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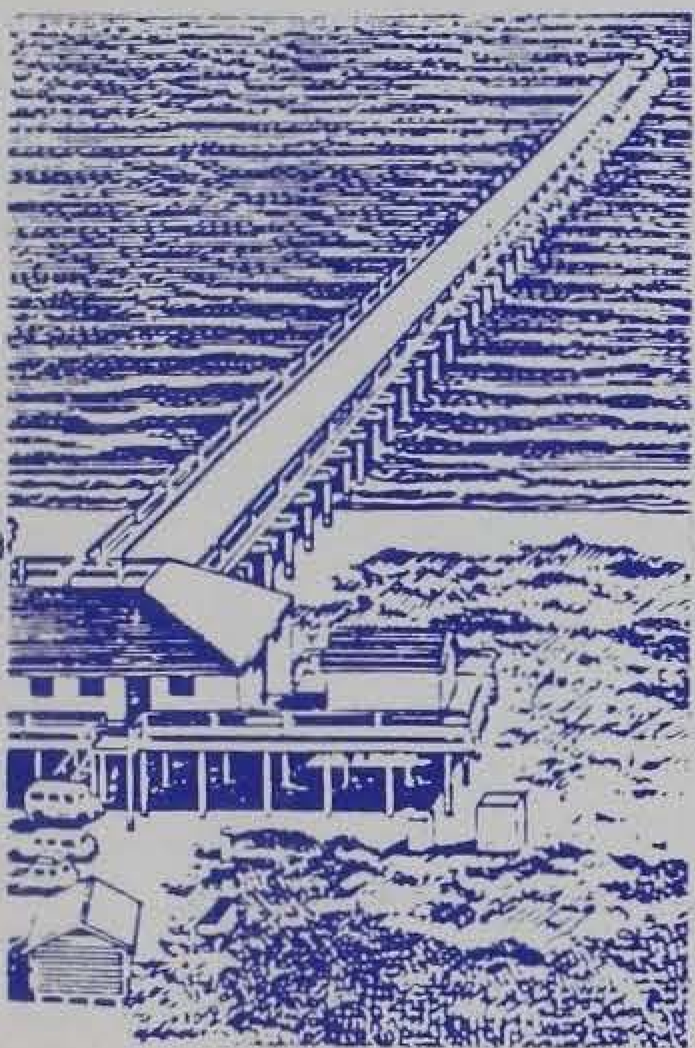

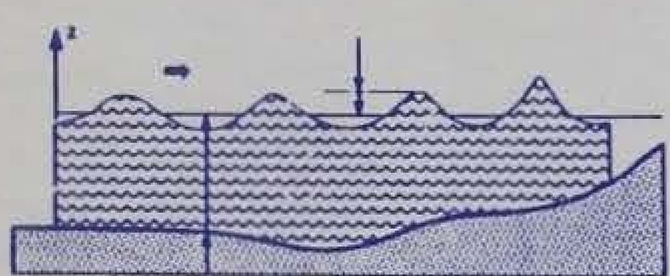

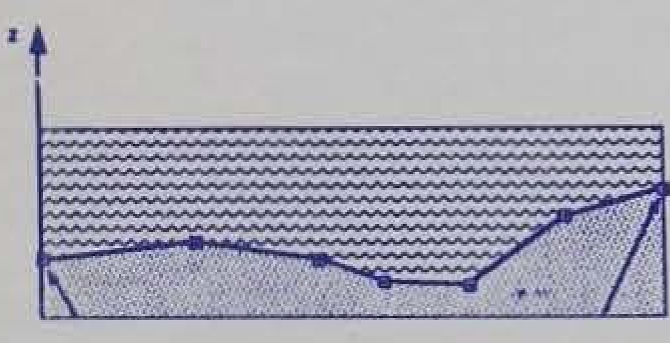

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APPLICATION OF THE GREEN-NAGHDI THEORY OF FLUID SHEETS TO SHALLOW-WATER WAVE PROBLEMS

Report 1
Model Development

by

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September 1992
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13. ABSTRACT *(Maximum 200 words)*

This report presents the mathematical formulation for the development of a numerical model that simulates wave transformation in shallow waters. The model is intended both for military and civil works projects involving propagation of time-dependent and nonlinear waves where existing models may either be inapplicable or use of simple analytic or numerical solutions is infeasible. The theory detailed in this report introduces a new-generation water wave model for shallow to moderate water depths where the seabed varies rapidly. The Green-Naghdi Level II theory, hereafter referred to simply as the GN theory, has been significantly modified in this research and a powerful, general-purpose numerical model, called GNWave, is developed for water wave problems.

The theory and ensuing model incorporate some of the most important mathematical features of the water wave equations. These include non-approximating of the governing Euler's field equations and imposing the proper boundary conditions necessary for capturing the bulk physical characteristics of

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wave trains in the shallow-water regime. The GN approach, which is fundamentally different from the perturbation method based on developments in classical wave theory begun by Stokes and Boussinesq in the last century, can do this only because it does not introduce any simplifications of the velocity variation in the vertical direction across the fluid layers or sheets.

In contrast to the Stokes and Boussinesq theories, the equations of motion in the GN theory are derived by enforcing exact kinematic and dynamic boundary conditions on the free surface and on the bottom, and by enforcing conservation of mass and of the 0th and 1st moments of momentum in the vertical direction. These conditions yield 11 coupled partial differential equations, which can be reduced to 3 complicated governing equations by elimination of many of the variables. In summary, the GN theory is different from the perturbation approach in that the free surface and bottom boundary conditions are met exactly, whereas the field equation is implicitly approximated. The result is a theory that can predict the shape and behavior of waves up to almost breaking conditions. The GN theory breaks down when the particle velocity at the crest equals the wave speed, the criterion for breaking in the exact theory.

By developing the unrestricted Green-Naghdi theory of fluid sheets, this research presents a new wave theory consisting of a coupled, nonlinear set of partial differential equations and integrates these in time and space to simulate either regular or irregular real waves. The theory and model have been shown to reproduce with engineering accuracy the evolution of a wave of permanent form, from small amplitudes up to almost breaking conditions. The theory presented is for a nonlinear numerical wave tank in which the seabed topography profile can be arbitrary and very irregular, and up to 20 wave gages can be positioned at will inside the computational domain to obtain snapshots and profiles of wave records.

The types of projects to which the theory can be applied are many, and include problems of both military and civil interest. The theory is purposely made to be versatile to permit decision makers, designers, and analysts to assess the various aspects of waves and wave-structure interaction problems arising in Army applications. One can evaluate, for instance, the effect of submerged obstacles during military landings on the train of waves approaching a beach or landing zone, or the reflection of waves and forces on sea walls or spillway hydraulic gates, and the time history of bottom-mounted pressure gage measurements for estimation of surface wave conditions in coastal design projects. The theory is particularly suited for the violent collision of waves with natural and man-made structures, and their impact on preventive and defensive hydraulic structures.

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Coastal processes
Hydraulic structures
Logistics-over-the-shore (LOTS)
Numerical simulation
Water surface profile
Wave-structure interaction
Wave theories

PREFACE

This research study was conducted jointly by the Coastal Engineering Research Center (CERC), US Army Corps of Engineers Waterways Experiment Station (WES), and Ship Research Incorporated (SRI), affiliated with the University of California at Berkeley. The study was authorized under the Time-Dependent Nonlinear Free-Surface Wave Modeling Project as part of the In-House Laboratory Independent Research Program (ILIR), Project A91D.

The study was performed and the report prepared over the period 1 May 1991 through 1 September 1992 by Dr. Zeki Demirbilek, Research Division (RD), Coastal Oceanography Branch (COB), CERC, and Dr. William C. Webster, Professor of Naval Architecture and Ocean Engineering at the University of California, Berkeley and President of SRI of Kensington, California, under Purchase Order DACA39-91-M-5515. This is the first in a series of reports presenting the mathematical basis for the development of a new-generation water wave numerical model.

This study was performed under the general supervision of Dr. James R. Houston and Mr. Charles C. Calhoun, Jr., Director and Assistant Director, CERC, respectively; Mr. H. Lee Butler, Chief, RD, CERC; and under the direct supervision of Dr. Martin C. Miller, Chief, COB, CERC. WES Project Manager during the conduct of this study and the publication of this report was Ms. Mary Vincent.

At the time of publication of this report, Director of WES was Dr. Robert W. Whalin. Commander and Deputy Director was COL Leonard G. Hassell, EN.

CONTENTS

APPLICATION OF THE GREEN-NAGHDI THEORY OF FLUID SHEETS TO SHALLOW-WATER WAVE PROBLEMS

PART I: THEORETICAL BASIS

Introduction

This is the first report in a series prepared for the Technical Director, U.S. Army Engineer Waterways Experiment Station (WES) for the project entitled Time-Dependent Nonlinear Free-Surface Wave Modeling conducted under the Director's auspices as part of the In-House Laboratory Independent Research Program (Project A91D per Office of Technical Planning and Programs guidelines). This report is a joint effort between researchers at the WES Coastal Engineering Research Center (CERC) and the University of California, Berkeley, under Purchase Order DACA39-91-M-5515. It is intended to provide a discussion and critique of various approaches for simplifying complex hydrodynamic boundary value problems, a derivation of the general Green-Naghdi (GN) theory of fluid sheets, theoretical documentation of the equations of motion of two-dimensional (2-D) shallow-water waves for GN levels 1 and 2, and a concise description of the numerical methods used to integrate these governing equations.

The intent of the authors was to assemble a document that a fluid mechanician can use to understand this relatively new method and its application to complex shallow-water wave problems. The entire theoretical formulation presented in this report is new, greatly improving upon several previously published papers or Ph.D. dissertations completed at the University of California, Berkeley. The derivation of the theory is in the form of a tutorial in which all of the intermediate steps are included, since no textbook or article is available in this level of detail.

During the two decades since its introduction (Green, Laws, and Naghdi 1974), the **theory of fluid sheets** has been applied to a variety of fluid flow problems. These include studies of waves in shallow and deep water (Green and Naghdi 1986 and 1987; Shields and Webster 1988), the flow beneath planing boats (Naghdi and Rubin 1981), the waves created by a moving pressure disturbance (Ertekin 1984; Ertekin et al. 1984 and 1986) and wave reflection by obstacles (Marshall and Naghdi 1990), to name a few. In particular, the development of fluid sheet theory in a Eulerian frame (Green and Naghdi 1984) made this theory much easier to apply to fluid flow problems. The reader is referred to the pair of papers by Green and Naghdi (1986 and 1987) for a definitive exposition of the theory, and especially for its application to water wave problems.

Overview of Theory

Alternative approaches, with a variety of approximations and assumptions, exist for calculating wave motion in coastal waters. The classical equations of motion for fluid flow in three dimensions are a continuum model which embodies many assumptions. For ordinary fluids, such as water, the Navier-Stokes equations are universally recognized as a good model for the resulting flows. However, these equations are not "exact" equations but are an idealization similar in spirit to the idealization of space by Euclidean geometry. Even for simple free-surface problems, these equations and their simpler inviscid counterparts, the Euler equations, are difficult to solve. One popular approach has been to systematically simplify the three-dimensional equations and their boundary conditions through a formal perturbation analysis until the resulting system can be solved. The theories of water waves developed by Stokes, Boussinesq, and others follow this type of development.

