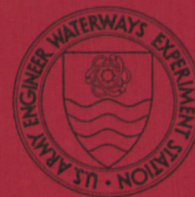US-CE-C Property of the United States Government

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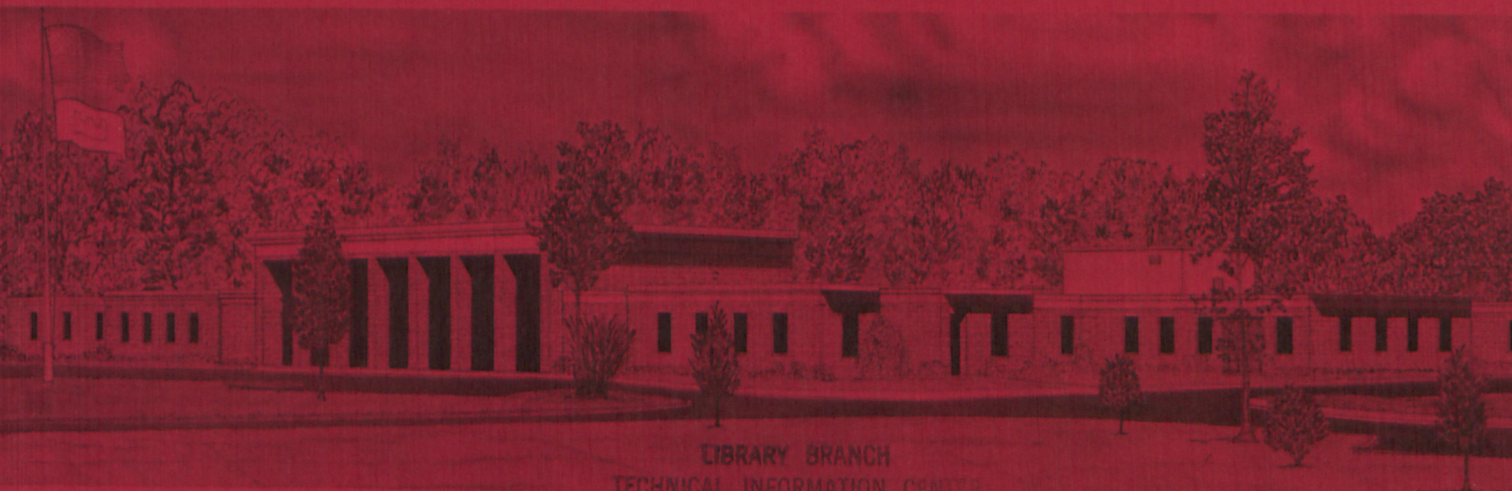# SATURATED SAND AS AN INELASTIC TWO-PHASE MEDIUM

by

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August 1976
Final Report

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Contract Report S-76-7	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE *(and Subtitle)* SATURATED SAND AS AN INELASTIC TWO-PHASE MEDIUM		5. TYPE OF REPORT & PERIOD COVERED Final Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR*(s)* Zdenek P. Bazant Raymond J. Krizek		8. CONTRACT OR GRANT NUMBER*(s)* DACW39-75-M-3739
9. PERFORMING ORGANIZATION NAME AND ADDRESS The Technological Institute Department of Civil Engineering Northwestern University, Evanston, Ill. 60201		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office, Chief of Engineers, U. S. Army Washington, D. C. 20314		12. REPORT DATE August 1976
		13. NUMBER OF PAGES 28
14. MONITORING AGENCY NAME & ADDRESS*(if different from Controlling Office)* U. S. Army Engineer Waterways Experiment Station Soils and Pavements Laboratory P. O. Box 631, Vicksburg, Miss. 39180		15. SECURITY CLASS. *(of this report)* Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Liquefaction (Soils)
Pore water pressure
Sands
Saturated soils

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

The inelastic densification produced by shear straining saturated sands is opposed by the elasticity of the pore water and leads to a pore pressure increase, which causes a decrease in the intergranular frictional forces and consequent liquefaction of the sand mass. This inelastic densification is accompanied by an inelastic strain of the fluid phase, and the magnitude of the developed pore water pressure is the product of the inelastic densification and the densification compliance, the latter being approximately equal to the drained compressibility of the sand. The tangent (incremental)   (Continued)

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20. ABSTRACT (Continued). elastic moduli are expressed in terms of the drained and undrained compressibilities of the two-phase medium and the compressibilities of water and the solid matter forming the grains. It is demonstrated that the volume change of the grains due to intergranular stresses has a negligible effect on the material parameters, even though it roughly equals the volume change of the grains due to the pore water pressure, which has an appreciable effect. Typical values are calculated for the material parameters.

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Preface

This report was prepared by Professors Zdenek P. Bazant and Raymond J. Krizek of Northwestern University under Contract No. DACW39-75-M-3739 as part of ongoing work at the U. S. Army Engineer Waterways Experiment Station (WES) under CWIS 31145 work unit entitled "Liquefaction Potential of Earth Dams and Foundation."

The contract was monitored by Dr. William F. Marcuson III of the Earthquake Engineering and Vibrations Division (EE&VD), Soils and Pavements Laboratory (S&PL). General guidance was provided by Mr. James P. Sale, Chief of S&PL; Mr. Stanley J. Johnson, Special Assistant to the Chief of S&PL; and Mr. Walter C. Sherman, Jr., and Dr. Francis G. McLean, former Chief and Chief of EE&VD, respectively.

Contracting Officer was COL G. H. Hilt, CE, Director of WES.

Contents

INTRODUCTION

Inelastic shear strains in saturated sands occur essentially without interference of pore water, but inelastic volumetric strains caused by the rearrangement of grain configurations in the process of shear straining are resisted elastically by the water filling the pores. Therefore, a coupling between the solid skeleton of sand grains and the pore water exists, and a saturated sand must be treated as a two-phase medium. The linear elastic and linear viscoelastic theory of such a medium has been developed by Biot (1956, 1957), but an extension of this theory into the nonlinear inelastic range is necessary to formulate inelastic densifications. This extension has acquired considerable importance in recent years due to the increased concern with the dynamic densification and associated liquefaction of saturated sand deposits as a result of seismic disturbances.

