A.1) to (A.3) of the Appendix, yields the matrix equation

$$\begin{bmatrix} \emptyset-1 & -(\emptyset-1)\frac{gh_o\Delta t^2}{4\Delta s^2}\sin\sigma_1\Delta s.\sin\sigma_2\Delta s & i(\emptyset+1)\frac{g\Delta t}{2\Delta s}\sin\sigma_1\Delta s \\ 0 & \emptyset-1 & i(\emptyset+1)\frac{g\Delta t}{2\Delta s}\sin\sigma_2\Delta s \\ i(\emptyset+1)\frac{h_o\Delta t}{2.\Delta s}\sin\sigma_1\Delta s & i(\emptyset+1)\frac{h_o\Delta t}{2.\Delta s}\sin\sigma_2\Delta s & \emptyset-1 \end{bmatrix} . \begin{bmatrix} u^* \\ v^* \\ h^* \end{bmatrix} = 0, \qquad (43)$$

the solution of which gives the following roots

$$\emptyset_1 = 1$$

$$\text{i.e.} \quad i\beta'_1\Delta t = 0 \text{ or } \beta'_1 = 0, \qquad (44)$$

and

$$\emptyset_{2,3} = \frac{-b_A \pm \sqrt{b_A^2 - 4}}{2}, \qquad (45)$$

where

$$b_A = \frac{2(A_A - 1)}{A_A + 1}, \qquad (46)$$

$$A_A = \frac{p_1^2}{4} + \frac{p_1^2 p_2^2}{16} + \frac{p_2^2}{4}, \qquad (47)$$

and p_1, p_2 are defined by equations (35).

It is apparent that the computed wave of the scheme proposed by Abbott is identical to that of Leendertse. Hence equation (45) reduces to

$$\beta'_{2,3}\Delta t = \pm \sin^{-1}\sqrt{1 - \left(\frac{b_A}{2}\right)^2} . \qquad (48)$$

Section VI. PROPAGATION FACTORS

Following Leendertse (Reference 1), a complex propagation factor, T, is defined as the ratio of the computed wave (solution of difference equations) to the real wave (solution of differential equations) after a time interval in which the real or physical wave propagates over its wave length, L.

$$\text{i.e.} \quad T = \frac{e^{i(\beta'.t + \sigma.s)}}{e^{i(\beta.t + \sigma.s)}}$$

$$\text{for} \quad t = \frac{2\pi}{\beta} \quad \text{and} \quad s = \frac{2\pi}{\sigma} ,$$

$$\text{which becomes} \quad T = e^{i2\pi\left(\frac{\beta'}{\beta} - 1\right)} \qquad (49)$$

It should be noted that a propagation factor exists for each wave in the system. In the system under consideration there are three waves. The first wave represents the steady-state flow of the whole field (Reference 1) and for all four cases considered here it has been shown (by equations (29), (32), (38) and (44)) that

$$\beta'_1 \equiv 0,$$

and hence it is apparent that also

$$T_1 \equiv 1. \qquad (50)$$

The second and third waves represent, respectively, the positive and negative characteristics of the system, and in all four cases considered here, it is apparent that when the scheme is stable,

$$T_2 \equiv T_3 \qquad (51)$$

and this is the T that will be subsequently considered.

It is convenient to consider separately the amplitude error of the propagated wave, represented by the modulus of T, and the phase error of the propagated wave, represented by the argument of T.

It can be shown that

$$|T| = \left| e^{-\mathrm{Im}(\beta'.\Delta t)} \right|^{\nu} \tag{52}$$

where ν is the number of time steps necessary for propagation of a real wave over its wave length,

$$\text{i.e.} \quad \nu = \frac{\text{Period}}{\Delta t}$$

$$\nu = \frac{2\pi}{\beta} \cdot \frac{1}{\Delta t}$$

$$\nu = \frac{2\pi}{\sigma.\Delta s} / (\sqrt{gh_o} \cdot \frac{\Delta t}{\Delta s}), \tag{53}$$

and that

$$\arg T = 2\pi \left[\frac{\mathrm{Re}(\beta'.\Delta t)}{\beta.\Delta t} - 1\right] \tag{54}$$

1. <u>Heaps</u>

In this case, equation (52) becomes

$$|T| = [1 + \frac{gh_o \Delta t^2}{2.\Delta s^2} (A_1^2 + A_2^2)]^{\frac{1}{2}\nu} \tag{55}$$

and equation (54) becomes

$$\arg T = 2\pi \left\{ \frac{\tan^{-1}\sqrt{\frac{gh_o\Delta t^2}{2.\Delta s^2} (A_1^2 + A_2^2)}}{\sqrt{gh_o} \frac{\Delta t}{\Delta s} .\sigma.\Delta s} - 1 \right\}. \tag{56}$$

2. <u>Reid and Bodine</u>

Again the two cases of Section V, paragraph 2 must be considered.

<u>Case 1</u> $b_R^2 - 4 > 0.$

In this case, equation (52) becomes

$$|T_{2,3}| = \left| -\frac{b_R}{2} \pm \sqrt{(\frac{b_R}{2})^2 - 1} \right|^{\nu}, \tag{57}$$

this being the exception to the generality of equation (51). The negative wave, having the larger magnitude, has been considered in the subsequent numerical comparisons.

Equation (54) becomes

$$\arg T = -2\pi \tag{58}$$

Case 2 $\quad b_R^2 - 4 \leq 0.$

In this case, equation (52) becomes

$$|T| = 1, \tag{59}$$

and equation (54) becomes

$$\arg T = 2\pi \left[\frac{\sin^{-1}\sqrt{1 - (\tfrac{1}{2}b_R)^2}}{\sqrt{gh_o}\,\frac{\Delta t}{\Delta s}\,.\sigma\Delta s} - 1 \right]. \tag{60}$$

3. Leendertse

In this case, equation (52) becomes

$$|T| = 1, \tag{61}$$

and equation (54) becomes

$$\arg T = 2\pi \left[\frac{\sin^{-1}\sqrt{1 - (\tfrac{1}{2}b_L)^2}}{\sqrt{gh_o}\,\frac{\Delta t}{\Delta s}\,.\sigma\Delta s} - 1 \right]. \tag{62}$$

4. Abbott

In this case, equation (52) becomes

$$|T| = 1, \tag{63}$$

and equation (54) becomes

$$\arg T = 2\pi \left[\frac{\sin^{-1}\sqrt{1 - (\tfrac{1}{2}b_A)^2}}{\sqrt{gh_o}\,\frac{\Delta t}{\Delta s}\,.\sigma\Delta s} - 1 \right]. \tag{64}$$

5. Numerical Comparison

Two dimensionless parameters have been used to facilitate the numerical comparison. These are a dimensionless celerity, $\sqrt{gh_o} \frac{\Delta t}{\Delta s}$, and a dimensionless grid size, $\frac{L}{\Delta s}$. This latter parameter, which represents the number of finite difference grid steps per wavelength, L being the wavelength, can also be written as

$$\frac{L}{\Delta s} = \frac{2\pi}{\sigma . \Delta s}$$

where $\sigma . \Delta s$ is a dimensionless wave number.