Green-Naghdi fluid sheet theory offers an alternative in the form of a new model, that of a 2-D continuum of unsteady 3-D flows. Although the examples which will be cited here involve inviscid fluids, the development of GN theory is not at all limited to such fluids. The discussion below is aimed at exploring the difference between these two very different paths to simplification of the analysis of fluid flow problems. In either case, it is anticipated that the solutions obtained are approximate ones, since there really is no substitute for solving the three-dimensional (3-D) equations exactly. Both approaches are called approximations, although it is clear that the meaning is not the same for each.

Before introducing details on the nature of GN theory, it is useful to first discuss the notion of approximation in general. An approximation approach for analyzing a given problem is usually chosen based on its ability to predict the phenomena that one is interested in and on its ease of use. The selection of an approximation scheme can be viewed as a type of non-zero-sum game where one attempts to make assumptions which will have a greater impact on the simplification of the analysis than on the accuracy of the prediction of the phenomena of interest.

Two observations from this discussion are significant. First, the choice of the approximation scheme depends on the specific answers that one is looking for (i.e., the choice depends on the context of the problem rather then its generic type). Second, the means of analysis change in time; that is, computations which 20 years ago would have required the world's largest computers can now be accomplished faster and for a minimal cost on a personal computer. It is proper to think in terms of approximation schemes "appropriate for the current time." Since the evolution of a new computer generation appears to take only a few years, it seems natural that we will see a corresponding evolution in approximation schemes which will take advantage of these new resources. It is a thesis of this report that GN theory and, in particular, higher-level GN theory is appropriate for our time. Two developments lead to this conclusion: the emergence of low-cost, high-speed computation, and the emergence of sophisticated symbolic manipulation software which allows one to accurately perform calculus and algebraic manipulations on rather large systems of equations.

Approximation schemes can be separated into different categories. Perturbation methods, both ordinary and singular, introduce some mathematical approximation to reduce the complexity of the model to the point where it can be solved. One advantage of these methods is that one obtains governing equations for the flow and from these, both specific solutions can be obtained and generalizations of the behavior of the flow can be made. On the other end of the spectrum, the original problem can be solved by purely numerical techniques. Finite difference, finite element, and panel methods are such schemes. These methods are comparable to physical experiments in that each computation yields another result corresponding to a single realization of the flow. Generalization about the behavior of the flow requires induction from many of these specific solutions. GN fluid sheet theory lies in the middle of this spectrum. It achieves simplification by reducing the dimensionality from three dimensions to two. This theory yields governing equations for the flow, which are solved numerically in a more efficient manner than those from the three-dimensional model.

In perturbation analyses, reference scales appropriate for the particular problem at hand are introduced. These scales are used to nondimensionalize the variables and to identify a nondimensional perturbation parameter (or parameters) which can be considered small (or large). For time invariant problems, the flow is decomposed into a sequence of flows of presumably decreasing importance, each of which is a correction to the sum of the previously computed flows. The assumed sequence is inserted into the field equations and boundary conditions and the perturbation parameters are used to segregate these into a corresponding sequence of perturbation problems. Typically, each of these problems is linear in the unknowns at its level, although it may involve higher-order terms of quantities determined already in previous (lower-order) solutions.

An implicit assumption is made that this sequence is convergent, but this is almost never proven. In some flow problems, such as two-dimensional water waves in both shallow and deep water, there is ample evidence of the convergence (Schwartz 1974). In problems such as the flow about thin airfoils, the lack of convergence is well known. These methods, which date back to Stokes and other early researchers, are often called "rational methods" because the assumptions are clear and testable, and the details can be embodied into a mathematical process through which theoretically one can obtain solutions to whatever level of accuracy one chooses, <u>if</u> the perturbation sequence converges. A particular advantage of the perturbation approach is that, since perturbation parameters are used to size quantities, the ingredients of this parameter give one insight into the types of problems for which the approximation is appropriate. However, the perturbation approach does not yield quantitative measures of the accuracy to be expected for a particular problem. This information can only be obtained from an analysis of higher order problems or from comparison with experiments.

Unsteady fluid problems are rather different. It is usually not feasible to consider the flow as a sum of linear perturbation problems that one can solve sequentially until sufficient accuracy has been obtained, unless some additional limiting assumption such as periodic motion is introduced. Generally, one must solve a single set of governing equations in time. Perturbation methods have also been used, for instance, by Wu (1981) for the formulation of approximations appropriate for time-domain wave problems. For these problems, the introduction of scales permits grouping of terms of like size for a particular problem. One can obtain a variety of different sets of governing equations depending on the order of terms retained. That is, one can obtain a sequence of sets of governing equations for a given problem, each of which contains all of the terms of the previous sets plus those due to the retention of the next order of smaller terms. Presumably, this sequence of increasing complexity will produce solutions of increasing accuracy. Generally, all of these sets of governing equations will be nonlinear with the exception perhaps of the first. However, since we are throwing away parts of the exact problem, we can expect that some quantities, such as mass and momentum, may not exactly be conserved (although, if a consistent analysis has been performed, the errors should be of a size comparable to the first neglected order).

The limitations resulting from such an analysis can be subtle. Many perturbation schemes consider the fluid velocity to be a small perturbation to a reference velocity. The typical result is that any order of the perturbation theory is not Galilean invariant, since the terms which are needed to make it so are spread amongst several orders. One such example is the Korteweg de Vries equations for nonlinear, shallow-water flow.

In problems where viscosity is not important (or can be ignored) and the flow is initially quiescent, the field equation becomes Laplace's equation, which is linear. For these flows much of the focus of perturbation analysis is therefore on the boundary conditions. Both the kinematic boundary condition on material surfaces, as well as the dynamic boundary condition on the free surface, are nonlinear. Expansions of these nonlinear conditions and grouping of terms by orders of the perturbation parameter(s) lead to a basis for selecting or discarding terms. For these flows, then, the perturbation method solves a problem where the exact field equation is satisfied, but where the boundary conditions are satisfied only approximately. In some sense, the uniqueness of these flow situations stems solely from the imposed conditions at the boundaries and it is worrisome to concentrate the approximations there.

Green-Naghdi theory is, however, quite the opposite in nature. The boundary conditions are met exactly, but the field equations are approximated. In this approach the dependence of the kinematic structure of the solutions along one coordinate direction is prescribed. This direction is the vertical (or x^3) direction for the theories discussed in this paper. In many problems, such as shallow water problems, the depth of the fluid in the x^3 direction may be quite small; in others the fluid domain can be infinite in this direction. In either case the resulting theories are called fluid sheet theories. The assumed variation of the fluid velocity across x^3 will be expressed here as a finite sum of products. The first term in each product is a coefficient that depends on the remaining two horizontal coordinates (x^1 and x^2) and time, and the second term is a function of x^3 alone.

Approach

The governing equations for the Green-Naghdi theories presented here are composed of: an exact statement of the conservation of mass, an approximate statement of the conservation of momentum, and exact statements for the various boundary conditions. As a result of the formulation, x^3 no longer appears in the governing equations and all quantities therein are functions of x^1, x^2, and time. No scales are introduced and no terms are thrown out. Development of GN theory takes place in two steps: postulation of a set of governing equations, and verification that these equations satisfy certain physical requirements. The process presented below is one which is unique to this project, and although it closely follows research efforts in the subject area at the University of California, Berkeley, it is substantially different from that used by Green and Naghdi.

The first step is a procedure for identifying a candidate fluid sheet model. In fact, there is no preferred method for this identification and at this point the model could just as well have been induced from the results of model tests. The same variational approach of Kantorovich and Krylov (1958) used previously by Shields and Webster (1988) is used to derive the candidate model from standard three-dimensional equations. This approach is a variation of the method of weighted residuals, and is therefore similar in nature to the procedure used in the development of finite elements. In this procedure, the dimensionality of the system of partial differential equations is reduced, rather than the system being replaced by the system of algebraic equations, as it would be in a Galerkin procedure.

PART II: FORMULATION WITH GENERAL WEIGHT FUNCTIONS

Governing Equations

Let $\mathbf{x} = x^i$ ($i = 1,2,3$) be a system of fixed Cartesian coordinates in Euclidian space with base vectors $\mathbf{e}_i$, where $\mathbf{e}_3$ is oriented vertically upward. For convenience, x^3 is denoted by ζ in the subsequent development because this dimension plays a much different role than the other two dimensions. In the following standard Cartesian tensor notation is used, with the summation convention implied for repeated indices. In many instances, however, the summation will be stated explicitly for clarity. Latin indices are used for quantities having three spatial components and take on values of 1, 2, 3; Greek indices take on the values of 1 and 2 only. A comma in the subscript denotes differentiation by the following variable or that corresponding to the subsequent index.

The fluid velocity vector at a point $\mathbf{x}$ and time t is given by $\boldsymbol{v} = \boldsymbol{v}(\mathbf{x}, t) = v_i \mathbf{e}_i$. The fluid is assumed to be bounded by two smooth and non-intersecting material surfaces. The material surfaces are given by $\zeta = \alpha(x^1, x^2, t)$ and $\zeta = \beta(x^1, x^2, t)$, $\beta > \alpha$, respectively. Since α and β are material surfaces, on these surfaces the kinematic ("no leak") boundary condition is

$$\frac{D\alpha}{Dt} = \left[v_3 - \alpha_{,t} - v_\gamma \alpha_{,\gamma}\right]\Big|_{\zeta=\alpha} = 0$$
$$\frac{D\beta}{Dt} = \left[v_3 - \beta_{,t} - v_\gamma \beta_{,\gamma}\right]\Big|_{\zeta=\beta} = 0 \qquad (1)$$

This discussion will be concerned only with an incompressible ideal fluid, and in this case the stress vector $\mathbf{t} = -p\,\mathbf{e}_i$ and the mass density of the fluid ρ are assumed to be constant. The body force is given by $-\rho g \mathbf{e}_3$. (In a more general theory, t would have the form $t_i = \tau_{ij} n_j$, where τ_{ij} is the stress tensor, and the body force would be a general vector $\rho\mathbf{f}$.) Unit outward normal vectors on the top and the bottom surfaces are denoted by $\hat{\mathbf{n}}$ and $\bar{\mathbf{n}}$, respectively. The three-dimensional (Euler) equations resulting from the conservation laws of mass and momentum are

$$v_{i,i} = 0$$
$$\rho \boldsymbol{v}_{,t} + (\rho v_i \boldsymbol{v})_{,i} = -p_{,i}\boldsymbol{e}_i - \rho g \boldsymbol{e}_3 \qquad (2)$$

The fundamental kinematic assumption that the velocity field can be approximated is introduced as

$$\boldsymbol{v}(x^1,x^2,\zeta,t) = \sum_{n=0}^{K} \boldsymbol{W}_n(x^1,x^2,t)\,\lambda_n(\zeta) \qquad (3)$$

where

$$\boldsymbol{W}_n(x^1,x^2,t) = W_n^i(x^1,x^2,t)\,\boldsymbol{e}_i$$

$\lambda_n(\zeta)$ is a "shape" function that depends upon ζ only. The coefficients $\mathbf{W}_n$ are unknown time-dependent vectors to be determined as a part of the solution. The $\mathbf{W}_n$ correspond with the "directors" in the original work (Green and Naghdi 1986 and 1987). For each choice of K a complete, closed set of equations is developed that is independent from those for a different value of K. Thus, the kinematic models form a hierarchy depending on K and increasing in complexity with K. Since this hierarchy is different from a perturbation expansion, the terminology adopted is that suggested later (Shields and Webster 1988) to describe the complexity of the theory. A particular member of this hierarchy is referred to as the "K[th] level approximation."