Despite the recognized need for such a two-phase medium formulation from a theoretical point of view, its practical usefulness requires that the elastic moduli be expressed in terms of easily and directly measurable quantities, such as the compressibility of water C_w and the compressibilities C_b and C_t of saturated sand under drained or undrained conditions. Although studies devoted to the determination of elastic moduli for a two-phase medium have been reported by Biot (1957) and Ishihara (1967), appropriate expressions are not available. Omitting C_t and C_w, Biot (1957) expressed these moduli in terms of the so-called unjacketed compressibilities of the solid skeleton and the pore water due to a pressure applied to the pore water, but these parameters are not particularly useful for practical applications. In addition, certain small terms (such as porosity increments due to deformation) were neglected without offering justification. The derivation of modulus expressions in terms of C_b, C_t, and C_w has been attempted by Ishihara (1967), but several of his basic equations are incorrect (for

example, Equation (9) is correct only in spatial (Euler) coordinates while all remaining equations are written in material (Lagrangian) coordinates, in which case this equation should read $n(de_\rho - de_s + de) = de_n$, and the assumption that precedes Equation (14), as well as the assumption that the volume change of the grains is caused only by pore water pressure, are unjustified). The stress-strain relations considered herein will be expressed in an incremental form, so that the present analysis is applicable to nonlinear behavior that is typical of sands.

ASSUMPTIONS

Homogeneity and Isotropy

The granular solid (sand) is assumed to be statistically homogeneous on a sufficiently large scale and incrementally isotropic. The assumption of isotropy is hardly ever true in the strict sense, even in a sand which is perfectly isotropic in the unstressed state, because the contact areas between grains depend on the stress and those contact areas which are normal to the major principal stress are larger, so that the sand must be stiffer in this direction. Nevertheless, the effect of the associated anisotropy is probably small and negligible. As a consequence of isotropy, the incremental stress-strain relations may be treated as separate relations for the deviatoric and volumetric components of the stress and strain tensors.

Shear Stresses

The shear stresses are assumed to be carried totally by the solid skeleton (as for dry sands) and the portion of the macroscopic shear stresses carried by the fluid phase is neglected (see Figure 1). Consequently, the incremental stress-strain relations for nonlinear inelastic deviatoric deformations involve only deviatoric stress and strain increments of the solid skeleton and not the fluid phase.

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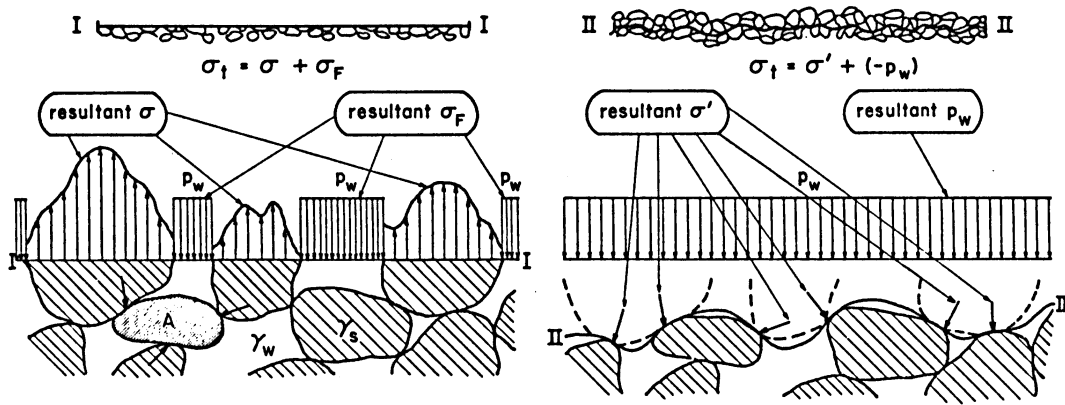
Figure 1. Sketch of Sand Cross section Elucidating Various
Macroscopic Stress Components

Volumetric Deformations

The volumetric deformations of the fluid within the pores are assumed
to be perfectly elastic and given by

$$\frac{d\gamma_w}{\gamma_w} = C_w \, dp_w \tag{1}$$

in which γ_w is the mass density of the pore water and p_w is the fluid
pressure, which will be taken as positive in the case of compression. Nor-
mal stresses, denoted σ (with a subscript of superscript), will be positive
for tension.

Contact Areas

The contact areas between the sand grains are assumed to be negligibly
small so that the pore water and associated pressure p_w completely sur-
rounds each grain. This assumption also precludes the existence of appreci-
able bond forces between the grains. As a consequence of this assumption,
the macroscopic resultant of the intergranular contact forces, called the
effective stress σ', may be determined by considering a macroscopically
planar, but microscopically sinuous, unit cross section which passes only
through interparticle contacts (Section II-II in Figure 1). On such a

cross section the pore water pressure p_w acts over the entire area and must be subtracted from the total macroscopic volumetric stress σ to obtain the resultant of the intergranular forces:

$$\sigma' = \sigma_t - (-p_w) \qquad (2)$$

Compressibility of Grains

Each individual grain is assumed to be perfectly elastic with a bulk compressibility C_s. Since each grain is loaded by pressure p_w over its entire boundary, there is a uniform hydrostatic pressure p_w within the grain, and this causes a volumetric compression $C_s p_w$ of the grain. In addition, each grain (such as A in Figure 1) is also loaded by the inter-granular forces which are in excess of p_w and characterized precisely and solely by σ'. Thus, the additional volume change is proportional to σ', so that

$$\frac{d\gamma_s}{\gamma_s} = C_s \, dp_w - C_s' \, d\sigma' \qquad (3)$$

in which C_s' is the compressibility of grains due to intergranular stress. C_s' is an average statistical property of the sand, whereas C_s and C_w are properties of homogeneous substances and are not of a statistical nature. In previous studies, C_s' has not been taken into account; there is no a priori reason for neglecting C_s', but it will be seen subsequently in Table 1 that the effect of C_s' is small compared to C_s, even when $C_s' \, d\sigma'$ is larger than $C_s \, dp_w$.

BASIC RELATIONS

Conservation of Mass

The kinematic variables needed to describe the behavior of a two-phase medium are most conveniently chosen as the macroscopic (statistical average) displacements of the solid u_k and fluid U_k, where subscript k (k = 1,2,3)

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refers to cartesian axes x_k. The volumetric deformations are then charac-terized by the volumetric strain of the solid skeleton $\epsilon = \partial u_k / \partial x_k = \text{div } u_k$ and the volumetric (macroscopic) strain of the pore fluid $\epsilon_F = \partial u_k / \partial x_k = \text{div } U_k$, where the subscript F refers to the macroscopic fluid phase. The strains may be expressed with the help of the (macroscopic) bulk mass densities of the solid phase ρ and fluid phase ρ_F which are related to γ_s and γ_w by

$$\rho = (1-n)\gamma_s \tag{4a}$$

and

$$\rho_F = n\gamma_w \tag{4b}$$

The condition of conservation of mass yields the mass continuity relations $\dot{\rho} = -\rho \text{ div } \dot{u}_k$ and $\dot{\rho}_F = -\rho_F \text{ div } \dot{U}_k$, in which the dots represent material (rather than spatial) time derivatives. From these equations it follows that

$$d\rho = -\rho \, d\epsilon \tag{5a}$$

and

$$d\rho_F = -\rho_F \, d\epsilon_F \tag{5b}$$

Note that the formulation developed herein, including that of the fluid phase, is expressed in terms of material (Lagrangian) coordinates.