The numerical comparisons were performed by an Algol-60 computer program written for the Telefunken TR-4 machine of the Technological University, Delft, The Netherlands. A graphical summary of the results is presented as Figures 4 to 12, where the behavior of the respective propagation factors is shown for dimensionless celerities of 0.1, 1 and 5 with γ values of 45, 22.5 and 0 degrees. Figures 4 to 12 begin on page 18.

Section VII. COMMENTS

There are three aspects of this study worthy of comment. The first, and largely unexpected, aspect is the usefulness and slight advantage of the primitive explicit scheme of Reid and Bodine, while this scheme remains stable. This is demonstrated by Figures 4 to 9 and especially by Figures 7, 8, and 9, where the dimensionless celerity is unity. This result is in some regard analogous to the surprising success of linearizing (and consequently simplifying) assumptions that have such widespread application in analytical treatment of differential equations in applied mechanics.

The second aspect is the positive manner in which the instability of the explicit schemes is manifested. This is demonstrated by the propagation-factor amplitude behavior in Figures 10 to 12. The scheme of Reid and Bodine, in particular, results in rather extreme amplitude distortions when unstable, but immediately reverts to the no-amplitude-distortion situation when the stability conditions are satisfied.

In contrast, the explicit scheme of Heaps exhibits a type of creeping instability within the range of the comparison; the deformation of the wave becomes more severe with increasing dimensionless celerity. However, further comment on this scheme would be of little value as it was not proposed as a direct alternative to the other three. Firstly, the spatial solution field is different (Figure 2) and secondly, Heaps proposed his scheme and solved his difference equations in a spherical-coordinate system, whereas an equivalent Cartesian form has been considered for the purposes of the present study.

The third aspect is the overall suitability of the implicit schemes proposed by Leendertse and Abbott. The satisfactory wave-deformation characteristics of these schemes throughout the range of the comparison, coupled with their unconditional stability, establishes their overall utility.

As a general conclusion, the present study has demonstrated the usefulness of the propagation-factor behavior as a measure of the suitability of difference schemes.

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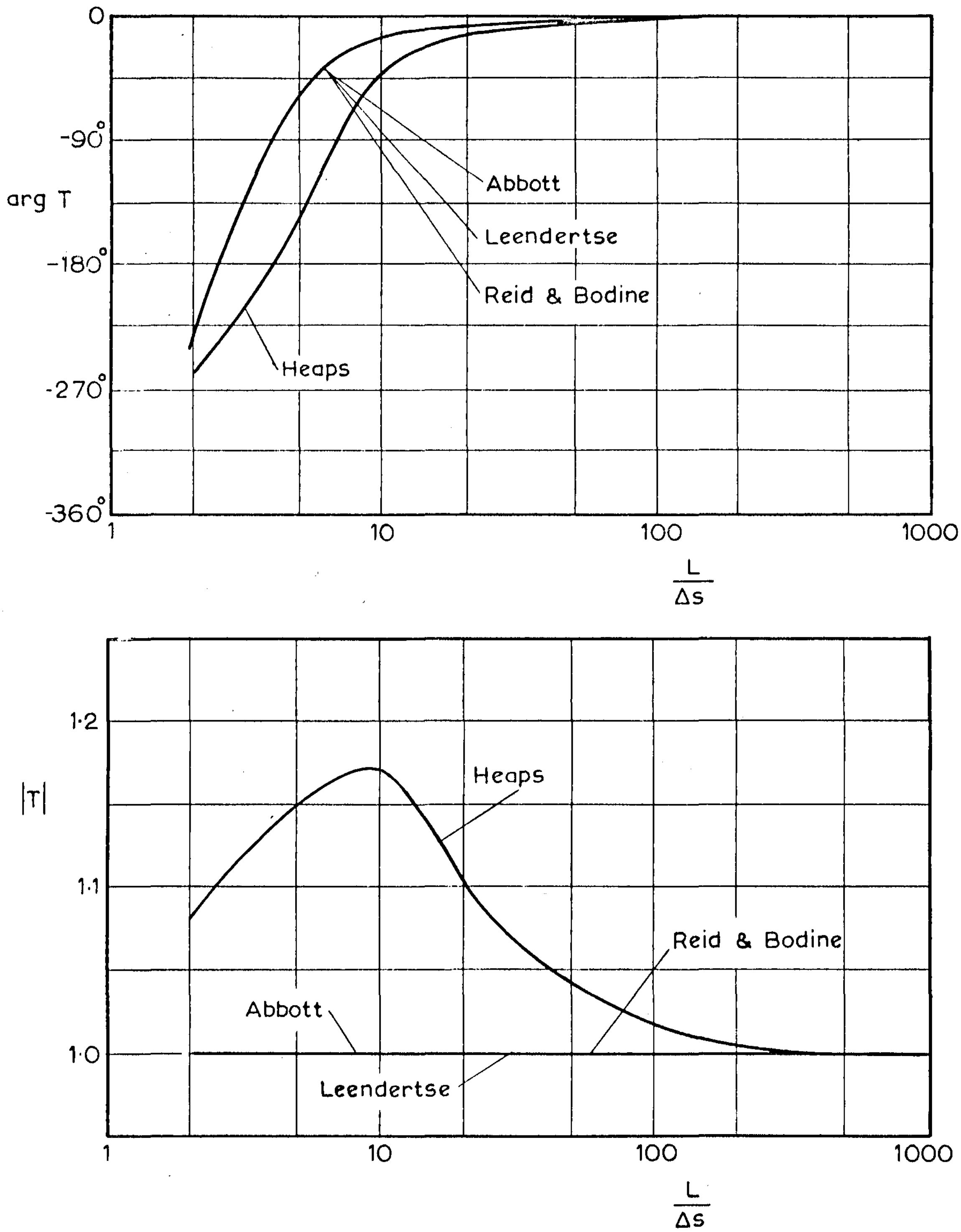
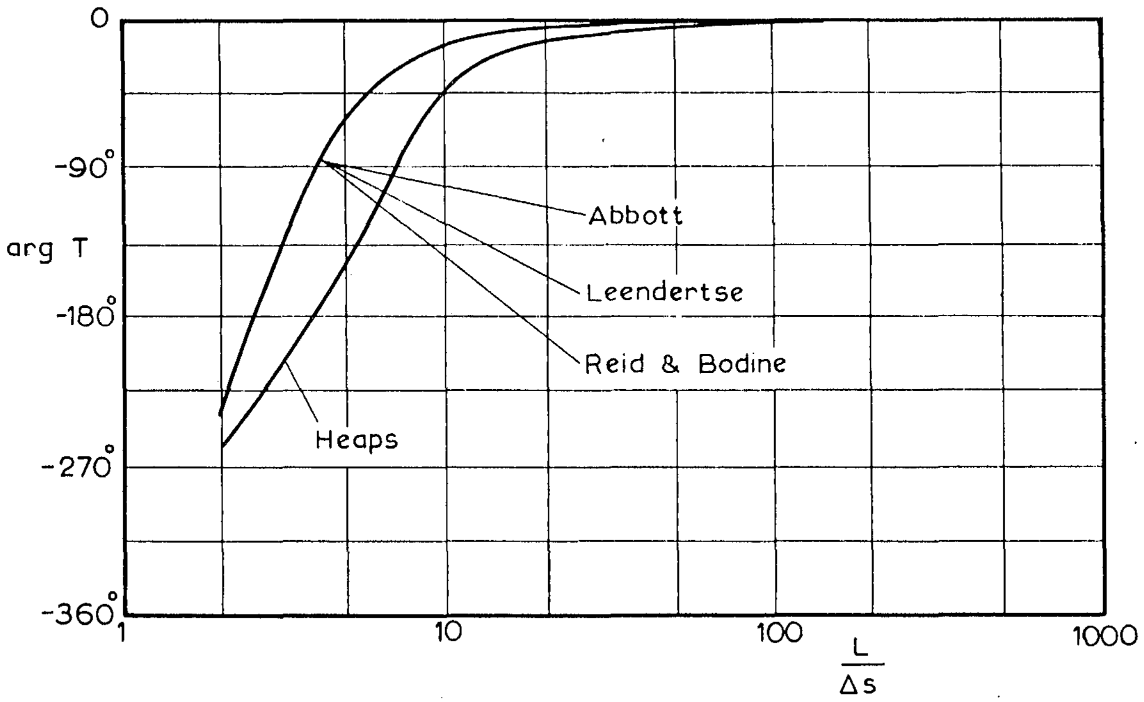