The kinematic boundary conditions in Equation 1 may be rewritten using Equation 3 as

$$\sum_{n=0}^{K} W_n^3 \lambda_n(\alpha) = \alpha_{,t} + \sum_{n=0}^{K} W_n^\gamma \lambda_n(\alpha)\alpha_{,\gamma}$$
$$\sum_{n=0}^{K} W_n^3 \lambda_n(\beta) = \beta_{,t} + \sum_{n=0}^{K} W_n^\gamma \lambda_n(\beta)\beta_{,\gamma} \tag{4}$$

The continuity equation, Equation 2a, likewise becomes

$$\sum_{m=0}^{K} W_{m,\gamma}^\gamma \lambda_m + \sum_{m=0}^{K} W_m^3 \lambda_{m,\zeta} = 0 \tag{5}$$

In addition to the kinematic boundary conditions, 4K scalar equations are needed. K scalar equations derived from conservation of mass are chosen and K vector equations derived from conservation of momentum (corresponding to 3K scalar equations for three-dimensional problems) to provide closure for the fluid sheet theory. It is convenient at this point to restrict the weighting functions to those that possess the following property

$$\lambda_{m,\zeta} = \sum_{r=0}^{n} a_r^m \lambda_r \qquad (n \le m) \tag{6}$$

where the a_r are constants. The function set $\{\lambda_m\}$ is therefore a finite closed set under differentiation. Inserting Equation 6 into Equation 5, the continuity equation can be expressed as

$$\sum_{r=0}^{K} \left\{ W_{r,\gamma}^\gamma + \sum_{m=0}^{K} W_m^3 a_r^m \right\} \lambda_r(\zeta) = 0 \tag{7}$$

or, since the terms in braces are not a function of ζ

$$\left\{ W_{r,\gamma}^\gamma + \sum_{m=0}^{K} W_m^3 a_r^m \right\} = 0 \tag{8}$$

for r = 1,..., K .

Equation 8 is therefore an exact statement of conservation of mass for the flow given by the kinematic approximation, Equation 3. Note that if the derivative of the r^{th} weighting function with respect to ζ is not expressible in terms of the previous orders of the weighting function set, more than K conditions will result from this procedure and the Krylov-Kantorovich method described below could be used to determine approximate equations for the conservation of mass.

There are many function sets that satisfy Equation 6 (for instance, exponential functions where m = n). Polynomial functions also satisfy Equation 6, like those previously used for shallow-water fluid sheet theory. Similarly the sets $\{\sinh(a\zeta), \cosh(a\zeta)\}$ and $\{\sin(a\zeta), \cos(a\zeta)\}$, $a = a_0, a_1, \ldots, a_n$ also satisfy Equation 6.

If Equation 3 is substituted into momentum Equation 2b, and the resulting equation is required to be satisfied everywhere in the fluid domain (as for continuity), many more equations than the desired K vector equations would be obtained. This difficulty is due to the presence of the quadratic terms in Equation 2b. A weak formulation due to Kantorovich is employed where the "shape functions" λ_m are used as weighting functions to develop K approximate equations which express the conservation of momentum in some integral sense. Multiplying Equation 2b by each $\lambda_n(\zeta)$ and integrating through the vertical direction results in

$$\int_\alpha^\beta [(\rho\, v)_{,t} + (\rho\, v_i\, v)_{,i}]\, \lambda_n(\zeta) d\zeta = \int_\alpha^\beta [\, -p_{,i} e_i - \rho\, g\, e_3]\, \lambda_n(\zeta) d\zeta \qquad (9)$$

for n = 1,...,K.

Using the product rule of differentiation, and noticing that λ_n is not a function of t, and that γ ranges from 1 to 2, Equation 9 can be expressed as

$$\int_\alpha^\beta [(\,\rho\, v)_{,t} = (\rho v_i v)_{,i}]\, \lambda_n\, d\zeta$$

$$= \int_\alpha^\beta [\rho\, v\, \lambda_n]_{,t} d\zeta + \int_\alpha^\beta (\rho v_\gamma v)_{,\gamma} \lambda_n d\zeta + \int_\alpha^\beta (\rho v_3 v)_{,3} \lambda_n d\zeta$$

which further reduces to

$$= \int_\alpha^\beta [\rho\, v\, \lambda_n]_{,t} d\zeta + \int_\alpha^\beta [\rho\, v_\gamma\, v\, \lambda_n]_{,\gamma} d\zeta - \int_\alpha^\beta \rho\, v_\gamma\, v\, \lambda_{n,\gamma} d\zeta +$$

$$+ \int_\alpha^\beta [\rho\, v_3\, v\, \lambda_n]_{,3} d\zeta - \int_\alpha^\beta \rho\, v_3\, v\, \lambda_{n,3} d\zeta$$

It is noted that λ_n is only a function of ζ and this means that $\lambda_{n,\gamma} = 0$. With this relation and recognizing that the fourth integral is an exact integral, one obtains:

where $\lambda_n' = \partial\lambda_n/\partial\zeta$.

Similarly, the right-hand side of Equation 9 becomes (with application of Leibnitz' rule)

$$\int_\alpha^\beta [\,(\rho\, v)_{,t} + (\rho\, v_i\, v)_{,i}]\, \lambda_n d\zeta$$

$$= \int_\alpha^\beta [\rho\, v\, \lambda_n]_{,t} d\zeta + \int_\alpha^\beta [\rho\, v_\gamma\, v\, \lambda_n]_{,\gamma} d\zeta + \rho\, v_3\, v\, \lambda_n \Big|_{\zeta=\alpha}^{\zeta=\beta} - \int_\alpha^\beta \rho\, v_3\, v\, \lambda'_n d\zeta \tag{10}$$

$$\int_\alpha^\beta [-\, p_{,i} e_i - \rho\, g\, e_3]\, \lambda_n\, d\zeta$$

$$= -\, e_\gamma \int_\alpha^\beta p_{,\gamma}\, \lambda_n\, d\zeta - e_3 \int_\alpha^\beta p_{,3}\, \lambda_n\, d\zeta - \rho\, g\, e_3 \int_\alpha^\beta \lambda_n\, d\zeta$$

$$= -\, e_\gamma \int_\alpha^\beta [p\, \lambda_n]_{,\gamma}\, d\zeta - e_3\, p\, \lambda_n \Big|_{\zeta=\alpha}^{\zeta=\beta} + e_3 \int_\alpha^\beta p\, \lambda'_n d\zeta - \rho\, g\, e_3 \int_\alpha^\beta \lambda_n\, d\zeta \tag{11}$$

$$= -\, e_\gamma \Big(\int_\alpha^\beta p\, \lambda_n\, d\zeta \Big)_{,\gamma} + e_\gamma\, p\, \lambda_n \big|_{\zeta=\beta}\, \beta_{,\gamma} - e_\gamma\, p\, \lambda_n \big|_{\zeta=\alpha}\, \alpha_{,\gamma}$$

$$-\, p\, e_3\, \lambda_n \Big|_{\zeta=\alpha}^{\zeta=\beta} + e_3 \int_\alpha^\beta p\, \lambda'_n d\zeta - \rho\, g\, e_3 \int_\alpha^\beta \lambda_n\, d\zeta$$

Combining Equations 10 and 11 gives

$$\int_\alpha^\beta [\rho v \lambda_n]_{,t} d\zeta + \int_\alpha^\beta [\rho v_\gamma v \lambda_n]_{,\gamma} d\zeta - \int_\alpha^\beta \rho v_3 v \lambda'_n d\zeta + \rho v_3 v \lambda_n \Big|_{\zeta=\alpha}^{\zeta=\beta}$$

$$= \big(-P_{n,\gamma} + \hat{p}\lambda_n(\beta)\,\beta_{,\gamma} - \bar{p}\lambda_n(\alpha)\,\alpha_{,\gamma}\big) e_\gamma + \Big(P'_n - \rho g \int_\alpha^\beta \lambda_n d\zeta - \hat{p}\lambda_n(\beta) + \bar{p}\lambda_n(\alpha)\Big) e_3 \tag{12}$$

for n=0,...,K, where $\bar{p}$ and $\hat{p}$ are the pressures on the top and bottom surfaces respectively, and P_n and P_n' are the n[th] integrated pressures:

$$P_n = \int_\alpha^\beta p \lambda_n d\zeta \qquad\qquad P'_n = \int_\alpha^\beta p \lambda'_n d\zeta$$

The expression for the velocity field in Equation 3 is now inserted into the left-hand side of Equation 12:

$$\int_\alpha^\beta \rho \sum_{m=0}^{K} W_{m,t}\lambda_m\lambda_n d\zeta + \int_\alpha^\beta \rho\left(\sum_{r=0}^{K} W_r^\gamma \lambda_r \sum_{m=0}^{K} W_m\lambda_m\lambda_n\right)_{,\gamma} d\zeta$$

$$- \int_\alpha^\beta \rho \sum_{m=0}^{K} W_m\lambda_m \sum_{r=0}^{K} W_r^3\lambda_r\lambda_n' d\zeta + \sum_{m=0}^{K}\sum_{r=0}^{K} \rho\left(W_m W_r^3 \lambda_m\lambda_r\lambda_n\right)\Big|_{\zeta=\alpha}^{\zeta=\beta}$$

=

$$\sum_{m=0}^{K} \rho W_{m,t} \int_\alpha^\beta \lambda_m\lambda_n d\zeta + \sum_{m=0}^{K}\sum_{r=0}^{K} \rho\left(W_m W_r^\gamma\right)_{,\gamma} \int_\alpha^\beta \lambda_r\lambda_m\lambda_n d\zeta$$

$$- \sum_{m=0}^{K}\sum_{r=0}^{K} \rho\, W_m W_r^3 \int_\alpha^\beta \lambda_m\lambda_r\lambda_n' d\zeta + \sum_{m=0}^{K}\sum_{r=0}^{K} \rho\, W_m W_r^3 \left(\lambda_m\lambda_r\lambda_n\right)\Big|_{\zeta=\alpha}^{\zeta=\beta}$$

=

$$\sum_{m=0}^{K} \rho W_{m,t} y_{mn} + \sum_{m=0}^{K}\sum_{r=0}^{K} \rho\left(W_m W_r^\gamma\right)_{,\gamma} y_{mrn} - \sum_{m=0}^{K}\sum_{r=0}^{K} \rho\, W_m W_r^3\, y_{mr}^n + \sum_{m=0}^{K}\sum_{r=0}^{K} \rho\, W_m W_r^3 \left(\lambda_m\lambda_r\lambda_n\right)\Big|_{\zeta=\alpha}^{\zeta=\beta} \tag{13}$$

where

$$y_{mn} = \int_\alpha^\beta \lambda_m\lambda_n d\zeta\ , \quad y_{mrn} = \int_\alpha^\beta \lambda_m\lambda_r\lambda_n d\zeta\ , \quad y_{mr}^n = \int_\alpha^\beta \lambda_m\lambda_r\lambda_n' d\zeta \tag{13a}$$

Thus, the equation for fluid sheets (Equation 12) for an inviscid fluid can be written as

$$\sum_{m=0}^{K}\left\{\rho W_{m,t} y_{mn} + \sum_{r=0}^{K} \rho\left(W_m W_r^\gamma\right)_{,\gamma} y_{mrn} - \sum_{r=0}^{K} \rho\, W_m W_r^3\, y_{mr}^n\right\} + \sum_{m=0}^{K}\sum_{r=0}^{K} \rho\, W_m W_r^3 \left(\lambda_m\lambda_r\lambda_n\right)\Big|_{\zeta=\alpha}^{\zeta=\beta}$$

$$= \left(-P_n + \hat{p}\lambda_n(\beta)\beta_{,\gamma} - \bar{p}\lambda_n(\alpha)\alpha_{,\gamma}\right)e_\gamma + \left(P_n' - \rho g y_{n0} - \hat{p}\lambda_n(\beta) + \bar{p}\lambda_n(\alpha)\right)e_3 \tag{14}$$

for n=0,...,K.