Stress Resultants

The force variables that are associated with ϵ and ϵ_F by means of a work expression are not σ', σ_t, or p_w, but the macroscopic volumetric stresses in the solid phase σ and the fluid phase σ_F defined such that $\sigma\delta\epsilon + \sigma_F\delta\epsilon_F$ is the correct work expression. The work done by the fluid flow per unit material element at $\delta\epsilon = 0$ equals $-p_w\delta V_F$, where $\delta V_F = n\delta\epsilon_F$ is the volume of water that flows out of the unit element. Thus, $\sigma_F\delta\epsilon_F = -p_w(n\delta\epsilon_F)$, which yields

$$\sigma_F = -np_w \tag{6}$$

as the only possible definition of σ_F. Equation (6) represents the resultant of the pore water pressure p_w over a unit cross section of porosity n; this cross section is perfectly planar (not microscopically sinuous) and passes through the grains (Section I-I in Figure 1). The stress in the solid phase σ. must then represent the resultant of the stresses acting over the remaining area, 1-n, of the unit cross section (this is equivalent to the resultant of the microscopic stresses in the solids on this cross-section). However, it makes no sense to write for σ an expression analogous to Equation (6), because, unlike p_w, the microscopic stresses in the solids are not uniformly distributed within the grains (unless $\sigma' = 0$). In terms of the stress and strain tensors σ_{ij} and ϵ_{ij} for the solid phase, σ and ϵ may be written as $\sigma = \frac{1}{3}\sigma_{kk}$ and $\epsilon = \frac{1}{3}\epsilon_{kk}$. By equilibrium of the macroscopic cross section, the total volumetric stress is $\sigma_t = \sigma + \sigma_F$, which allows σ' to be written as

$$\sigma' = \sigma + \sigma_F + p_w = \sigma + (1-n)p_w = \sigma - \frac{1-n}{n}\sigma_F \tag{7}$$

Elastic Strains

Due to the nonlinearity of sand, the stress-strain relations will be expressed in an incremental form. The elastic strain increments of the two-phase medium $d\epsilon = d\epsilon^{el}$ and $d\epsilon_F = d\epsilon_F^{el}$ are defined as the strain increments that are perfectly reversible upon removal of the stress incre- ment. This implies the existence of an incremental strain energy density, W, as a function of $d\epsilon^{el}$ and $d\epsilon_F^{el}$, and since $\Delta\sigma = \partial W/\partial\epsilon = (\partial^2 W/\partial\epsilon^2)\Delta\epsilon + (\partial^2 W/\partial\epsilon\partial\epsilon_F)\Delta\epsilon_F$ and $\Delta\sigma_F = \partial W/\partial\epsilon_F = (\partial^2 W/\partial\epsilon_F\partial\epsilon)\Delta\epsilon + (\partial^2 W/\partial\epsilon_F^2)\Delta\epsilon_F$, the volu- metric stress-strain relation must have the form (Ishihara, 1967)

$$\begin{Bmatrix} d\sigma \\ d\sigma_F \end{Bmatrix} = \begin{bmatrix} P & Q \\ Q & R \end{bmatrix} \begin{Bmatrix} d\epsilon^{el} \\ d\epsilon_F^{el} \end{Bmatrix} \tag{8}$$

in which P, Q, and R are tangent (incremental) volumetric elastic moduli of the two-phase medium ($P = \partial^2 W/\partial \epsilon^2$; $Q = \partial^2 W/\partial \epsilon \partial \epsilon_F$; $R = \partial^2 W/\partial \epsilon_F^2$). The symmetry of this matrix is a consequence of the existence of an incremental potential.

Definition of Inelastic Strains

In general, the strain increments are composed of reversible and irreversible components, the latter of which are called inelastic strain increments $d\epsilon''$ and $d\epsilon_F''$ and defined as the strains that occur without any change in stress (i.e. at $d\sigma = d\sigma_F = 0$). According to this definition, $d\epsilon^{e\ell} = d\epsilon - d\epsilon''$ and $d\epsilon_F^{e\ell} = d\epsilon_F - d\epsilon_F''$, so that Equation (8) becomes

$$d\sigma = P(d\epsilon - d\epsilon'') + Q(d\epsilon_F - d\epsilon_F'') \tag{9a}$$

and

$$d\sigma_F = Q(d\epsilon - d\epsilon'') + R(d\epsilon_F - d\epsilon_F'') \tag{9b}$$

The inelastic strain or densification strain of the solid phase $d\epsilon''$ represents the inelastic densification that results from the slip of grains over each other and the consequent rearrangement of particles into denser configurations (at $d\sigma = 0$). The strain increment $d\epsilon''$ is a functional of the deviatoric strains and stresses in the solid skeleton itself. One of the advantages of postulating a two-phase medium is that $d\epsilon''$ may be considered to depend on the shear strains in the same manner as for unsaturated sands. Actually, of course, the same densification can not occur in a saturated sand, but the difference from $d\epsilon''$, representing the elastic resistance of the pore fluid against densification, is handled by Equations (1), (3), and (5).

Although the introduction of an inelastic strain in the fluid phase might seem illogical, the fluid phase should not be confused with the fluid (water) itself. In general, the fluid phase displacements U_k may result

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from both the volume compression of the fluid (which is perfectly elastic by virtue of Equation (1)) and the change in pore space (which may be inelastic) due to the slip of grains over each other and the consequent rearrangement of particles into denser configurations. At this point the possibility that $de_F'' = 0$ is not excluded, but it will be shown later that this is impossible, because the inelastic densification of the solid phase ($d\epsilon$ at $d\sigma = 0$) is accompanied by a contraction of the pore space, and this can not occur at $d\sigma_F = 0$ (or $d\gamma_W \approx 0$) unless water is expelled from the pore space. To expel water, a divergent flux is generated (i.e. div $U_k = 0$) which means that $d\epsilon_F \neq 0$ at $d\sigma_F = d\sigma = 0$. The notion of an inelastic strain in the fluid phase and the necessity of its dependence on $d\epsilon''$ were originally suggested by Z. J. Bazant (1967) without establishing the proper relationship for $d\epsilon_F''/d\epsilon''$, as given subsequently in Equation (43).