Figure 4. Propagation Factors for

$\sqrt{gh_o}\,\frac{\Delta t}{\Delta s} = 0.1$ & $\gamma = 45°$



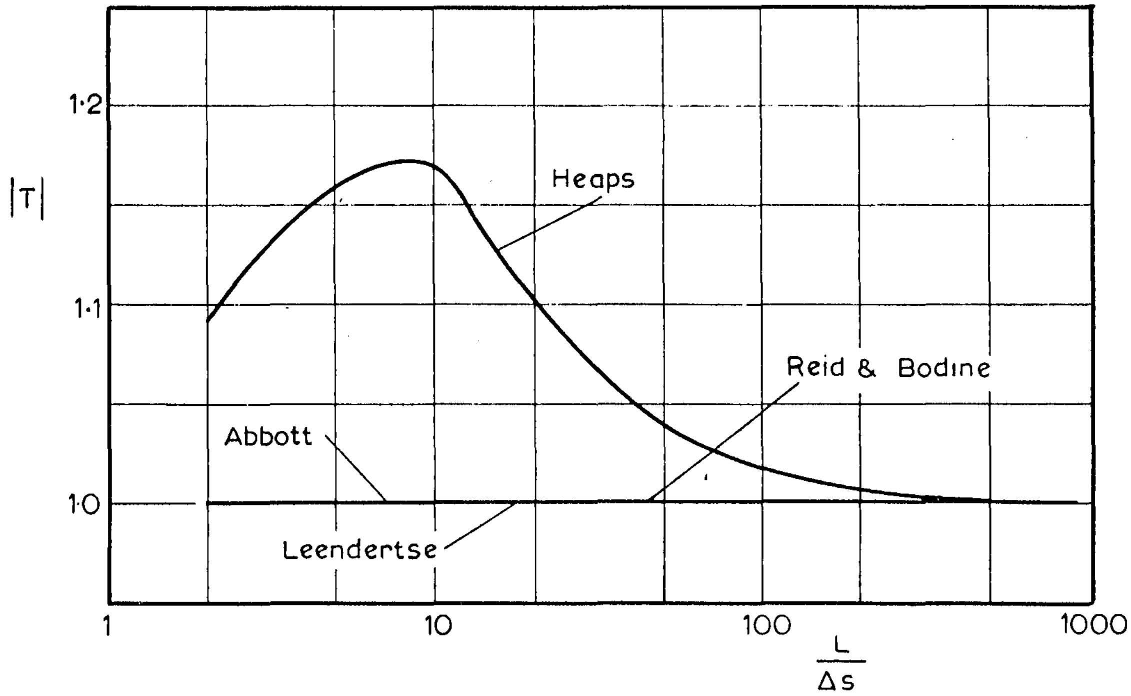


Figure 5. Propagation Factors for