Equation 14 can be reduced further. Prior to this reduction, some intermediate results should be recorded for later use. In the continuity equation, Equation 5, the dummy index m is changed to r, and after multiplying this equation with λ_m and λ_n, it should be summed over m (m=0,..,K).

This equation is integrated through the vertical direction to yield

$$\int_\alpha^\beta \sum_{m=0}^{K} \sum_{r=0}^{K} W_r^\gamma,_\gamma \lambda_r \lambda_m \lambda_n d\zeta + \int_\alpha^\beta \sum_{m=0}^{K} \sum_{r=0}^{K} W_r^3 \lambda_r{}' \lambda_m \lambda_n d\zeta = 0 \tag{15}$$

After interchange of the order of summation and integration and using Equation 13a, we have

$$\sum_{m=0}^{K} \sum_{r=0}^{K} W_{r,\gamma}^\gamma \, y_{mrn} + \sum_{m=0}^{K} \sum_{r=0}^{K} W_r^3 \, y_{mn}^r = 0 . \tag{16}$$

The left-hand side of Equation 14 is now considered and the chain rule is used for differentiation to expand the second term:

$LHS_{(14)}$ =

$$\begin{aligned} &\sum_{m=0}^{K} \left\{ \rho\, W_{m,t} y_{mn} + \sum_{r=0}^{K} \rho \; W_{m,\gamma} \, W_r^\gamma \, y_{mrn} + \sum_{r=0}^{K} \rho \; W_m \, W_{r,\gamma}^\gamma \, y_{mrn} \right. \\ &\left. - \sum_{r=0}^{K} \rho \; W_m \; W_r^3 \; y_{mr}^n \right\} + \sum_{m=0}^{K} \sum_{r=0}^{K} \rho \; W_m \; W_r^3 \; (\lambda_m \lambda_r \lambda_n) \Big|_{\zeta=\alpha}^{\zeta=\beta} \end{aligned} \tag{17}$$

After use of Equation 17 to replace the third term in Equation 18, the LHS of Equation 14 becomes

$$\begin{aligned} &= \sum_{m=0}^{K} \left\{ \rho\, W_{m,t} \, y_{mn} + \sum_{r=0}^{K} \rho \; W_{m,\gamma} \; W_r^\gamma \; y_{mrn} - \sum_{r=0}^{K} \rho \; W_m W_r^3 \; y_{mn}^r \right. \\ &\qquad \left. - \sum_{r=0}^{K} \rho \; W_m \; W_r^3 \; y_{mr}^n \right\} + \sum_{m=0}^{K} \sum_{r=0}^{K} \rho \; W_m \; W_r^3 \; (\lambda_m \lambda_r \lambda_n) \Big|_{\zeta=\alpha}^{\zeta=\beta} \\ &= \sum_{m=0}^{K} \left\{ \rho\, W_{m,t} \, y_{mn} + \sum_{r=0}^{K} \rho \; W_{m,\gamma} \; W_r^\gamma \; y_{mrn} - \sum_{r=0}^{K} \rho \; W_m \, W_r^3 \left[y_{mn}^r + y_{mr}^n \right] \right. \\ &\qquad \left. + \sum_{m=0}^{K} \sum_{r=0}^{K} \rho \; W_m W_r^3 \; (\lambda_m \lambda_r \lambda_n) \Big|_{\zeta=\alpha}^{\zeta=\beta} \right. . \end{aligned} \tag{18}$$

The value of $[y_{mn}{}^r + y_{mr}{}^n]$ can be determined by using integration by parts:

$$\left[y^r_{mn} + y^n_{mr}\right] = \int_\alpha^\beta \lambda_m \lambda_n \lambda_r' d\zeta + \int_\alpha^\beta \lambda_m \lambda_r \lambda_n' d\zeta = \int_\alpha^\beta \lambda_m (\lambda_n \lambda_r)' d\zeta$$
$$= \left(\lambda_m \lambda_r \lambda_n\right) \Big|_{\zeta=\alpha}^{\zeta=\beta} - \int_\alpha^\beta \lambda_n \lambda_r \lambda_m' d\zeta$$
$$= \left(\lambda_m \lambda_r \lambda_n\right) \Big|_{\zeta=\alpha}^{\zeta=\beta} - y^m_{rn} \,. \tag{19}$$

Inserting this identity into Equation 18 and using this to replace the left-hand side of Equation 14 yields

$$\sum_{m=0}^{K} \left\{ \rho\, W_{m,t}\, y_{mn} + \sum_{r=0}^{K} \rho\, W_{m,\gamma}\, W_r^\gamma\, y_{mrn} + \sum_{r=0}^{K} \rho\, W_m W_r^3 y^m_{rn} \right\} =$$
$$\left(-P_n + \hat{p}\lambda_n(\beta)\beta_{,\gamma} - \bar{p}\lambda_n(\alpha)\alpha_{,\gamma}\right) e_\gamma + \left(P_n' - \rho g y_{n0} - \hat{p}\lambda_n(\beta) + \bar{p}\lambda_n(\alpha)\right) e_3 \tag{20}$$

for n=0, ...,K.

The derivation of the general Green-Naghdi equations for an inviscid, incompressible fluid have now been completed . The governing equations for inviscid flow are then: the two kinematic boundary conditions in Equation 4, the K conservation of mass equations in Equation 8, and the K approximate conservation of momentum (vector) equations in Equations 14 or 20. Note that the conservation of momentum equations are therefore 3K scalar equations for three-dimensional flows. The variables include 3K unknown components of W_n, K integrated pressures P_n, and two conditions on the bounding surfaces. On the top surface either β or $\hat{p}$ is unknown, depending on the problem. Similarly, on the bottom surface either α or $\bar{p}$ is unknown. Thus, we have 4K+2 unknowns and the same number of equations, and the system is closed.

In some previous work and in our examples, a so-called "restricted" theory is used. In these theories the λ functions which are used are often polynomials, in which the last term W_K is restricted to have no components in the x^1 or x^2 directions. This situation will be discussed separately below.

We can make several observations about the results so far. The equations depend only on x^1, x^2 and t and do not have any explicit dependence on the variable ζ. After the initial assumption of the form of the velocity distribution in the z direction was made, no terms were thrown out. The governing equations, like the conservation laws from which they were derived, are Galilean invariant. Because no scale was introduced, there is no explicit flow situation for which this theory is most applicable. The governing equations derived this way are, to be sure, an approximation. However, the limits of this approximation are implicit and must be determined by numerical or physical experiment. Even for the lowest level theory, the governing equations are nonlinear because both the conservation of momentum laws and the boundary conditions are. These are:

Velocity profile:

$$v(x^1,x^2,\zeta,t) = \sum_{n=0}^{K} W_n(x^1,x^2,t)\ \lambda_n(\zeta) \tag{3}$$

where

$$W_n(x^1,x^2,t) = W_n^i(x^1,x^2,t)\, e_i$$

Kinematic boundary conditions:

$$\begin{aligned} \sum_{n=0}^{K} W_n^3 \lambda_n(\alpha) &= \alpha_{,t} + \sum_{n=0}^{K} W_n^\gamma \lambda_n(\alpha)\alpha_{,\gamma} \\ \sum_{n=0}^{K} W_n^3 \lambda_n(\beta) &= \beta_{,t} + \sum_{n=0}^{K} W_n^\gamma \lambda_n(\beta)\beta_{,\gamma} \end{aligned} \tag{4}$$

Conservation of mass:

$$\left\{ W_{r,\gamma}^\gamma + \sum_{m=0}^{K} W_m^3\, a_r^{\ m} \right\} = 0 \tag{8}$$

for r=0,...,K

Conservation of momentum:

$$\begin{aligned} &\sum_{m=0}^{K} \left\{ \rho\, W_{m,t}\, y_{mn} + \sum_{r=0}^{K} \rho\ W_{m,\gamma}\ W_r^\gamma\, y_{mrn} + \sum_{r=0}^{K} \rho\ W_m W_r^3\, y_{rn}^{\ m} \right\} \\ &= \\ &\left(-P_n + \hat{p}\lambda_n(\beta)\beta_{,\gamma} - \bar{p}\lambda_n(\alpha)\alpha_{,\gamma}\right)e_\gamma \\ &+ \left(P_n' - \rho g y_{n0} - \hat{p}\lambda_n(\beta) + \bar{p}\lambda_n(\alpha)\right) e_3 \end{aligned} \tag{20}$$

for n=0,...,K.

Because the governing equations are approximate, they do not exactly satisfy Kelvin's theorem and the flow computed from these equations does not remain irrotational. Recall that irrotationality is not an property of an inviscid fluid, but rather a consequence of an assumption that the fluid is initially quiescent and is acted upon by conservative forces. Shields and Webster (1989) showed that the n^{th} level shallow water fluid sheet theory did satisfy conservation of circulation in an average sense across the fluid domain and the flow does remain approximately irrotational in an initial value problem when the initial state was quiescent. However, the treatment of steady flow (time invariant) problems does require some additional specifications of the average circulation (or of the vorticity distribution).

The appearance of the momentum equations in Equation 20 is deceptively simple but note that there are two levels of implied summation from index repetition. Actual evaluation of the equations is sufficiently tedious that it is impractical to carry out any but the first one or two levels without the use of computer programs to perform the calculus and the algebra.

Discussion

The result of the derivation above has been to reduce the dimensionality of the three-dimensional equations to a set of two dimensional equations in x^1, x^2, and t. As such, these equations are reminiscent of equations for a membrane although, unlike a membrane, this "fluid sheet" has a much greater kinematic complexity. For instance, a membrane has only one kinematic variable, the location of the membrane for a given x^1 and x^2. The fluid sheet has vectors $\mathbf{W}_n$, one of which may be identified as the "location" of the sheet, but the others of which are clearly kinematical ingredients that have no counterpart as a membrane.

Using a development that is an analog of the development of the three-dimensional equations (for instance, the Navier-Stokes equations), Green and Naghdi developed a continuum model of two-dimensional sheets with kinematic complexity. This development specifies the general form to be expected with arbitrary complexity. In this development, the kinematic ingredients are called "directors" and the sheet is a "directed fluid sheet" or "Cosserat" surface.

In their treatment, Green and Naghdi regard a set of governing equations, such as those developed in the previous section as a postulated set of equations motivated by the three dimensional equations. The equations are to be validated by comparison with the general theory or, if required, modified to reflect the physical principals embodied in the general fluid sheet model. It is fortunate that an ideal, incompressible fluid has such a simple constitutive relation (its internal stresses are only pressures and these do not depend on the rate of strain of the fluid) that the variational procedure above does yield a set of governing equations that fits the mold provided by the GN fluid sheet model.