The change of state of saturated sand is fully specified by $d\epsilon$, $d\epsilon_F$, and $d\epsilon_F''$. Thus, since Equations (1), (3), (4), (5), (6), (7), and (9) represent a system of 10 equations, it should be possible to determine the 10 unknowns ($d\sigma$, $d\sigma_F$, $d\epsilon_F''$, $d\sigma'$, dp_w, dn, $d\gamma_w$, $d\gamma_s$, $d\rho$, and $d\rho_F$) when $d\epsilon$, $d\epsilon_F$, and $d\epsilon''$ are given. If $d\epsilon_F''$ were omitted, there would be only 9 unknowns versus 10 equations to be satisfied, and the problem would then be unsolvable. Hence, it is seen that $d\epsilon_F''$ must be included in Equation (9), and the solution of the 10 equations will relate $d\epsilon_F''$ to $d\epsilon''$. The establishment of this relationship will be one objective of the subsequent analysis. Another objective will be the determination of P, Q, and R in terms of readily measurable or known quantities, such as C_w, C_s, and n.

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INCREMENTAL ELASTIC MODULI

Auxiliary Relations

To develop relationships for the incremental moduli, it is expedient to first eliminate some of the variables. Differentiation of Equations (4) gives $d\rho = (1-n)d\gamma_s - \gamma_s\,dn$ and $d\rho_F = n\,d\gamma_w + \gamma_w\,dn$, which, upon substitution for $d\rho$ and $d\rho_F$ from Equations (5) and for ρ and ρ_F from Equations (4), allows ρ and ρ_s to be eliminated, this yielding

$$d\epsilon = \frac{dn}{1-n} - \frac{d\gamma_s}{\gamma_s} \tag{10a}$$

and

$$d\epsilon_F = -\frac{dn}{n} - \frac{d\gamma_w}{\gamma_w} \tag{10b}$$

The further use of Equations (1) and (3) to eliminate γ_s and γ_w gives

$$d\epsilon = \frac{dn}{1-n} - C_s\,dp_w + C'\,d\sigma' \tag{11a}$$

and

$$d\epsilon_F = -\frac{dn}{n} - C_w\,dp_w \tag{11b}$$

Differentiation of Equation (6) and substitution for dp_w from Equation (1) yields $d\sigma_F = -p_w\,dn - n\,d\gamma_w/\gamma_w\,C_w$, which, upon replacement of $d\gamma_w$ from Equation (10b), can be written as $dn = (C_w\,d\sigma_F - nd\epsilon_F)/(1-C_w p_w)$. Inserting this expression for dn into Equation (11a), together with $dp_w = \sigma_F dn/n^2 - d\sigma_F/n$ and $d\sigma' = d\sigma - (1-n)d\sigma_F/n + \sigma_F\,dn/n^2$ from Equations (7) and (6), gives

$$C_1 d\sigma_F + (1-n)C_s'\,d\sigma = (1-n)d\epsilon + n\beta\,d\epsilon_F \tag{12}$$

in which

$$\beta = \frac{1 + \frac{1-n}{n}(C_s - C_s')p_w}{1 - C_w p_w} \tag{13a}$$

$$C_1 = C_o - \frac{(1-n)^2}{n}C_s' \tag{13b}$$

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$$C_o = \beta C_w + \frac{1-n}{n} C_s \qquad (13c)$$

In most cases of practical interest, $p_w \leq 0.5 N/mm^2$, which corresponds to a depth of less than 50 m below the surface. The use of this value for p_w, together with typical values for C_w and C_s ($C_w = 0.49$ mm^2/kN; $C_s = 0.028$ mm^2/kN) gives $C_w p_w \leq 3 \times 10^{-4}$ and $C_s p_w (1-n)/n \leq 3 \times 10^{-5}$, which indicates that all terms with p_w are negligible and $\beta \doteq 1$.

Substituting for $d\sigma$ from Equation (9) and considering the special case of elastic deformations ($d\epsilon'' = d\epsilon_F'' = 0$) leads to

$$d\sigma_F = \frac{1}{C_1} \left\{ (1-n)(1-C_s'P)d\epsilon + [\beta n - (1-n)C_s'Q] \, d\epsilon_F \right\} \qquad (14)$$

which holds for any values of $d\epsilon$ and $d\epsilon_F$. Since the coefficients of $d\epsilon$ and $d\epsilon_F$ must be the same as those in Equations (9), Q and R may be written as

$$Q = \frac{1-n}{C_1} (1-C_s'P) \qquad (15a)$$

and

$$R = \frac{\beta n - (1-n)C_s'Q}{C_1} \qquad (15b)$$

The basic equations will now be expressed in terms of directly measurable quantities, such as p_w or σ' instead of σ and σ_F. Equations (7) and (6) yield $d\sigma = p_w dn - (1-n)dp_w + d\sigma'$ and $d\sigma_F = -n \, dp_w - p_w \, dn$, respectively, and the substitution for dn and $d\epsilon_F$ from Equations (11) and the use of Equations (9) with $d\epsilon''$ and $d\epsilon_F'' = 0$ gives

$$d\sigma = d\sigma' - (1-n)dp_w - p_w n(d\epsilon_F + C_w \, dp_w) = Pd\epsilon + Qd\epsilon_F \qquad (16a)$$

and

$$d\sigma_F = p_w n(d\epsilon_F + C_w \, dp_w) - n \, dp_w = Qd\epsilon + Rd\epsilon_F \qquad (16b)$$

Upon elimination of $d\epsilon_F$, it follows for elastic deformations that

$$(R + np_w)d\sigma' + \{nQ - (1-n)R - [C_w(Q + R) - 1] np_w\} dp_w$$

$$= [PR - Q^2 - (P + Q)np_w] d\epsilon \qquad (17)$$

The mass of water which is expelled from a unit volume of sand is

$dm = \rho_F \, div(dU_k - du_k) = \rho_F(d\epsilon_F - d\epsilon)$, where ρ_F is the bulk mass density of the fluid phase. Expressing $d\epsilon_F$ from Equation (9) for the elastic case gives $d\epsilon_F = (d\sigma_F - Qd\epsilon)/R$, and replacement of $d\sigma_F$ yields $d\epsilon_F = -(p_w \, dn + n \, dp_w + Qd\epsilon)/R$. Then, substituting $dn = -n(d\epsilon_F + C_w \, dp_w)$ from Equation (11b), provides

$$\frac{dm}{\gamma_w} = n(d\epsilon_F - d\epsilon) = \frac{n(Q - R + np_w)d\epsilon - (1 - C_w p_w)n^2 dp_w}{R - np_w} \qquad (18)$$

Bulk Compressibility

Consider a drained compression test in which a saturated soil sample is encased within an impermeable, but flexible, membrane, and the pore water pressure p_w is maintained constant (i.e. $dp_w = 0$). In most tests of this type $p_w = 0$, and all subsequent relations simplify. The load is applied on the external surface of the membrane as a fluid pressure p_e which exceeds the constant pore water pressure p_w by a value denoted as p' and balances the total stress σ_t in the specimen (i.e. $\sigma_t = -p_e = -p' - p_w$). Consequently, the external excess fluid pressure p' represents the negative intergranular or effective stress $-\sigma'$, not the stress in the solid phase $-\sigma$, as assumed in Equation (5) by Biot (1957).