$\sqrt{gh_o}\,\frac{\Delta t}{\Delta s} = 0.1$ & $\gamma = 22.5°$

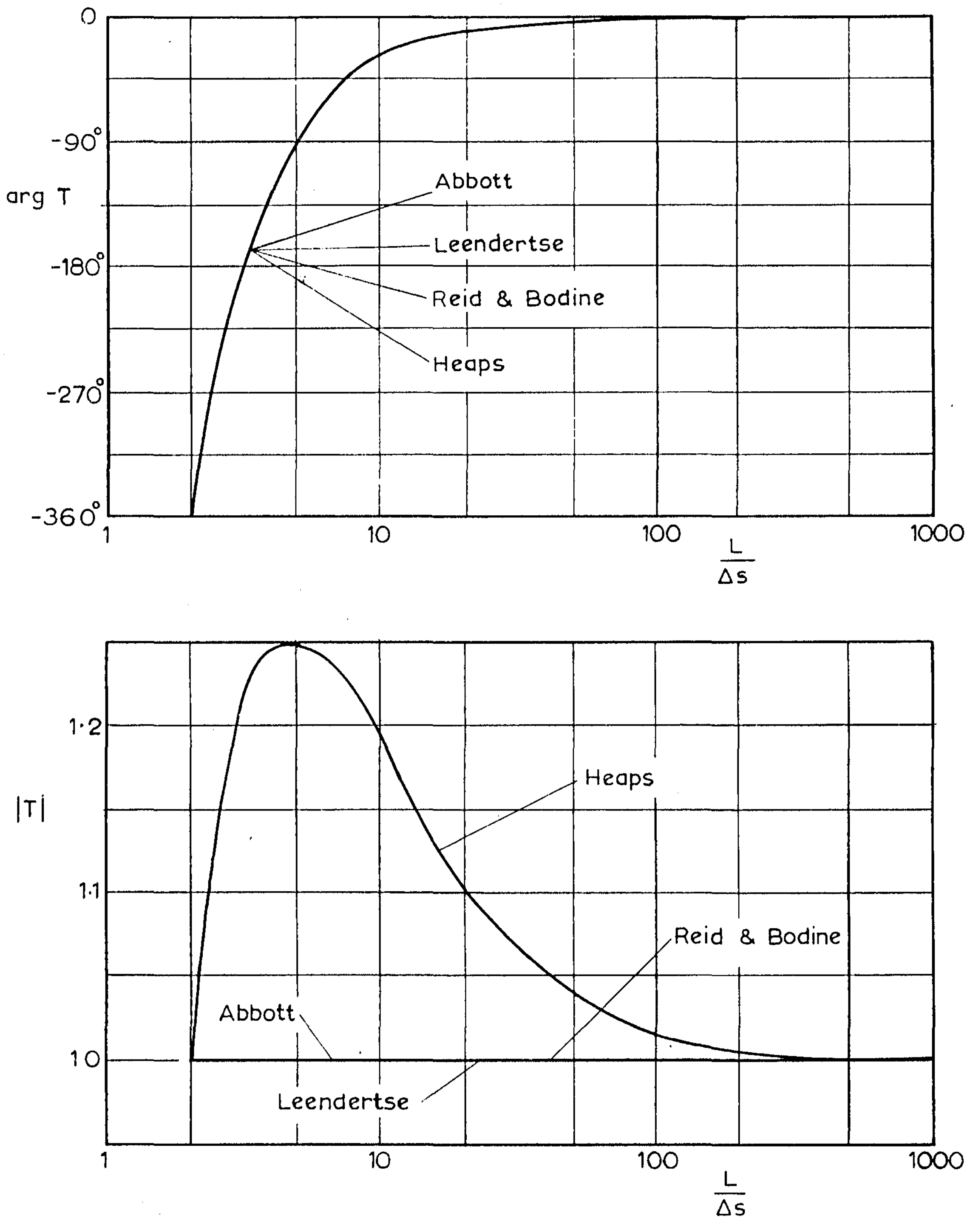


Figure 6. Propagation Factors for

$\sqrt{gh_o}\,\frac{\Delta t}{\Delta s} = 0.1$ & $\gamma = 0°$