One of the distinct advantages of fluid sheet theory is that it always results in approximate governing equations for unsteady, three-dimensional flows. The specialization of these equations to either two dimensions or to steady flows presents no difficulty. However, many of the specialized numerical techniques used in fluid mechanics are developed with appeal to specialized methods which depend on the flow being steady or two-dimensional (or both), and are therefore limited in their applications.

Finite Depth Applications

In this section, a special case of the theory for shallow water is provided. In this case further reduction of the equations is possible due to the choice polynomial weighting functions. The equations for shallow-water are previously given by Green, Laws and Naghdi (1974) and Green and

Naghdi (1984, 1986 and 1987) and Shields and Webster (1988). In this chapter these equations are re-derived based on the general derivation above. The present work yields equations that are identical to those given by Green and Naghdi (1984 and 1986); that is, these equations can algebraically be transformed into those of Green and Naghdi. In this research, constraints of the restricted theory that existed in the earlier work of Shields and Webster have been removed, and a generalized set of equations for an arbitrary level of the Green and Naghdi theory is presented. The complete set of equations for the first two level theories for shallow-water applications will next be presented.

Equations for Shallow Water

With the choice of the polynomial weighting functions, $\lambda_n(\zeta) = \zeta^n$, it is possible to reduce the general equations further. In this case the various y_{mn}, y_{mrn}, etc. can be expressed using a single function:

$$H_n = \int_\alpha^\beta \zeta^n d\zeta = \frac{1}{n+1}\left(\beta^{n+1} - \alpha^{n+1}\right) \tag{21}$$

With the use of Equation 21, one obtains

$$\begin{aligned} y_{mn} &= H_{(m+n)} \\ y_{mrn} &= H_{(m+r+n)} \\ y_{rn}^{m} &= m H_{(m+r+n-1)} \end{aligned} \tag{22}$$

The velocity field is given by

$$v_\gamma = \sum_{n=0}^{K} W^\gamma{}_n \zeta^n , \qquad v_3 = \sum_{n=0}^{K} W^3{}_n \zeta^n \tag{23}$$

With the use of Equation 23, the equations for the kinematic boundary conditions and the conditions of the conservation of mass and momentum may be obtained. The kinematic boundary conditions are given by

$$\sum_{n=0}^{K} W^3{}_n \alpha^n = \alpha_{,t} + \sum_{n=0}^{K} W^\gamma{}_n \alpha^n \alpha_{,\gamma} \tag{24}$$

$$\sum_{n=0}^{K} W^3{}_n \beta^n = \beta_{,t} + \sum_{n=0}^{K} W^\gamma{}_n \beta^n \beta_{,\gamma} \tag{25}$$

The continuity equation becomes

$$\sum_{n=0}^{K} W^{\gamma}{}_{n,\gamma}\, \zeta^{n} + \sum_{n=0}^{K} W^{3}{}_{n} n \zeta^{n-1} = 0 \tag{26}$$

Separating the K^{th} term in the first summation and changing the index n to n+1 results in

$$W^{\gamma}{}_{K,\gamma}\, \zeta^{K} + \sum_{n=0}^{K-1} \left\{ W^{\gamma}{}_{n,\gamma} + (n+1)\, W^{3}{}_{n+1} \right\} \zeta^{n} = 0 \tag{27}$$

If Equation 27 is to hold everywhere, each coefficient of ζ^{n} must be set to zero as:

$$W^{\gamma}{}_{K,\gamma} = 0 \tag{28}$$

$$W^{\gamma}{}_{n,\gamma} + (n+1) W^{3}{}_{n+1} = 0 \qquad for\ n=0,1,\ldots,K-1 \tag{29}$$

Finally, the conditions of the conservation of momentum become

$$\sum_{m=0}^{K} \left\{ \rho\, W_{m,t}\, H_{(m+n)} + \sum_{r=0}^{K} \rho\, H_{(m+r+n)}\, W_{m,\gamma}\, W_{r}^{\gamma} + \sum_{r=0}^{K} \rho\, W_{m} W_{r}^{3}\, H_{(m+r+n-1)} \right\} =$$
$$\left(-P_{n} + \hat{p}\lambda_{n}(\beta)\, \beta_{,\gamma} - \bar{p}\lambda_{n}(\alpha)\, \alpha_{,\gamma}\right) e_{\gamma} + \left(P'_{n} - \rho\, g\, y_{n0} - \hat{p}\, \lambda_{n}(\beta) + \bar{p}\, \lambda_{n}(\alpha)\right) e_{3} \tag{30}$$

for n=0,...,K.

Equations 23 through 30 are equivalent to those previously given by Green and Naghdi (1984) and Shields and Webster (1988). It is noted that Equation 28 is the equation that is related to the so-called "restricted theory." The details of the discussion about the restricted theory will be given in the following sections.

PART III: IRROTATIONAL FLOW AND THE RESTRICTED THEORY

Discussion

When Green and Naghdi introduced their theory, they restricted the last component of the director so that it remains vertical at all times. In the theory of Green and Naghdi, specific constitutive equations are required for the 3-D response functions; that is, the terms on the right-hand side of Equation 20, which for a more general fluid can be considerably more complicated. These constitutive equations represent the material properties of the fluid and its particular geometry. Also the inertia coefficients y_{mn} and the relationship of the velocity fields $\mathbf{v}$ to the director velocities $\mathbf{W}_m$ need to be specified. Green and Naghdi chose the response functions so that the pressure is the only component that determines the mechanical power. The constraint responses are found such that the corresponding mechanical power is zero. They also assign force vectors to obtain proper responses. The above procedure is central to Green and Naghdi's approach and is what makes their theory self-consistent in its internal structure. Notably in the theory of plates and shells, their theory shows its self-consistency in its ability to satisfy both dynamic boundary conditions and kinematic boundary conditions. Many competing approaches used to form the three-dimensional equations ended up having inconsistencies, and more specifically, both boundary conditions were not satisfied at the same time. If one starts with the right kinematic conditions, one ended up with the wrong dynamic conditions, and vice versa. The approach of Green and Naghdi can model a general fluid by specifying its constitutive equation without any conflict.

Shields (1986) set $W_K^{\gamma} = 0$ (corresponding to a restricted director) in order to satisfy the continuity equation. This was necessary in order to obtain $K+1$ conditions from the continuity equation because of the transformation that had been introduced. Still, no clear meaning of the constraint is offered. It appears to exclude the solution which may be possible otherwise.

The condition with no constraint is investigated in this research. Originally the restricted theory meant the first level of the direct theory with a constrained director. This concept is now extended to the K^{th} level theory. It is called a restricted theory if the K^{th} components of the two-dimensional velocity components are constrained (corresponds to constraint of the K^{th} component of the directors). In this work it will be shown that this constraint has a simple meaning.

For a two-dimensional flow (Equation 28 in the previous section), the most general solution of Equation 28 becomes

$$W^1{}_K = constant \tag{31}$$

Consider now two-dimensional, steady periodic waves. In determining the wave celerity of steady periodic waves an additional assumption is needed in order to ensure that solutions are unique. Traditionally this is accomplished by either of two definitions of irrotational wave speed introduced by Stokes. Cokelet (1977) defined the circulation per unit length, C, by

$$C = \frac{1}{\lambda} \int_0^{\lambda} u \, dx \qquad (32)$$

where λ is a wave length and u is a horizontal component of the velocity. In the work of Cokelet (1977), Equation 32 is satisfied by the choice of reference frame which travels with the wave speed c. Because the flow is assumed irrotational, Equation 32 holds at every vertical location in the fluid. According to Stokes' theorem, the vertical gradient of the averaged horizontal velocity is zero if the flow is irrotational. The wave speed defined by Cokelet is then that according to Stokes' first definition. It is noted that Stokes' first definition is based on the prior assumption of an irrotational flow.

Stokes' first definition has been used by most researchers. However, in the direct theory no assumption of an irrotational flow is made a priori. In the direct theory it is more natural to adopt Stokes' second definition of the wave speed. However, to be consistent with previous work, Stokes' first definition is used here, and accordingly additional irrotational requirements are needed. They can be treated analogously in the direct theory. Equation 32 will be used for the condition of irrotationality. We recall that each weighting function represents different vertical dependence. Since there is no vertical gradient of the averaged horizontal velocity for an irrotational flow, we may obtain K+1 conditions for the requirements of an irrotational flow if Equation 32 is to be satisfied at any vertical location. These are:

$$\int_0^{\lambda} (W^1{}_0 + c) \, dx = 0 \qquad (33)$$

$$\int_0^{\lambda} W^1{}_j \, dx = 0 \qquad for\ j=1,2,...,K \qquad (34)$$

where λ is a wave length and c is the speed of a moving frame which is the same as the wave speed. Equation 33 is the definition of the wave celerity and is analogous to the first definition of Stokes. Equation 34 is an expression for global irrotational requirements, and may be used for the measure of the vorticity. If Equation 34 is not met, then the solution is not an irrotational solution in these integral senses. If Equation 34 is satisfied, then Equation 33, the definition of the wave speed, is independent of vertical location within the fluid field and is therefore equivalent to the first definition of Stokes.

We now go back to Equation 31. In order to satisfy Equation 34, it is necessary that $W_K{}^1 \equiv 0$. Consider for example, two-dimensional solitary waves. Since $W_K{}^1$ vanishes far upstream and downstream, $W_K{}^1$ becomes identically zero in this case. Recall that $W_K{}^1 = 0$ corresponds to the statement of the restricted theory. From this it becomes clear that the restriction is an implicit assumption of an irrotational flow. This is considered as a necessary condition for the irrotational flow although it is not a sufficient condition. Shields stated that Theory I does not admit shear flow solutions. This is not true in general, but it is true if one uses the restricted theory. It is possible to model waves with shear flow (or current) with Theory I which is unrestricted. Higher level theories are capable of modeling shear flow solution with the restricted theory.

It becomes clear that the restricted theory is needed in order to model shallow-water problems whose fluid field is considered as irrotational. In modeling rotational flow, either a higher level theory or the unrestricted first level theory may be used depending on the accuracy of the solution desired and other conditions, if necessary. From now on, the restricted theory will be used in shallow-water problems unless otherwise stated.

Two-Dimensional Unsteady Flow - Theory I (General Formulation)

In this section two-dimensional equations for a free surface flow over an even bottom are provided. This section presents a basic overview of the treatment of these equations. The notation will be changed from tensor notation to a component notation in which $x = x^1$ is the flow direction. But usage of ζ is retained to represent the vertical direction together with z for convenience. The coordinate system is taken so that the bottom is expressed by $\alpha(x,t) = 0$ with $\bar{p}$, the unknown pressure on the bottom. The upper surface $\beta(x,t)$ is a free-surface. On this surface, $\hat{p}$ needs to be specified. With $K=1$, Theory I equations for this specific case will next be presented.