According to Equation (17) for $dp_w = 0$, the bulk compressibility of the solid skeleton in the presence of fluid in the pores is

$$C_b = \left[\frac{\partial \epsilon}{\partial \sigma'}\right]_{p_w} = \left[\frac{\partial \epsilon}{\partial \sigma_t}\right]_{p_w} = \frac{R + np_w}{PR - Q^2 - (P + Q)np_w} \qquad (19)$$

For typical values of n and for $p_w \leq 0.5$ N/mm^2, $np_w \leq 0.0003R$ and $(P + Q)np_w \leq 0.002 (PR - Q^2)$; hence, the terms containing p_w can be neglected,

and Equation (19) reduces to

$$C_b = \frac{R}{PR - Q^2} \tag{20}$$

Although it is quite reasonable to neglect the indicated terms in problems involving relatively shallow depths (less than 50 m) of natural sand, these terms may be important in situations involving very large depths or highly compressible granular materials which are much different from natural sands.

Another expression for C_b can be obtained as follows. Expressing $dn = -nd\epsilon_F$ from Equation (10b) and substituting into Equation (9a) for $d\epsilon'' = d\epsilon_F'' = 0$ gives $d\sigma = Pd\epsilon - Qdn/n$. Insertion of $d\sigma = d\sigma' + p_w\,dn$, which follows from Equation (7) for $dp_w = 0$, yields $d\sigma' = Pd\epsilon - (Q/n + p_w)dn$, and further substitution of $dn = (1-n)(d\epsilon - C_s'\,d\sigma')$ from Equation (11a) leads to $d\sigma' = Pd\epsilon - (Q/n + p_w)(1-n)(d\epsilon - C_s'\,d\sigma')$. This may be rewritten as $d\epsilon = C_b\,d\sigma'$, in which

$$C_b = \frac{n - (1-n)(Q + np_w)C_s'}{nP - (1-n)(Q + np_w)} \tag{21}$$

Since $np_w \leq 0.0003\,Q$ for typical values of n and for $p_w \leq 0.5$ N/mm^2, Equation (21) reduces to

$$C_b = \frac{n - (1-n)QC_s'}{nP - (1-n)Q} \tag{22}$$

According to Equation (18) the mass of water expelled from the pores of a unit volume of sand is characterized by the coefficient

$$w_b = \frac{1}{\gamma_w}\left[\frac{\partial m}{\partial \epsilon}\right]_{p_w} = n\,\frac{Q - R + np_w}{R - np_w} \tag{23}$$

which, for small values of p_w, simplifies to

$$w_b = n\left(\frac{Q}{R} - 1\right) \tag{24}$$

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Undrained Bulk Compressibility

To determine the undrained bulk compressibility, a saturated sand specimen is enclosed within an impermeable, but flexible, membrane and loaded externally with no drainage allowed. The displacement of the solid and the water are assumed equal at the specimen-membrane interface, and, because of the assumed homogeneity of the specimen, they are equal everywhere (i.e. $u_k = U_k$ in this type of test). Thus, div u_k = div U_k and $d\epsilon = d\epsilon_F$, and from Equation (1) for the elastic component of stress it follows that $d\sigma_t = d\sigma + d\sigma_F = (P + 2Q + R)d\alpha$. Hence, the bulk of total compressibility of the undrained material can be expressed as

$$C_t = \left[\frac{\partial e}{\partial \sigma_t}\right]_{\epsilon - \epsilon_F} = \frac{1}{P + 2Q + R} \tag{25}$$

Since the pore water pressure p_w can be measured in an undrained test, Equation (16b) yields

$$Q_p = -\left[\frac{\partial p_w}{\partial \epsilon}\right]_{\epsilon - \epsilon_F} = \frac{Q + R - np_w}{n(1 - C_w p_w)} \tag{26}$$

for $d\epsilon - d\epsilon_F = 0$. For small values of p_w, Equation (26) simplifies to

$$Q_p = \frac{Q + R}{n} \tag{27}$$

Note that the strain in the solid due to a change in the pore water pressure $\partial \epsilon / \partial p_w$ at a condition of no flow ($\epsilon = \epsilon_F$) equals $1/Q_p$.

Compressibility Due to Pore Fluid Pressure

To determine the compressibility due to pore fluid pressure, a saturated sand specimen is enclosed within an impermeable, but flexible, membrane, and loaded with an external cell pressure p_e and an internal back pressure p_w such that $p_e - p_w = -\sigma' = $ constant; although it is possible to conduct such a test, the resulting deformations would probably be too small to measure accurately. Nevertheless, since $d\sigma' = 0$ for such a test, Equation (17) yields

$$C_p = -\left[\frac{\partial \epsilon}{\partial p_w}\right]_{\sigma'} = \frac{nQ - (1-n)R - [(Q + R)C_w - 1]np_w}{PR - Q^2 - (P + Q)np_w} \tag{28}$$

which for small values of p_w reduces to

$$C_p = \frac{nQ - (1-n)R}{PR - Q^2} \tag{29}$$

An alternative expression may be obtained by differentiating Equations (4a) and (4b), substituting for $d\rho$ and $d\rho_F$ from Equation (5a) and (5b) and for $d\gamma_w$ and $d\gamma_s$ from Equations (1) and (3), and eliminating dn from the resulting two equations; this provides

$$(1-n)d\epsilon + nde_F + [nC_w + (1-n)C_s]dp_w + (1-n)C_s'd\sigma' = 0 \tag{30}$$

which, upon setting $d\sigma' = 0$ and substituting for de_F from Equation (16b), gives

$$C_p = -\left[\frac{\partial \epsilon}{\partial p_w}\right]_{\sigma'} = \frac{n^2 - R[nC_w + (1-n)C_s] + n\,C_s p_w}{(1-n)R - nQ - n(1-n)p_w} \tag{31}$$

For small values of p_w, Equation (31) simplifies to

$$C_p = \frac{n^2 - R[nC_w + (1-n)C_s]}{(1-n)R - nQ} \tag{32}$$

Similarly, the alternative use of Equation (16a) instead of Equation (16b) gives

$$C_p = \frac{n(1-n) - Q[nC_w + (1-n)C_s]}{(1-n)Q - nP} \tag{33}$$

for small values of p_w. The mass of water that is forced into a unit volume of sand at $d\sigma' = 0$ is a quantity that can be measured and is characterized, according to Equation (18), by the coefficient

$$C_{pw}' = -\frac{1}{\gamma_w}\left[\frac{\partial m}{\partial p_w}\right]_{\sigma'} = \frac{n^2(1 - C_w p_w)}{R - np_w} \tag{34}$$

which for small values of p_w becomes

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$$C'_{pw} = \frac{n^2}{R} \qquad (35)$$

In a different type of test a sand specimen can be placed in a perfectly rigid container ($d\epsilon = 0$) and the mass of water m that is forced into the container under pressure p_w can be measured. In this case we obtain the coefficient $C''_{pw} = -(\partial m/\partial p_w)_\epsilon / \gamma_w$, which can be easily determined from Equation (18).