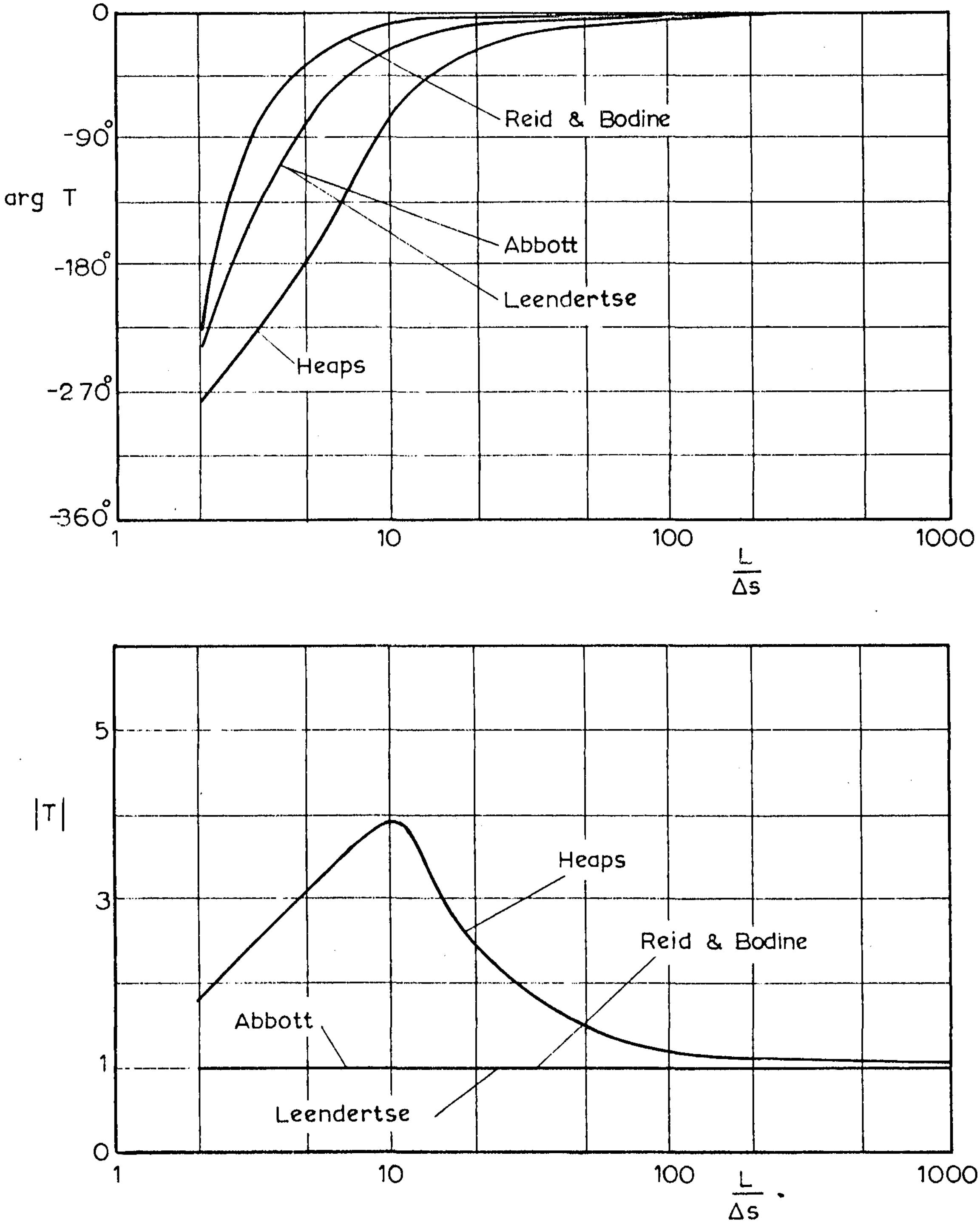


Figure 7. Propagation Factors for

$\sqrt{gh_o}\,\frac{\Delta t}{\Delta s} = 1$ & $\gamma = 45°$

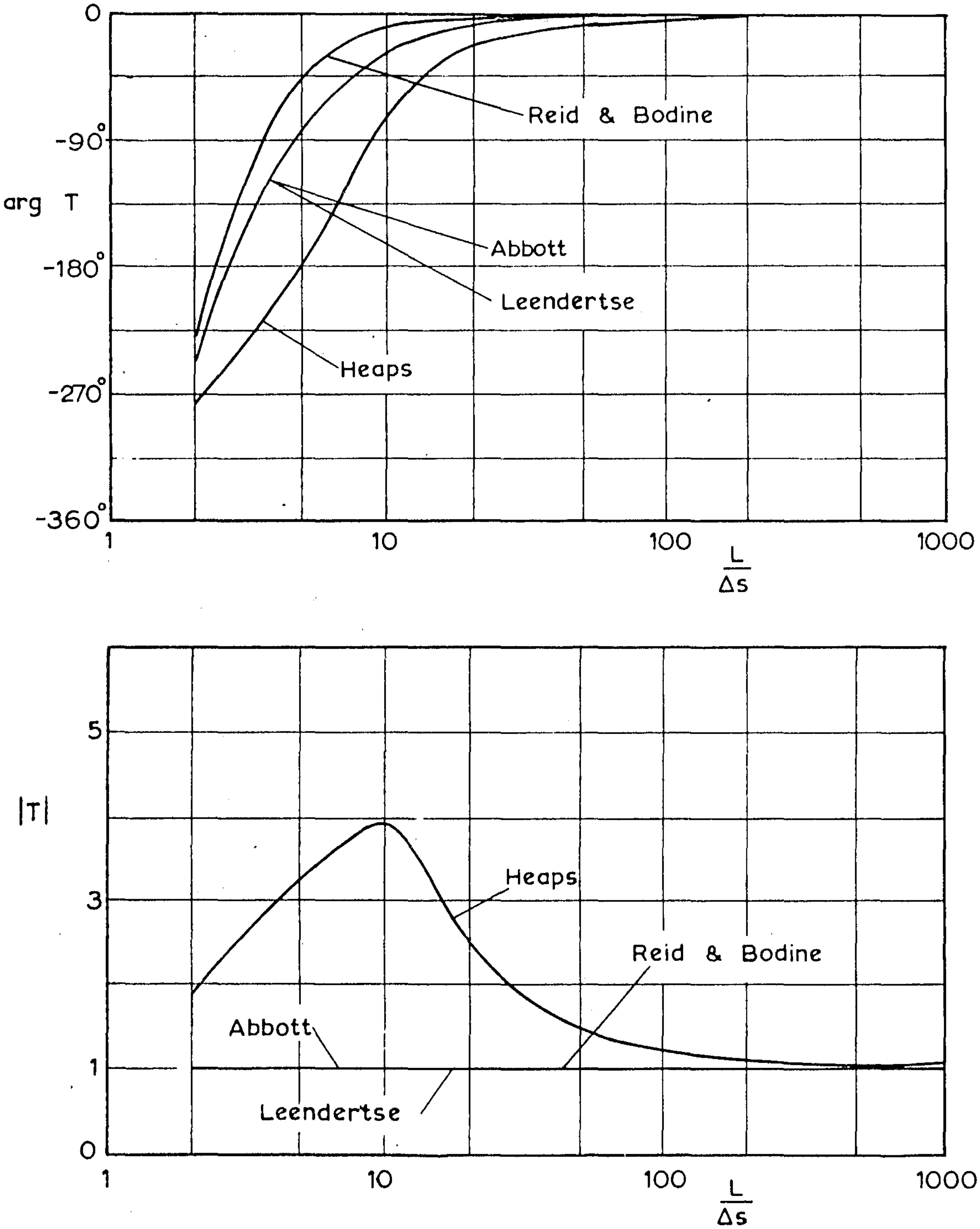
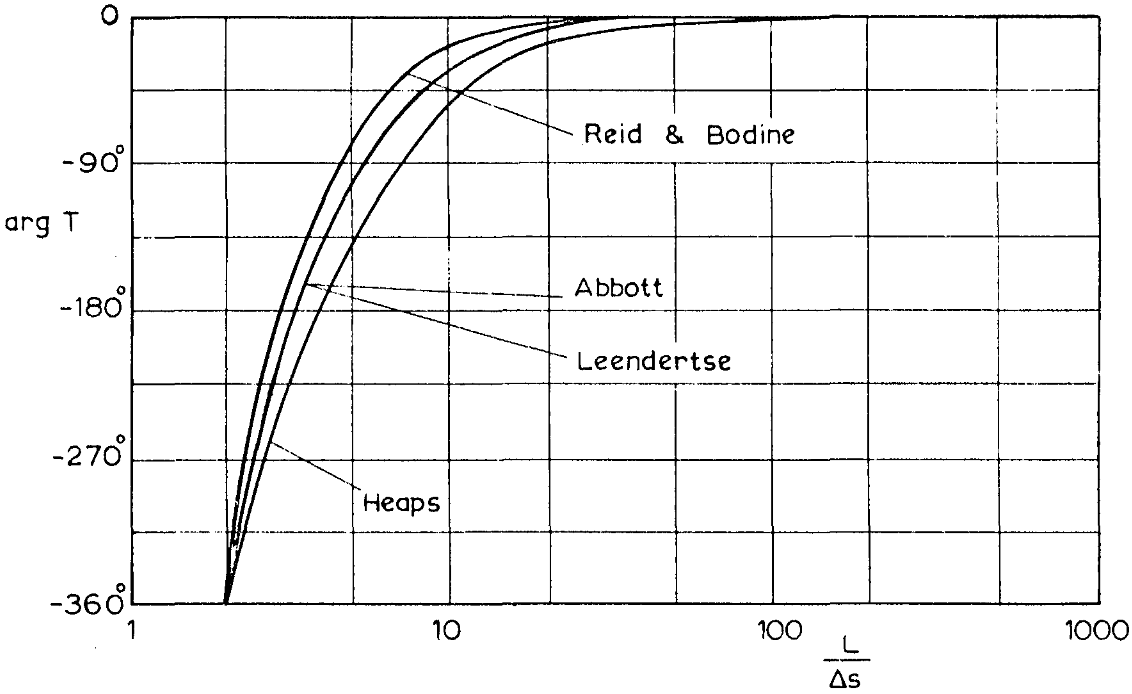