In Theory I the velocity profile is given by

$$u = u_0 + u_1 \zeta \tag{35}$$

$$w = w_0 + w_1 \zeta \tag{36}$$

where u is the horizontal velocity component and w is the vertical velocity component. The kinematic boundary conditions are

$$w_0 = 0 \tag{37}$$

$$w_1 \beta = \beta_t + u_0 \beta_x \tag{38}$$

The continuity equation yields the following conditions:

$$u_1 = 0 \tag{39}$$

$$u_{0x} + w_1 = 0 \tag{40}$$

Equation 39 means that the restricted theory is used. The conditions of the conservation of momentum yield the following equations:

$$\beta u_{0t} + \beta u_0 u_{0x} = \hat{p}\frac{\beta_x}{\rho} - \frac{P_{0x}}{\rho} \tag{41}$$

$$\frac{1}{2}\beta^2 u_{0t} + \frac{1}{2}\beta^2 u_0 u_{0x} = \beta\beta_x \frac{\hat{p}}{\rho} - \frac{P_{1x}}{\rho} \tag{42}$$

$$\frac{1}{2}\beta^2 w_{1t} + \frac{1}{2}\beta^2 u_0 w_{1x} + \frac{1}{2}\beta^2 w_1^2 = -\frac{\hat{p}}{\rho} + \frac{\bar{p}}{\rho} - g\beta \tag{43}$$

$$\frac{1}{3}\beta^3 w_{1t} + \frac{1}{3}\beta^3 u_0 w_{1x} + \frac{1}{3}\beta^3 w_1^2 = -\frac{\hat{p}\beta}{\rho} + \frac{P_0}{\rho} - \frac{1}{2}g\beta^2 \tag{44}$$

Equations 41 and 42 are statements of conservation of the horizontal component of the momentum equations for $n=0$ and 1, respectively, and Equations 43 and 44 are similar statements of the vertical component of the momentum equation.

Because there are many unknown variables in this theory, it is convenient for computational purposes to reduce the system of equations to one with fewer unknowns and equations. The reduction is usually done by expressing all of the vertical components of the velocity in terms of the horizontal components of the velocity through the continuity equation and the kinematic boundary conditions. It is noted that this can be done for any level of the theory. Moreover, some terms are decoupled from other variables. For instance P_1 (P_K in the K^{th} level theory) is decoupled and occurs in only one place, Equation 42. Since there is no particular interest in this unknown, it may be possible to discard this variable. The parameter $\bar{p}$ occurs only in Equation 43, and therefore, this equation is used to express $\bar{p}$ and can be removed. If one wishes, it is always possible to compute $\bar{p}$ from the solutions obtained. Also P_0 (P_0, P_1,...,P_{k-1} for the K^{th} level theory) may be eliminated. As a result, the original set of eight equations may be reduced to a system with two equations subject to the variables β and u_0. For the second level theory it is possible to reduce the system to three equations with three unknowns. For steady flow problems one of the components of the horizontal velocity may be further reduced.

The number of unknowns is now reduced. As was mentioned above, the vertical components of velocity can be expressed in terms of the horizontal components of the velocity to give

$$w_0 = 0\ , \qquad w_1 = -u_{0x} \ . \tag{45}$$

From Equations 43 and 45, an expression for pressure on the bottom surface is obtained as

$$\frac{\bar{P}}{\rho} = -\frac{1}{2}\beta^2 u_{oxt} - \frac{1}{2}\beta^2 u_0 u_{0xx} + \frac{1}{2}\beta^2 u_{ox}^{\ 2} + \frac{\hat{p}}{\rho} + g\,\beta \tag{46}$$

It is noted that P_1 occurs only in Equation 42. Since there is no interest in P_1, Equation 42 will be omitted here. A reduced set of differential equations with three unknowns may thus be obtained. These three equations are:

$$\beta_t = -[u_0\,\beta]_x \tag{47}$$

$$\beta\, u_{0\,t} + \beta\, u_0\, u_{0\,x} = \hat{p}\,\frac{\beta_x}{\rho} - \frac{P_{0\,x}}{\rho\,\beta} \tag{48}$$

$$-\frac{1}{3}\beta^3 u_{0\,xt} - \frac{1}{3}\beta^3 u_0\, u_{0\,xx} + \frac{1}{3}\beta^3 u_{0\,x}^{\ 2} = -\hat{p}\,\frac{\beta}{\rho} + \frac{P_0}{\rho} - \frac{1}{2} g\,\beta^2 \tag{49}$$

P_0 may be eliminated using Equations 48 and 49 to yield the following two equations with two unknowns, β and u_0:

$$\beta_t = -[u_0\,\beta]_x$$

$$3\frac{\partial u_0}{\partial t} - 3\beta\frac{\partial\beta}{\partial x}\frac{\partial^2 u_0}{\partial x\partial t} - \beta^2\frac{\partial^3 u_0}{\partial x^2\partial t} = -3g\frac{\partial\beta}{\partial x} - 3u_0\frac{\partial u_0}{\partial x} - 3\beta\frac{\partial\beta}{\partial x}\left(\frac{\partial u_0}{\partial x}\right)^2$$
$$+\,3\beta\, u_0\frac{\partial\beta}{\partial x}\frac{\partial^2 u_0}{\partial x^2} - \beta^2\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2} + \beta^2 u_0\frac{\partial^3 u_0}{\partial x^3} \tag{50}$$

<u>Two-Dimensional Steady Flow - Theory I (General Formulation)</u>

In this section, equations for steady flow will be obtained by setting every time derivative equal to zero. Since there are three governing equations, Equations 47 through 49, and three unknowns (β, u_0, and P_1), these three equations will be used here. The pressure on the upper surface is assumed

$$\hat{p} = p_a - q \tag{51}$$

where

$$q = \frac{T \beta_{xx}}{\sqrt{(1+\beta_x^2)^3}} \tag{52}$$

The variable T represents the constant surface tension on the surface and p_a is the atmospheric pressure. Since the fluid is assumed incompressible, $p_a = 0$ without loss of generality for convenience.

Among these three equations, two can be integrated. Equation 50 can be integrated with respect to x to yield

$$u_0 \beta = Q \tag{53}$$

where Q is a constant of integration. A meaning of this constant can be easily interpreted.

The mass flux per unit span is expressed as

$$\int_0^\beta u \, d\zeta = \int_0^\beta u_0 \, d\zeta = u_0 \beta = Q \tag{54}$$

Therefore Q represents the mass flux per unit span.

Integration of Equations 48 and 49 yields

$$\beta u_0 u_{0x} = -\frac{P_{0x}}{\rho} - \frac{T \beta_x \beta_{xx}}{\rho \sqrt{(1+\beta_x^2)^3}} \tag{55}$$

$$\frac{1}{3} \beta^3 u_{0x}^2 - \frac{1}{3} \beta^3 u_0 u_{0xx} = \frac{P_0}{\rho} - \frac{1}{2} g \beta^2 + \frac{T \beta \beta_{xx}}{\rho \sqrt{(1+\beta_x^2)^3}} \tag{56}$$

The pressure on the bottom is determined by

$$\frac{\bar{P}}{\rho} = -\frac{1}{2} \beta^2 u_0 u_{0xx} + \frac{1}{2} \beta^2 u_{0x}^2 + \frac{T \beta_{xx}}{\rho \sqrt{(1+\beta_x^2)^3}} + g \beta \tag{57}$$

Equation 55 may be integrated with respect to x as

$$P_0/\rho = -Q u_0 + \frac{T}{\rho \sqrt{1+\beta_x^2}} + S \tag{58}$$

S is a constant of integration, and the meaning of this constant can be found. The momentum flux per unit span S^* is given by

$$S^* = \int_0^\beta [p/\rho + u^2]\, d\zeta = P_0/\rho + Q\, u_0 \tag{59}$$

The relation between S and S* is now obvious; S may be interpreted as the momentum flux per unit span less the momentum defect due to the effect of surface tension.

With the use of Equations 54 and 58, a single governing equation inclusive of β may be obtained from Equation 56:

$$\frac{1}{3} Q^2 \beta_{xx} - \frac{1}{3} Q^2 \frac{\beta_x^2}{\beta} + \frac{Q^2}{\beta} + \frac{1}{2} g\, \beta^2 - \frac{T}{\rho\sqrt{1+\beta_x^2}} - \frac{T\, \beta_{xx}}{\rho\, \sqrt{(1+\beta_x^2)^3}} + S = 0 \tag{60}$$

Equation 60 may be integrated once more after multiplying by β_x/β^2. After integrating, multiply by β^2 to obtain

$$\frac{1}{3} Q^2\, \beta_x^2 + g\, \beta^3 + 2\, R\, \beta^2 + 2\, S\, \beta + \frac{2\, T\, \beta}{\rho\sqrt{1+\beta_x^2}} - Q^2 = 0 \tag{61}$$

where R is another constant of integration. Solutions to these equations for different values of Q, R, and S correspond to regular waves and solitary waves. Such waves have been computed by various researchers using these equations (for instance, see Ertekin 1984).

Two-Dimensional Unsteady Flow - Theory II (Flat Bottom)

In this section, the Theory II equations are given. Since most of the procedures are explained in the previous section, the equations are simply listed.

Velocity profiles:

$$u = u_0 + u_1\, \zeta + u_2\, \zeta^2 \tag{62}$$

$$w = w_0 + w_1\, \zeta + w_2\, \zeta^2 \tag{63}$$

Kinematic boundary conditions:

$$w_0 = 0 \tag{64}$$

$$w_1\, \beta + w_2\, \beta^2 = \beta_t + u_0\, \beta_x + u_1\, \beta\, \beta_x \tag{65}$$

Continuity equation:

$$u_2 = 0 \qquad \textit{(restricted theory)} \tag{66}$$

$$u_{0x} + w_1 = 0 \tag{67}$$

$$u_{1x} + 2\, w_2 = 0 \tag{68}$$

Conservation of momentum:

n=0, x-component :

$$\beta\, u_{0t} + \frac{1}{2}\beta^2 u_{1t} + \frac{1}{3}\beta^3 u_1 u_{1x} + \frac{1}{2}\beta^2 u_0 u_{1x} + \frac{1}{2}\beta^2 u_1 u_{0x} + \beta\, u_0 u_{0x} + \frac{1}{2}\beta^2 u_1 w_1 + \frac{1}{3}\beta^3 u_1 w_2 = \beta_x \frac{\hat{p}}{\rho} - \frac{P_{0x}}{\rho} \tag{69}$$

n=1, x-component :

$$\frac{1}{2}\beta^2 u_{0t} + \frac{1}{3}\beta^3 u_{1t} + \frac{1}{4}\beta^4 u_1 u_{1x} + \frac{1}{3}\beta^3 u_1 u_{0x} + \frac{1}{2}\beta^2 u_0 u_{0x} + \frac{1}{4}\beta^4 u_1 w_2 + \frac{1}{3}\beta^3 u_1 w_1 = \beta\,\beta_x \frac{\hat{p}}{\rho} - \frac{P_{1x}}{\rho} \tag{70}$$

n=2, x-component :

$$\frac{1}{3}\beta^3 u_{0t} + \frac{1}{4}\beta^4 u_{1t} + \frac{1}{5}\beta^5 u_1 u_{1x} + \frac{1}{4}\beta^4 u_1 u_{0x} + \frac{1}{3}\beta^3 u_0 u_{0x} + \frac{1}{5}\beta^5 u_1 w_2 + \frac{1}{4}\beta^4 u_1 w_1 = \beta^2 \beta_x \frac{\hat{p}}{\rho} - \frac{P_{2x}}{\rho} \tag{71}$$

n=0, ζ-component :

$$\frac{1}{2}\beta^2 w_{1t} + \frac{1}{3}\beta^3 w_{2t} + \frac{1}{4}\beta^4 u_1 w_{2x} + \frac{1}{3}\beta^3 u_0 w_{2x} + \frac{1}{3}\beta^3 u_1 w_{1x} + \frac{1}{2} u_0 w_1 x$$
$$+ \frac{1}{2}\beta^4 w_2^2 + \beta^3 w_1 w_2 + \frac{1}{2}\beta^2 w_1^2 = -\hat{p} + \frac{\bar{p}}{\rho} - g\beta \qquad (72)$$

n=1, ζ-component :

$$\frac{1}{3}\beta^3 w_{1t} + \frac{1}{4}\beta^4 w_{2t} + \frac{1}{5}\beta^5 u_1 w_{2x} + \frac{1}{4}\beta^4 u_0 w_{2x} + \frac{1}{4}\beta^4 u_1 w_{1x} + \frac{1}{3}\beta^3 u_0 w_1$$
$$+ \frac{2}{5}\beta^5 w_2^2 + \frac{3}{4}\beta^4 w_1 w_2 + \frac{1}{3}\beta^3 w_1^2 = -\beta\hat{p} + \frac{P_0}{\rho} - \frac{1}{2} g\beta^2 \qquad (73)$$

n=2, ζ-component :

$$\frac{1}{4}\beta^4 w_{1t} + \frac{1}{5}\beta^5 w_{2t} + \frac{1}{6}\beta^6 u_1 w_{2x} + \frac{1}{5}\beta^5 u_0 w_{2x} + \frac{1}{5}\beta^5 u_1 w_{1x} + \frac{1}{4}\beta^4 u_0 w_1 x$$
$$+ \frac{1}{3}\beta^6 w_2^2 + \frac{3}{5}\beta^5 w_1 w_2 + \frac{1}{4}\beta^4 w_1^2 = -\beta^2 \frac{\hat{p}}{\rho} + 2\frac{P_1}{\rho} - \frac{1}{3} g\beta^3 \qquad (74)$$