Compression of Dry Sand

In the case of dry sand $Q = R = 0$ and

$$P = 3K = 3/C \qquad (36)$$

where K is the bulk modulus of the solid skeleton and C is its bulk compressibility, which, in general, may be different from C_b. This value may be applied to the two-phase medium, provided the absence of water from grain contacts does not change appreciably the material properties.

CALCULATION OF ELASTIC MODULI FROM TEST RESULTS

By means of Equations (15), (20), (22), (24), (25), (27), (29), (32), (33), and (35), coefficients P, Q, R, C_w, C_s, and C'_s are related to various measurable test parameters. Most convenient for measurement are probably C_b and C_t, and advantage can be taken of the fact that C_w is quite accurately known without testing (0.49 mm^2/kN at $25°C$) and that C_s is approximately known when the mineralogical composition of grains is identified (typically, $C_s = 0.028$ mm^2/kN for quartz). Furthermore, it is not necessary to know C_s too accurately, because its value has only a minor effect on the calculated values of P, Q, and R.

A method for calculating P, Q, and R from C_b, C_t, C_w, C_s, and n will now be given. Solving Equation (22) for P and substituting the result into Equation (15a) gives a linear algebraic equation for Q, which, when combined with Equation (15b) ($\beta = 1$) and Equation (22), yields

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$$Q = \left(\frac{C_1}{1-n} \frac{C_b}{C_b - C'_s} + \frac{1-n}{n} C'_s \right)^{-1} \tag{37a}$$

$$R = \frac{n - (1-n)C'_s Q}{C_1} \tag{37b}$$

$$P = \frac{1}{C_b} + \frac{1-n}{n} \frac{C_b - C'_s}{C_b} Q \tag{37c}$$

in which $C_1 = C_w + C_s(1-n)/n - C'_s(1-n)^2/n$, according to Eqs. (13b) and (13c). These expressions may be evaluated if a value of C'_s is assumed. The resulting values must satisfy Equation (25), and so

$$F(C'_s) = 0 = P + 2Q + R - \frac{1}{C_t} \tag{38}$$

in which C_t is an experimentally determined value. Although Equation (38) will, in general, not be satisfied for an arbitrary value of C'_s, it may be regarded as a function $F(C'_s)$ of the chosen C'_s value and the iterative regula falsi method may be used to find the C'_s value which gives $F(C'_s) = 0$.

The required calculations may be shortened by deducing a suitable initial estimate of C'_s. Because of equilibrium in the unit cross section of sand, the statistical average $\hat{\sigma}$ of the volumetric component of the microscopic stresses produced within the grains by the intergranular stress σ' equals $\sigma'/(1-n)$. The corresponding relative change in the volume of the grains is $C_s \hat{\sigma}$ or $C_s \sigma'/(1-n)$, and this must equal $C'_s \sigma'$, as indicated by Equation (3); hence, C'_s may be estimated by

$$C'_s \approx \frac{C_s}{1-n} \tag{39}$$

which further implies that $C_1 \approx C_w$. Equation (39) also allows the approximate evaluation of P, Q, and R from Equations (37) when only C_b (and not C_t) has been measured.

The compressibility of the solid material C_s is always much less than

C_b. For this reason, it may be assumed for a crude approximation that $C_s \approx 0$ and $C_s' \approx 0$. Substituting these values into Equations (15) and (20) yields

$$P = \frac{1}{C_b} + \frac{Q^2}{R} \tag{40a}$$

$$Q \approx \frac{1-n}{C_w} \tag{40b}$$

$$R \approx \frac{n}{C_w} \tag{40c}$$

An alternative method of calculation may be based on Equations (21) and (25). Eliminating P from these two relations gives a quadratic equation for Q, and the only positive solution of this equation is

$$Q = -R + \left[R\left(\frac{1}{C_b} - \frac{1}{C_t}\right)\right]^{\frac{1}{2}} \tag{41}$$

Furthermore, eliminating C_s' from Equations (15a) and (15b) (in which C_1 also contains C_s') and substituting for P from Equation (25) leads to

$$0 = F(R) = \frac{(1-n)^2}{n} R - 2Q + (Q + 2R)QC_o + (n-RC_o)\left(\frac{1}{C_t} - R\right) \tag{42}$$

in which $C_o = C_w + C_s(1-n)/n$ and $\beta = 1$. Obviously, if Equation (41) were substituted here, a fourth degree algebraic equation for F would result and it would be necessary to solve for R numerically. After insertion of Equation (41), Equation (42) represents a function, F(R), which must be made equal to zero, and the solution for R may be found by the regula falsi method. For the initial estimate of R, Equation (40c) may be used. This method of solution does not involve C_s', which may be subsequently found, e.g., from Equation (22).

Equations (24), (27), (29), (33), and (35) represent redundant relations, which can be used when w_b, Q_p, C_p, and C_{pw}' have been measured. They can be employed as a check, and statistical averaging may be performed to reduce

the experimental error. Equations (32) or (33) may be used to calculate a
value for $[nC_w + (1-n)C_s]$, which allows a check on the assumed values of
C_w and C_s. Note that Equation (20) assures that $PR - Q^2 > 0$ because $R > 0$
and $C_b' > 0$. Hence, the matrix of elastic constants in Equation (8) is
positive definite, which guarantees local stability of the material.

Table 1. Material Parameters of Typical Sands

		Dense Sand			Loose Sand		
		Exact (1)	Approximate (2)	(3)	Exact (4)	Approximate (5)	(6)
Given	n	0.3	0.3	0.3	0.5	0.5	0.5
	C_b (mm^3/kN)	18.0	18.0	18.0	90.0	90.0	90.0
	C_w (mm^2/kN)	0.490	0.490	0.490	0.490	0.490	0.490
	C_s (mm^2/kN)	0.028	0.028	0	0.028	0.028	0
	C_s' (mm^2/kN)	$C_s/(1-n)$	0	0	$C_s/(1-n)$	0	0
Computed	P (kN/mm^2)	2.9844	2.9967	3.3889	0.9752	0.9764	1.0315
	Q (kN/mm^2)	1.2580	1.2605	1.4286	0.9653	0.9653	1.0204
	R (kN/mm^2)	0.5404	0.5402	0.6122	0.9647	0.9653	1.0204
	C_t (mm^2/kN)	0.1655	0.1651	0.1458	0.2584	0.2583	0.3267
	C_1 (mm^2/kN)	0.4900	0.5553	0.4900	0.4900	0.5180	0.4900
	C_d (mm^3/kN)	18.011	17.653	18.145	90.210	90.285	90.328

Note: Columns (1), (2), (4), and (5) are calculated from Equations
(37), (25), and (50), and columns (3) and (6) from Equations
(40), (25), and (51).