Figure 8. Propagation Factors for

$\sqrt{gh_o}\ \frac{\Delta t}{\Delta s} = 1$ & $\gamma = 22.5°$



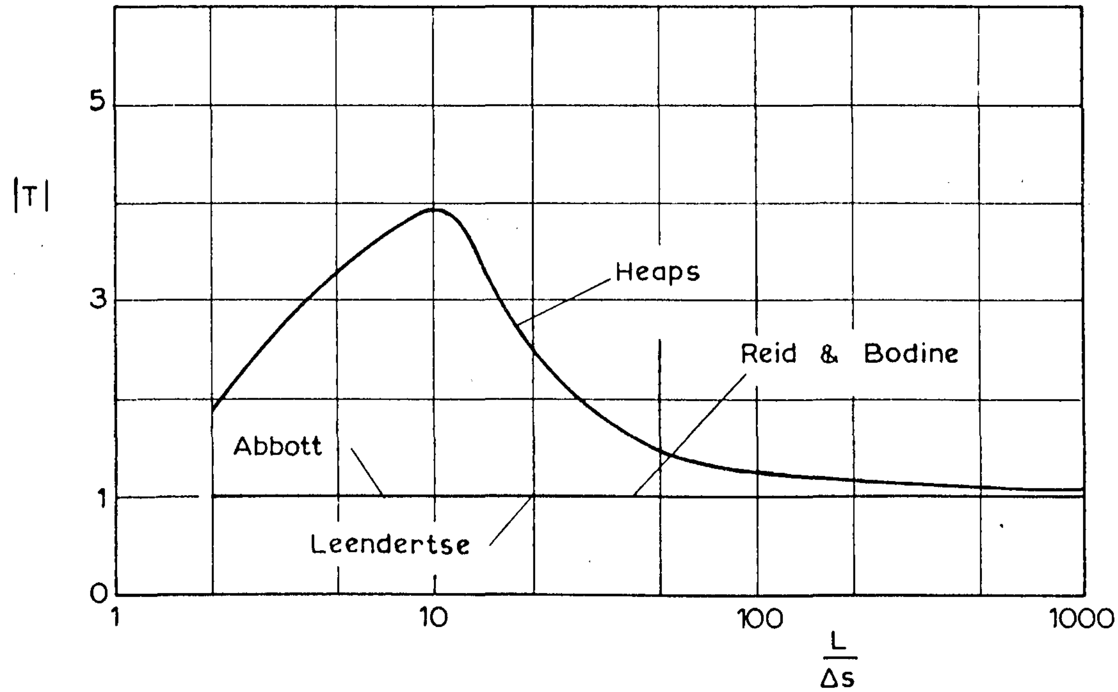


Figure 9. Propagation Factors for

$\sqrt{gh_o}\,\frac{\Delta t}{\Delta s} = 1$ & $\gamma = 0°$

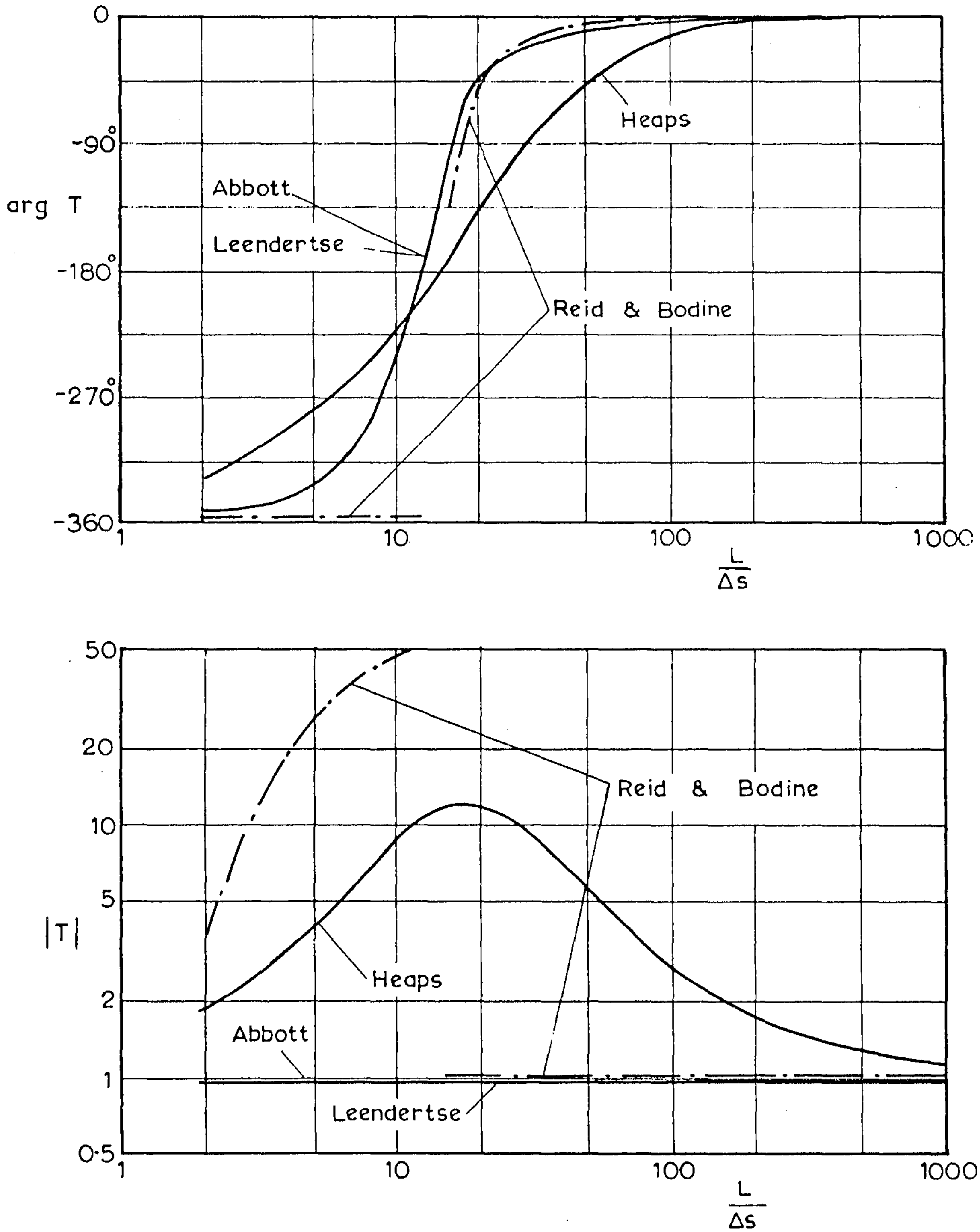


Figure 10. Propagation Factors for

$$\sqrt{gh_o}\,\frac{\Delta t}{\Delta s} = 5 \quad \& \quad \gamma = 45^\circ$$

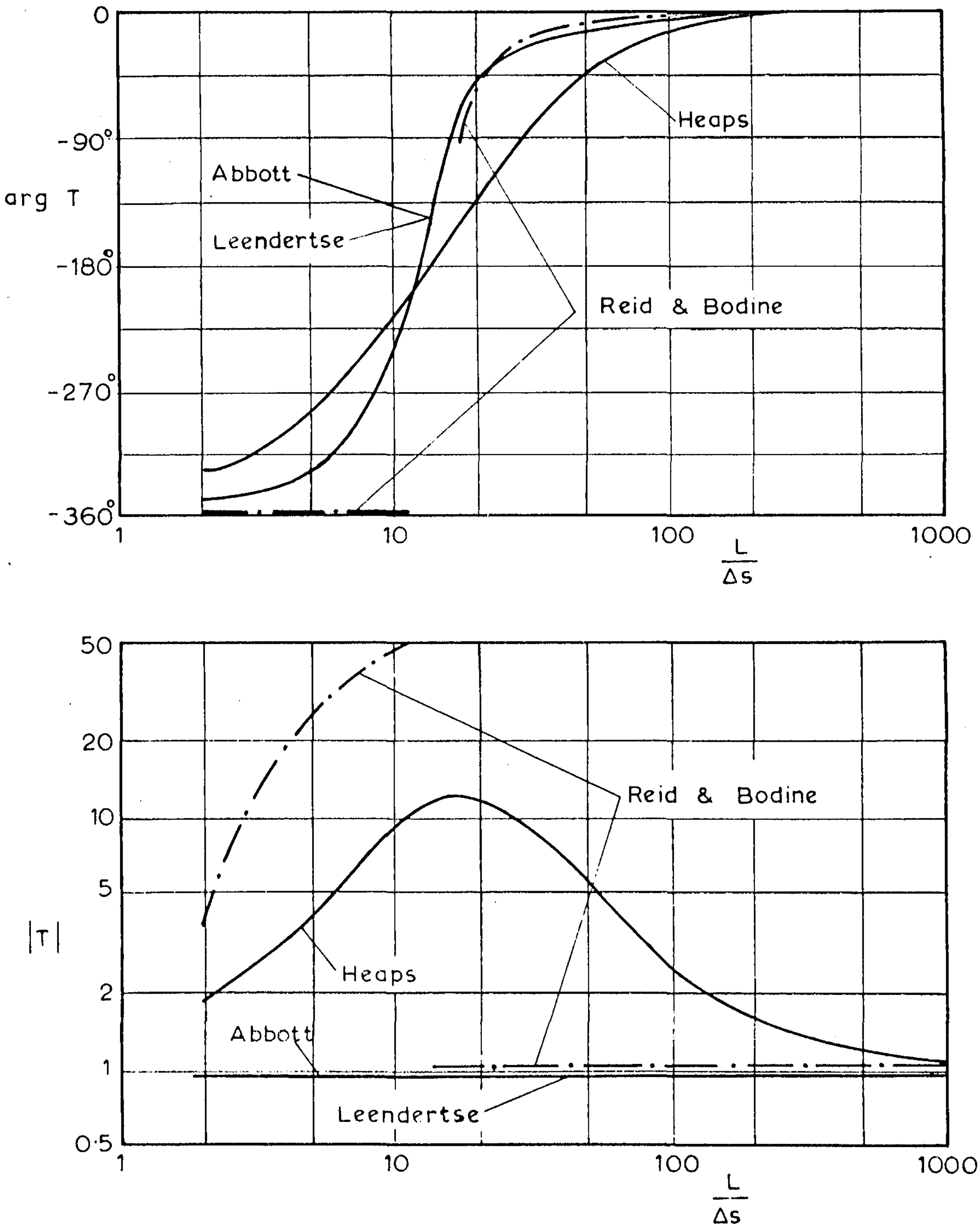


Figure 11. Propagation Factors for

$\sqrt{gh_o}\,\frac{\Delta t}{\Delta s} = 5$ & $\gamma = 22.5°$

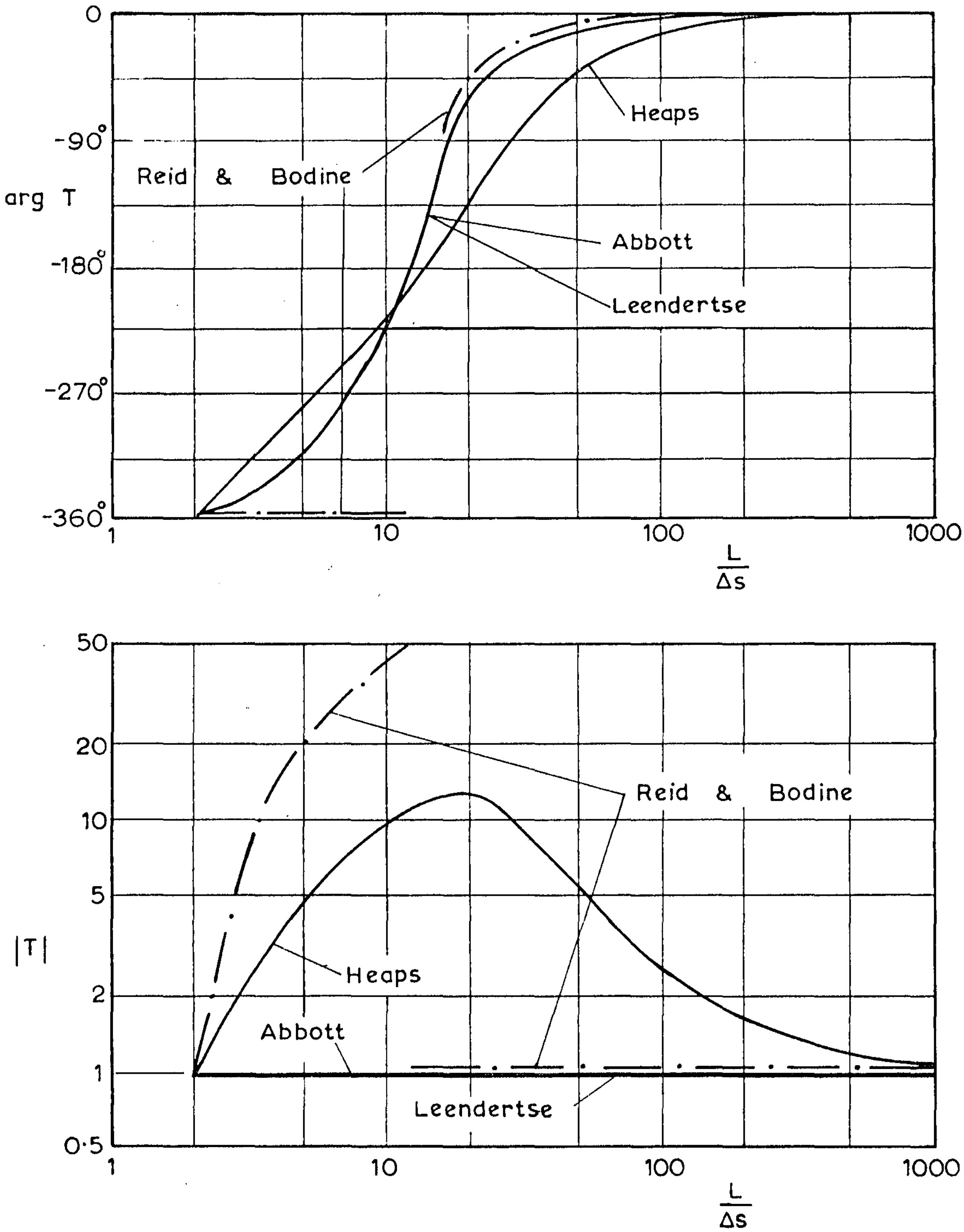


Figure 12. Propagation Factors for

$$\sqrt{gh_o}\,\frac{\Delta t}{\Delta s} = 5 \;\&\; \gamma = 0°$$

APPENDIX

ASPECTS OF AN ANALYSIS OF A DIFFERENCE SCHEME PROPOSED BY M. B. ABBOTT

The difference scheme detailed by equations (22) to (25) is written in three time levels, namely $n.\Delta t$, $(n + \frac{1}{2})\Delta t$ and $(n + 1)\Delta t$. For mathematical analysis of the scheme, it must be reduced to an equivalent two-level scheme.

This is achieved by elimination of the $(n + \frac{1}{2})\Delta t$ time level between the four equations, yielding the following three equations, which thus represent an equivalent two-level scheme:

$$u^{n+1}_{j,k} - u^{n}_{j,k} + \frac{g.\Delta t}{4.\Delta s}\left[(h^{n+1}_{j+1,k} - h^{n+1}_{j-1,k}) + (h^{n}_{j+1,k} - h^{n}_{j-1,k})\right]$$

$$+ \frac{gh_o\Delta t^2}{16.\Delta s^2}\left[(v^{n+1}_{j+1,k+1} - v^{n+1}_{j+1,k-1}) - (v^{n}_{j+1,k+1} - v^{n}_{j+1,k-1})\right.$$

$$\left. - (v^{n+1}_{j-1,k+1} - v^{n+1}_{j-1,k-1}) + (v^{n}_{j-1,k+1} - v^{n}_{j-1,k-1})\right] = 0, \tag{A.1}$$

$$v^{n+1}_{j,k} - v^{n}_{j,k} + \frac{g.\Delta t}{4.\Delta s}\left[(h^{n+1}_{j,k+1} - h^{n+1}_{j,k-1}) + (h^{n}_{j,k+1} - h^{n}_{j,k-1})\right] = 0, \tag{A.2}$$