There are a total of 11 equations to solve, Equations 64 to 74. The system of equations is reduced as follows. The variables w_0, w_1, and w_2 are eliminated using Equations 66-68. The expressions of the vertical components of the velocity are given by

$$w_0 = 0 \qquad w_1 = -u_{0x} \qquad w_2 = -\frac{1}{2} u_{1x} \qquad (75)$$

With Equation 75, five governing equations for β, u_0, u_1, P_0, and P_1 are written here:

$$\beta_t = -\left(u_0\beta + \frac{1}{2} u_1 \beta^2\right)_x \qquad (76)$$

$$\beta\, u_{0t} + \frac{1}{2}\beta^2 u_{1t} + \frac{1}{6}\beta^3 u_1 u_{1x} + \frac{1}{2}\beta^2 u_0 u_{1x} + \beta\, u_0 u_{0x} = \hat{p}\,\frac{\beta_x}{\rho} - \frac{P_{0x}}{\rho} \tag{77}$$

$$\frac{1}{2}\beta^2 u_{0t} + \frac{1}{3}\beta^3 u_{1t} + \frac{1}{8}\beta^4 u_1 u_{1x} + \frac{1}{3}\beta^3 u_0 u_{1x} + \frac{1}{2}\beta^2 u_0 u_{0x} = \beta\,\beta_x\,\hat{p}/\rho - P_{1x}/\rho \tag{78}$$

$$\begin{aligned} -\frac{1}{3}\beta^3 u_{0xt} - \frac{1}{8}\beta^4 u_{1xt} - \frac{1}{10}\beta^5 u_1 u_{1xx} - \frac{1}{8}\beta^4 u_0 u_{1xx} + \frac{1}{10}\beta^5 u_{1x}^2 \\ + \frac{3}{8}\beta^4 u_{0x} u_{1x} - \frac{1}{4}\beta^4 u_1\; u_{0xx} - \frac{1}{3}\beta^3 u_0 u_{0xx} + \frac{1}{3}\beta^3 u_{0x}^2 \\ = -\beta\,\frac{\hat{p}}{\rho} + \frac{P_0}{\rho} - \frac{1}{2}g\,\beta^2 \end{aligned} \tag{79}$$

$$\begin{aligned} -\frac{1}{4}\beta^4 u_{0xt} - \frac{1}{10}\beta^5 u_{1xt} - \frac{1}{12}\beta^6 u_1 u_{1xx} - \frac{1}{10}\beta^5 u_0 u_{1xx} + \frac{1}{12}\beta^6 u_{1x}^2 \\ + \frac{3}{10}\beta^5 u_{0x} u_{1x} - \frac{1}{5}\beta^5 u_1\; u_{0xx} - \frac{1}{4}\beta^4 u_0 u_{0xx} + \frac{1}{4}\beta^4 u_{0x}^2 \\ = -\beta^2\,\frac{\hat{p}}{\rho} + 2\,\frac{P_1}{\rho} - \frac{1}{3}g\,\beta^3 \end{aligned} \tag{80}$$

It is noted that P_0 and P_1 may be eliminated. The result is the following set of three equations in u_0, u_1, and β, and assuming in this case that the surface tension is zero and thus $\hat{p}$ is zero:

$$\beta_t = -\left(u_0\,\beta + \frac{1}{2}u_1\,\beta^2\right)_x \tag{81}$$

$$
\begin{aligned}
&- 120\frac{\partial u_0}{\partial t} - 60\ \beta\frac{\partial u_1}{\partial t} + 120\beta\frac{\partial \beta}{\partial x}\frac{\partial^2 u_0}{\partial x \partial t} + 60\ \beta^2\frac{\partial \beta}{\partial x}\frac{\partial^2 u_1}{\partial x \partial t} + 40\ \beta^2\frac{\partial^3 u_0}{\partial x^2 \partial t} + 15\ \beta^3\frac{\partial^3 u_1}{\partial x^2 \partial t} \\
&= 120\ g\frac{\partial \beta}{\partial x} + 120\ u_0\frac{\partial u_0}{\partial x} + 120\ \beta\frac{\partial \beta}{\partial x}\frac{\partial u_{0_2}}{\partial x} + 20\ \beta\ (3\ u_0 + \beta\ u_1)\frac{\partial u_1}{\partial x} \\
&+ 180\ \beta^2\frac{\partial \beta}{\partial x}\frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x} + 60\ \beta^3\frac{\partial \beta}{\partial x}\frac{\partial u_{1_2}}{\partial x} - 120\ \beta(u_0 + \beta\ u_1)\frac{\partial \beta}{\partial x}\frac{\partial^2 u_0}{\partial x^2} + 40\ \beta^2\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2} \\
&+ 15\ \beta^3\frac{\partial u_1}{\partial x}\frac{\partial^2 u_0}{\partial x^2} - 60\ \beta^2(u_0 + \beta\ u_1)\frac{\partial \beta}{\partial x}\frac{\partial^2 u_1}{\partial x^2} + 30\ \beta^3\frac{\partial u_0}{\partial x}\frac{\partial^2 u_1}{\partial x^2} + 12\ \beta^4\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2} \\
&- 10\ \beta^2(4\ u_0 + 3\ \beta\ u_1)\frac{\partial^3 u_0}{\partial x^3} - 3\ \beta^3(5\ u_0 + 4\ \beta\ u_1)\frac{\partial^3 u_1}{\partial x^3}\ ,\ \textit{and}
\end{aligned}
\tag{82}
$$

$$
\begin{aligned}
&- 60\frac{\partial u_0}{\partial t} - 40\ \beta\frac{\partial u_1}{\partial t} + 60\beta\frac{\partial \beta}{\partial x}\frac{\partial^2 u_0}{\partial x \partial t} + 30\ \beta^2\frac{\partial \beta}{\partial x}\frac{\partial^2 u_1}{\partial x \partial t} + 15\ \beta^2\frac{\partial^3 u_0}{\partial x^2 \partial t} + 6\ \beta^3\frac{\partial^3 u_1}{\partial x^2 \partial t} \\
&= 60\ g\frac{\partial \beta}{\partial x} + 60\ u_0\frac{\partial u_0}{\partial x} + 60\ \beta\frac{\partial \beta}{\partial x}\frac{\partial u_{0_2}}{\partial x} + 5\ \beta\ (8\ u_0 + 3\beta\ u_1)\frac{\partial u_1}{\partial x} \\
&+ 90\ \beta^2\frac{\partial \beta}{\partial x}\frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x} + 30\ \beta^3\frac{\partial \beta}{\partial x}\frac{\partial u_{1_2}}{\partial x} - 60\ \beta(u_0 + \beta\ u_1)\frac{\partial \beta}{\partial x}\frac{\partial^2 u_0}{\partial x^2} + 15\ \beta^2\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2} \\
&+ 6\ \beta^3\frac{\partial u_1}{\partial x}\frac{\partial^2 u_0}{\partial x^2} - 30\ \beta^2(u_0 + \beta\ u_1)\frac{\partial \beta}{\partial x}\frac{\partial^2 u_1}{\partial x^2} + 12\ \beta^3\frac{\partial u_0}{\partial x}\frac{\partial^2 u_1}{\partial x^2} + 5\ \beta^4\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2} \\
&- 3\ \beta^2(5\ u_0 + 4\ \beta\ u_1)\frac{\partial^3 u_0}{\partial x^3} - \beta^3(6\ u_0 + 5\ \beta\ u_1)\frac{\partial^3 u_1}{\partial x^3}
\end{aligned}
\tag{83}
$$

Two-Dimensional Steady Flow - Theory II (Flat Bottom)

In this section, the second level of the theory for two-dimensional steady flow with surface tension is provided. For this case, surface tension is allowed and the pressure on the upper surface is described in Equations 51 and 52. All of the time derivatives in the governing equations, Equations 76 -80 are taken to be zero.

Equation 76 may be integrated to yield

$$
u_0\ \beta + \frac{1}{2}\ u_1\ \beta^2 = Q \tag{84}
$$

As before, Q may be interpreted as the mass flux per unit span. One additional term is included in Equation 84 compared with Equation 53. This shows the difference between the second level theory and the first level theory. The x-component of the momentum equation, Equation 77 can be integrated with the use of Equation 84 to yield

$$P_0/\rho = \frac{1}{12}\beta^3 u_1^2 - Q^2\frac{1}{\beta} + \frac{T}{\rho\sqrt{1+\beta_x^{\,2}}} + S \tag{85}$$

Again, S may be interpreted as the momentum flux per unit span less the momentum defect due to the effect of surface tension.