The same typical properties of natural sands as those considered by
Ishihara (1967) have been used to calculate elastic moduli by both exact and
approximate formulas, and the results are given in Table 1. It is seen that,
although the value of C_s' has a negligible effect and may be set equal to either
zero or the value given by Eq. (39), C_s has a more profound effect. Four
decimal places are given in Table 1 only to indicate the order of magnitude
of the differences between various methods of calculation. From a practical
point of view, of course, an error of less than 1% is meaningless because

of the uncertainty involved in determining values for the measured proper-
ties (e.g., C_b').

INELASTIC STRAINS IN SOLID AND FLUID PHASES

The inelastic strains are defined as those strains which occur at
$d\sigma = d\sigma_F = 0$. Then, Equations (9a) and (9b) yield $d\epsilon = d\epsilon''$ and $d\epsilon_F = d\epsilon_F''$,
which, when substituted into Equation (12), give

$$d\epsilon_F'' = -\frac{1-n}{n} d\epsilon'' \tag{43}$$

since $\beta = 1$ for small values of p_w. Perhaps surprisingly, however, the
inelastic strain in the fluid phase can not be taken as zero. Equation (43)
can also be derived in a more direct, though less rigorous, way by realizing
that the changes in unit weights γ_w and γ_s must be second-order small quan-
tities when $d\sigma = d\sigma_F = 0$. Since mass densities do not change appreciably,
the strain increment $d\epsilon = d\epsilon''$ causes the pore volume of the material to
expand by $dn = d\epsilon = d\epsilon''$. On the other hand, because the density of the
pore fluid does not change appreciably, the volume of the pore water which
is imbibed into the material is $n(d\epsilon - d\epsilon_F)$ or $n(d\epsilon'' - d\epsilon_F'')$. This must
equal dn, and so $dn = n(d\epsilon'' - d\epsilon_F'') = d\epsilon''$, which yields Equation (43). With
the use of Equation (43), Equations (9a) and (9b) reduce to the form

$$d\sigma = P\, d\epsilon + Q\, d\epsilon_F - \left(P - \frac{1-n}{n} Q\right) d\epsilon'' \tag{44a}$$

and

$$d\sigma_F = Q\, d\epsilon + R\, d\epsilon_F - \left(Q - \frac{1-n}{n} R\right) d\epsilon'' \tag{44b}$$

For the typical values of P and Q listed in Table 1, the quantities
$\left(P - \frac{1-n}{n} Q\right)$ and $\left(Q - \frac{1-n}{n} R\right)$ equal 49.0 N/mm^2 and 2.8 N/mm^2 for $n = 0.3$, and
10.5 N/mm^2 and 0.6 N/mm^2 for $n = 0.5$, respectively. Hence, as a rough
approximation, $\left(Q - \frac{1-n}{n} R\right) \approx 0$, (i.e. the inelastic strain in Equation (44b)
for the fluid phase is negligible) and Equations (44) may be written as

$$dσ ≈ P \, dε + Q \, dε_F - \frac{1}{C_b} \, dε''$$

(45a)

and

$$dσ_F ≈ Q \, dε + R \, dε_F$$

(45b)

PORE PRESSURE DUE TO DENSIFICATION IN UNDRAINED SAND

Consider now the sand to be in an undrained condition ($dε = dε_F$) and let the total stress be constant ($dσ_t = 0$), as, for example, when the overburden of a given sand element is constant. Summing Equations (9a) and (9b) and noting that $P + 2Q + R = 1/C_t$ (Equation (25)) and $Q + R ≈ 1/C_w$ (Equations (40b) and (40c)), we get

$$dσ_F = dσ + dσ_F = \frac{1}{C_t} \, dε + \left(\frac{Q + R}{n} - \frac{1}{C_t} \right) dε''$$

(46)

According to Equations (9b) and (43), the stress produced in the fluid phase by densification is

$$dσ_F = Q(dε - dε'') + R(dε_F - dε_F'') = \frac{1}{n} \left[R - C_t (Q + R)^2 \right] dε''$$

(47)

The change in the pore water pressure is $dp_w = -d(σ_F/n) = -dσ_F/n - p_w dn/n = -dσ_F/n - p_w(dε_F + C_w \, dp_w)$. Since $C_w p_w < 0.0001$ for $p_w ≤ 0.5 \ N/mm^2$ and since $p_w dε$ is a second-order small quantity, one has $dp_w ≈ -dσ_F/n$, and so

$$dp_w = -dε''/C_d$$

(48)

in which C_d, termed the densification compliance, is given by

$$\frac{1}{C_d} = - \left[\frac{∂p_w}{∂ε''} \right]_{σ_t, \, ε - ε_F} = \frac{R - C_t(Q + R)^2}{n^2}$$

(49)

For the case where $C_s' = 0$, $C_d = C_b^2/(C_b - C_t)$. Since the total stress $σ_t$ remains constant, the build-up of pore water pressure due to inelastic densification produces, according to Equation (2), a drop in the effective stress, $dσ' = -dp_w = dε''/C_d$, which leads to a reduction and ultimately the complete loss of the friction forces between grains. This is the underlying cause

behind the dynamic liquefaction of sand subjected to cyclic shear. A rough estimate of the magnitude of C_d may be obtained by putting $1/C_t \approx 1/C_b + 1/nC_w$, $R = n/C_w$, and $Q + R \approx 1/C_w$, which follows from Equations (40) for $C_s \approx C_s' \approx 0$; Equation (49) then gives

$$C_d \approx \left(\frac{1}{C_b} + \frac{1}{nC_w} - \frac{1}{nC_w}\right)^{-1}\left(1 + \frac{nC_w}{C_b}\right) \approx C_b \qquad (50)$$

because normally $C_w \ll C_b$. From the typical values of C_d indicated in Table 1 it is seen that this estimate is quite good; $1/C_d$ is orders of magnitude less than $1/C_w$.