$$h^{n+1}_{j,k} - h^{n}_{j,k} + \frac{h_o\Delta t}{4.\Delta s}\left[(u^{n+1}_{j+1,k} - u^{n+1}_{j-1,k}) + (u^{n}_{j+1,k} - u^{n}_{j-1,k})\right.$$

$$\left. + (v^{n+1}_{j,k+1} - v^{n+1}_{j,k-1}) + (v^{n}_{j,k+1} - v^{n}_{j,k-1})\right] = 0. \tag{A.3}$$

Stability in the large (i.e. remote from the boundaries) is investigated by assuming a Fourier series solution of equations (A.1) to (A.3) of form

$$\overline{U} = \sum_m \overline{U}^*_m e^{i\sigma_m s} \tag{A.4}$$

By using the validity of superposition and assuming a square difference grid of size Δs, the substitution yields an expression of form

$$\overline{U}^{n+1} = G\,\overline{U}^{n}, \tag{A.5}$$

where G is the amplification matrix, whose elements g_{ij} are listed below:

$$g_{11} = \frac{\left(1 - \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2\sigma_1 \Delta s\right)}{\left(1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2\sigma_1 \Delta s\right)}$$

$$g_{12} = \frac{-2 \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin\sigma_1 \Delta s. \quad \sin\sigma_2 \Delta s}{\left(1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2\sigma_1 \Delta s\right)}$$

$$g_{13} = \frac{-i \; 2 \frac{g.\Delta t}{2.\Delta s} \sin\sigma_1 \Delta s}{\left(1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2\sigma_1 \Delta s\right)}$$

$$g_{21} = \frac{-2 \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin\sigma_1 \Delta s \; \sin\sigma_2 \Delta s}{\left(1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2\sigma_1 \Delta s\right)\left(1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2\sigma_2 \Delta s\right)}$$