It is instructive to note that a much smaller coefficient $\partial p_w/\partial \epsilon''$ would be obtained, if, instead of $d\sigma_t = 0$, the condition were imposed that no elastic strain of the solid accompanies the densification (i.e. $d\epsilon = d\epsilon''$). In this hypothetical case Equation (47) yields $d\sigma_F = R(d\epsilon'' - d\epsilon_F'') \approx (R/n)d\epsilon''$, and substituting for R from Equation (40), $d\sigma_F \approx d\epsilon''/C_w$ or $dp_w = -C_d' \, d\epsilon''$, where $C_d' = -\left[\dfrac{\partial p_w}{\partial \epsilon''}\right]_{e\ell, \, \epsilon - \epsilon_F} = 1/(n \, C_w) \gg 1/C_b \approx C_d$. However, to enforce the condition that $d\epsilon = d\epsilon''$, an enormous change in the total stress σ_t which can be evaluated from Equation (46), would be necessary.

REMARK ON PRACTICAL APPLICATION

The ultimate goal of this work is primarily the development of a procedure for predicting the liquefaction of undrained saturated sands subjected to dynamic excitations, such as earthquakes or blasts. However, before this objective can be achieved, the law relating the inelastic densification, $d\epsilon''$, to the history of deviatoric strains must be established.

CONCLUSIONS

Within the context of this study the following conclusions can be advanced:

1. The inelastic densification of the solid skeleton $d\epsilon''$ is accompanied by an inelastic strain in the fluid phase $d\epsilon_F''$ which is related to

de" (Equation (43)). The fluid itself is perfectly elastic, and e_F'' represents the flow of water that is necessary to allow densification without change in stress.

2. The pore water pressure that is produced in an undrained element of saturated sand by a unit densification at constant total stress is characterized by the densification compliance C_d, the value of which is approximately equal to the drained compressibility of the solid skeleton C_b and is orders of magnitude higher than the compressibility of water C_w.

3. The tangent (incremental) elastic moduli of a two-phase granular material can be expressed in terms of the drained compressibility C_b and the undrained compressibility C_t of the medium and known values for the compressibilities of water and the solid matter forming the grains (Equations (37) and (38)).

4. The volume change of the grains due to intergranular stresses (characterized by C_s') has a negligible effect on the properties of a sand, even though it may be of the same order of magnitude as the volume change of the grains due to pore water pressure (characterized by C_s), which has an appreciable effect on the elastic moduli of the two-phase medium.

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APPENDIX A

PORE WATER COMPRESSIBILITY WITH DILATION
OF SKELETON AT CONSTANT OVERBURDEN

As is well known, the relation between the diffusivity and permeability of a porous medium involves the compressibility of the pore fluid, which is water in this case. In certain practical situations (e.g. when the overburden of a given sand element remains constant) changes in pore water pressure occur at approximately a constant value of total stress σ_t. Thus, it is of interest to determine the ratio of a change in water content to a change in pore water pressure at $d\sigma_t = 0$. Adding Equations (9) for $d\epsilon'' = d\epsilon_F'' = 0$, one gets $d\sigma_t = 0 = (P + Q)d\epsilon + (Q + R)d\epsilon_F$ or $d\epsilon = -d\epsilon_F(Q + R)/(P + Q)$, the latter of which may be substituted into Equation (9b), together with $d\sigma_F = -ndp_w - p_w dn$ and $dn = -n(d\epsilon_F + C_w dp_w)$ resulting from Equation (11b), to yield $d\epsilon_F = n(1 - C_w p_w)dp_w/[Q(Q + R)/(P + Q) - R + np_w]$ at $d\sigma_t = 0$. Equation (18) gives $dm/\gamma_w = n(d\epsilon_F - d\epsilon) = n[1 - (Q + R)/(P + Q)]d\epsilon_F$, which, upon substitution for $d\epsilon_F$, allows C_{pw} to be written as

$$C_{pw} = -\frac{1}{\gamma_w}\left[\frac{\partial m}{\partial p_w}\right]_{\sigma_t} = \frac{n^2(P - R)(1 - p_w C_w)}{PR - Q^2 - np_w(P + Q)} \doteq \frac{n^2(P - R)}{PR - Q^2} \qquad (A-1)$$

Furthermore, using the approximations from Equation (40), one gets

$$C_{pw} \approx (1 - 2n)C_b + nC_w \approx (1 - 2n)C_b \qquad (A-2)$$

The compressibility given by Equation (A-2) for constant total stress can be orders of magnitude different from that given by Equation (35) for constant effective stress. For sufficiently large porosities ($n > 0.5$), C_{pw} becomes negative; this means that the application of pressure on the pore water does not cause flow into the pore space, but rather out of it, because the solid

skeleton is dilated by p_w. Although the physical implications of this phenomenon require further investigation, it can be intuitively understood by noting that C_{pw} is a sum of 3 terms: (a) nC_w, which represents the compression of the pore water alone, (b) C_b, which represents the increase in pore water content due to dilation of the solid skeleton, and (c) $-2nC_b$, a negative term which represents the deformation of the solid skeleton necessary to produce a stress in the solid phase that offsets the decrease in σ_t due to dp_w (i.e. the stress required to maintain σ_t at its original value). The compressibility C_{pw} can be measured by enclosing a specimen in a flexible, impermeable membrane and varying the internal back-pressure p_w while maintaining the external cell pressure constant. By a similar procedure it can be shown that the strain in the solid due to a change in the pore water pressure at constant overburden is given by

$$C_{bp} = \left[\frac{\partial \epsilon}{\partial p_w}\right]_{\sigma_t} = \frac{n(Q + R)(1 - p_w C_w)}{PR - Q^2 - np_w(P + Q)} \doteq \frac{n(Q + R)}{PR - Q^2}$$

which, upon using the approximations of Equation (40), reduces to

$$C_{bp} \approx C_b$$

APPENDIX B: NOTATION

C_w, C_s, C_s' = compressibilities of water and solid grains due to pore pressure and intergranular stress (Equations (1) and (3));

C_b, C_t = compressibility of sand in drained or undrained conditions (Equations (20)-(22) and (25));

C_1 = compressibility in Equation (13);

C_d = densification compliance (Equations (49) and (50));

n = porosity;

P, Q, R = volumetric tangent (incremental) elastic moduli of two-phase medium (Equation (8));

P_w = pore water pressure;

β = parameter in Equation (13);

γ_s, γ_w = weight densities of solid and water phases;

ρ_s, ρ_w = mass densities of solid and water phases (Equation (4));

ϵ, ϵ_F = volumetric strains of solid and fluid phases;

ϵ'', ϵ_F'' = inelastic strains in solid and fluid phases (Equation (9));

σ, σ_F = volumetric stresses in solid and fluid phases;

σ' = effective (intergranular) volumetric stress (Equation (2), Figure 1):

σ_t = total stress ($\sigma + \sigma_F$)