$$g_{22} = \frac{\frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2\sigma_1 \Delta s \left(1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2\sigma_2 \Delta s\right) + \left(1 - \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2\sigma_2 \Delta s\right)}{\left(1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2\sigma_1 \Delta s\right) \left(1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2\sigma_2 \Delta s\right)}$$

$$g_{23} = \frac{-i \; 2 \frac{g.\Delta t}{2.\Delta s} \sin\sigma_2 \Delta s}{\left(1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2\sigma_1 \Delta s\right) \left(1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2\sigma_2 \Delta s\right)}$$

$$g_{31} = \frac{-i \;\; 2 \;\; \frac{h_o \Delta t}{2.\Delta s} \sin\sigma_1 \Delta s}{(1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2 \sigma_1 \Delta s) \;\; (1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2 \sigma_2 \Delta s)}$$

$$g_{32} = \frac{-i \;\; 2 \; \frac{h_o \Delta t}{2.\Delta s} \sin\sigma_2 \Delta s}{(1 \;\; + \;\; \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2 \;{}_1\; s) \;\; (1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2 \sigma_2 \Delta s)}$$

$$g_{33} = \frac{(1 - \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2 \sigma_1 \Delta s) - \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2 \sigma_2 \Delta s \;\; (1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2 \sigma_1 \Delta s)}{(1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2 \sigma_1 \Delta s) \;\; (1 + \frac{gh_o \Delta t^2}{4.\Delta s^2} \sin^2 \sigma_2 \; \Delta s)}$$

Because of the rather complex nature of this matrix, the behavior of its eigenvalues is conveniently investigated numerically. It can thus be shown that all eigenvalues λ satisfy the condition

$$|\lambda| \quad \leq \quad 1, \qquad \text{(A.6)}$$

which satisfies the "von Neuman necessary condition" for stability (Reference 5).

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