From Equation 72, $\bar{p}$ may be determined if other variables are known. As there is no interest in the variable P_2 in this research, the corresponding equation is omitted. Then it is possible to eliminate u_0 and P_0 using Equations 81 and 82 to obtain three equations in three unknowns, β, u_1, and P_1. These are:

$$\frac{T\,\beta\,\beta_x\,\beta_{xx}}{\rho\sqrt{(1+\beta_x^{\,2})^3}} + \frac{1}{12}\,\beta^4\,u_1\,u_{1x} + \frac{1}{12}\,Q\,\beta^2\,u_{1x} + \frac{1}{8}\,\beta^3\,\beta_x\,u_1^{\,2} - \frac{1}{2}\,Q^2\frac{\beta_x}{\beta} + \frac{P_{1x}}{\rho} = 0 \tag{86}$$

$$\left\{\frac{1}{24}\,\beta^4\,u_1^{\,2} + \frac{1}{4}\,Q\,\beta^2\,u_1 + \frac{1}{3}\,Q^2 - \frac{T\beta}{\rho\sqrt{(1+\beta_x^{\,2})^3}}\right\}\beta_{xx} + \left\{\frac{1}{120}\,\beta^5\,u_1 + \frac{1}{24}\,Q\,\beta^3\right\}u_{1xx}$$

$$- \frac{1}{240}\,\beta^5\,u_{1x}^2 + \frac{1}{16}\,\beta^4\,\beta_x\,u_1\,u_{1x} + \frac{7}{24}\,\beta^2\,\beta_x\,u_{1x} + \frac{1}{12}\,\beta^3\,\beta_x^{\,2}\,u_1^{\,2} + \frac{1}{6}\,Q\,\beta\,u_{1x}\,\beta_x^{\,2} \tag{87}$$

$$- \frac{1}{3}\,Q^2\,\frac{\beta_x^{\,2}}{\beta} + \frac{T}{\rho\sqrt{1+\beta_x^{\,2}}} - \frac{1}{12}\,\beta^3\,u_1^{\,2} + \frac{1}{2}g\,\beta^2 + Q\frac{1}{\beta} - S = 0$$

$$\left\{\frac{3}{80}\,\beta^5\,u_1^{\,2} + \frac{1}{5}Q\beta^3\,u_1 + \frac{1}{4}Q^2\,\beta - \frac{T\,\beta^2}{\rho\sqrt{(1+\beta_x^{\,2})^3}}\right\}\beta_{xx} + \frac{1}{240}\beta^6\,u_1 + \frac{1}{40}Q\,\beta^4\}\,u_{1xx}$$

$$- \frac{1}{240}\beta^6\,u_{1x}^{\,2} + \frac{1}{20}\beta^5\,\beta_x\,u_1\,u_{1x} + \frac{1}{5}Q\,\beta^3\,\beta_x\,u_{1x} + \frac{1}{16}\beta^4\,u_1^{\,2}\,\beta_x^{\,2} + \frac{1}{10}Q\,\beta^2\,u_1\,\beta_x^{\,2} \tag{88}$$

$$- \frac{1}{4}Q^2\,\beta_x^{\,2} - 2\frac{P_1}{\rho} + \frac{1}{3}g\,\beta^3 = 0$$

Further, P_1 may be eliminated between Equations 87 and 88, but for these steady flow equations there is no particular advantage in doing so. Shields and Webster (1988) present solutions to these equations for both solitary waves and for large-amplitude regular waves; their results were very satisfactory.

Two-Dimensional Unsteady Flow - Theory II (Uneven Bottom)

The governing equations for the situation of an unsteady flow over an uneven bed are extremely difficult to derive or present. As above, the equations can be reduced to three equations in u_0, u_1, and β. The governing equations were obtained using an algebraic manipulation program, Mathematica™. Since these equations will form the foundation for the research performed herein, a presentation of them in usual mathematical form is made below. These equations have a new (time-independent) variable, α, the vertical coordinate of the bottom. The solution involves derivatives of α and it is assumed that these derivatives through the third derivative in x are bounded. The set of applicable equations so obtained for an uneven seabed are listed next for completeness. These equations, excluding the surface tension effects, establish the basis of the numerical code in the implementation of mathematical theory.

Uneven bottom governing Equation 1:

$$-2\frac{\partial\beta}{\partial t} = -2(\alpha - \beta)\frac{\partial u_o}{\partial x} - 2\frac{\partial\alpha}{\partial x}(u_0 + \alpha\, u_1) + 2\frac{\partial\beta}{\partial x}(u_0 + \beta\, u_1) - (\alpha - \beta)(\alpha + \beta)\frac{\partial_{u_1}}{\partial x} \tag{89}$$

Uneven bottom governing Equation 2:

$$\begin{aligned}
&-120\,(\alpha - \beta)\frac{\partial u_0}{\partial t} + 6-\frac{\partial^2\alpha}{\partial x^2}(\alpha - \beta)^2\frac{\partial u_0}{\partial t} - 120\,\frac{\partial\alpha}{\partial x}(\alpha - \beta)\frac{\partial\beta}{\partial x}\frac{\partial u_0}{\partial t} \\
&+ 120\frac{\partial\alpha}{\partial x}(\alpha - \beta)^2\frac{\partial^2 u_0}{\partial t\partial x} - 120\,(\alpha - \beta)^2\frac{\partial\beta}{\partial x}\frac{\partial^2 u_0}{\partial t\partial x} + 40\,(\alpha - \beta)^3\frac{\partial^3 u_0}{\partial t\partial x^2} \\
&+ 60\,\frac{\partial\alpha^2}{\partial x}(\alpha - \beta)^2\frac{\partial u_1}{\partial t} + 60\,\alpha\frac{\partial^2\alpha}{\partial x^2}(\alpha - \beta)^2\frac{\partial u_1}{\partial t} - 60\,(\alpha - \beta)(\alpha + \beta)\frac{\partial u_1}{\partial t} \\
&- 120\,\alpha\frac{\partial\alpha}{\partial x}(\alpha - \beta)\frac{\partial\beta}{\partial x}\frac{\partial u_1}{\partial t} + 120\,\alpha\frac{\partial\alpha}{\partial x}(\alpha - \beta)^2\frac{\partial^2 u_1}{\partial t\partial x} \\
&- 60\,(\alpha - \beta)^2(\alpha + \beta)\frac{\partial\beta}{\partial x}\frac{\partial^2 u_1}{\partial t\partial x} + 5\,(\alpha - \beta)^3(5\,\alpha + 3\,\beta)\frac{\partial^3 u_1}{\partial t\partial x^2} =
\end{aligned} \tag{90}$$

$$120\,(\alpha-\beta)\frac{\partial\beta}{\partial x}g - 120\,(\alpha-\beta)^2\,\frac{\partial\beta}{\partial x}\frac{\partial u_0{}^2}{\partial x} + 40\,(\alpha-\beta)^3\frac{\partial u_0}{\partial x}\,\frac{\partial^2 u_0}{\partial x^2}$$

$$+\,120\,(\alpha-\beta)\,\frac{\partial u_0}{\partial x}\,(u_0+\alpha u_1) + 120\,\frac{\partial\alpha}{\partial x}\,(\alpha-\beta)\;u_1\,(u_0+\alpha\,u_1)$$

$$-\,120\,\frac{\partial\alpha}{\partial x}(\alpha-\beta)\,\frac{\partial\beta}{\partial x}\frac{\partial u_0}{\partial x}(-u_0+u_1\,(\alpha-2\beta)) - 120\frac{\partial\alpha}{\partial x}(\alpha-\beta)^2\,\frac{\partial^2 u_0}{\partial x^2}(u_0+\beta\,u_1)$$

$$+\,120(\alpha-\beta)^2\,\frac{\partial\beta}{\partial x}\,\frac{\partial^2 u_0}{\partial x^2}(u_0+\beta\,u_1) - 120\frac{\partial\alpha^2}{\partial x}(-\alpha+\beta)\frac{\partial\beta}{\partial x}u_1\,(u_0+\beta\,u_1)$$

$$+\,120\,\frac{\partial^2\alpha}{\partial x^2}(\alpha-\beta)\,\frac{\partial\beta}{\partial x}\,(u_0+\alpha\,u_1)\,(u_0+\beta\,u_1)$$

$$-\left(60\frac{\partial^2\alpha}{\partial x^2}\,\frac{\partial u_0}{\partial x} + 60\frac{\partial\alpha}{\partial x}\,\frac{\partial^2\alpha}{\partial x^2}u_1 + 20\frac{\partial^3\alpha}{\partial x^3}(u_0+\alpha\,u_1)\right)(\alpha-\beta)^2(3\,u_0+(\alpha+2\beta)u_1)$$

$$-10(\alpha-\beta)^3\frac{\partial^3 u_0}{\partial x^3}(4u_0+(\alpha+3\beta)\,u_1) - 60(\alpha-\beta)^2(\alpha+3\beta)\frac{\partial\beta}{\partial x}\,\frac{\partial u_0}{\partial x}\,\frac{\partial u_1}{\partial x}$$

$$+\,5(\alpha-\beta)^3\,(5\,\alpha+3\,\beta)\,\frac{\partial^2 u_0}{\partial x^2}\,\frac{\partial u_1}{\partial x} + 120\frac{\partial\alpha}{\partial x}(\alpha-\beta)\,\frac{\partial\beta}{\partial x}\,((2\,\alpha-\beta)\,u_0+\alpha\,\beta\,u_1)\,\frac{\partial u_1}{\partial x}$$

$$-\left(60\frac{\partial\alpha^2}{\partial x} + 60\,\alpha\,\frac{\partial^2\alpha}{\partial x^2}\right)(\alpha-\beta)^2\,(3\,u_0+(\alpha+2\,\beta)\,u_1)\,\frac{\partial u_1}{\partial x}$$

$$+\,20(\alpha-\beta)\,(3(\alpha+\beta)\,u_0+(4\,\alpha^2+\alpha\,\beta+\beta^2)u_1)\,\frac{\partial u_1}{\partial x}$$

$$- 60(\alpha - \beta)^2(\alpha + \beta)\beta \frac{\partial \beta}{\partial x}\frac{\partial u_1}{\partial x}^2 + 10(\alpha - \beta)^3(\alpha + 3\beta)\frac{\partial u_0}{\partial x}\frac{\partial^2 u_1}{\partial x^2}$$

$$+ 60(\alpha - \beta)^2(\alpha + \beta)\frac{\partial \beta}{\partial x}(u_0 + \beta u_1)\frac{\partial^2 u_1}{\partial x^2}$$

$$- 40\frac{\partial \alpha}{\partial x}(\alpha - \beta)^2((4\alpha - \beta)u_0 + \alpha(\alpha + 2\beta)u_1)\frac{\partial^2 u_1}{\partial x^2}$$

$$+ (\alpha - \beta)^3(7\alpha^2 + 21\alpha\beta + 12\beta^2)\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2}$$

$$- (\alpha - \beta)^3(5(5\alpha + 3\beta)u_0 + (7\alpha^2 + 21\alpha\beta + 12\beta^2)u_1)\frac{\partial^3 u_1}{\partial x^3}$$

Uneven Bottom Governing Equation 3:

$$- 60(\alpha - \beta)(\alpha + \beta)\frac{\partial u_0}{\partial t} + 20\frac{\partial^2 \alpha}{\partial x^2}(\alpha - \beta)^2(2\alpha + \beta)\frac{\partial u_0}{\partial t} \quad (91)$$

$$- 60\frac{\partial \alpha}{\partial x}(\alpha - \beta)(\alpha + \beta)\frac{\partial \beta}{\partial x}\frac{\partial^2 u_0}{\partial t} + 40\frac{\partial \alpha}{\partial x}(\alpha - \beta)^2(2\alpha + \beta)\frac{\partial^2 u_0}{\partial t\partial x}$$

$$- 60(\alpha - \beta)^2(\alpha + \beta)\frac{\partial \beta}{\partial x}\frac{\partial^2 u_0}{\partial t\partial x} + 5(\alpha - \beta)^3(5\alpha + 3\beta)\frac{\partial^3 u_0}{\partial t\partial x^2}$$

$$+ 20\frac{\partial \alpha^2}{\partial x}(\alpha - \beta)^2(2\alpha + \beta)\frac{\partial u_1}{\partial t} + 20\alpha\frac{\partial^2 \alpha}{\partial x^2}(\alpha - \beta)^2(2\alpha + \beta)\frac{\partial u_1}{\partial t}$$

$$- 40 (\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2)\frac{\partial u_1}{\partial t} - 60\alpha \frac{\partial \alpha}{\partial x} (\alpha - \beta) (\alpha + \beta)\frac{\partial \beta}{\partial x}\frac{\partial u_1}{\partial t}$$

$$+ 40\alpha\frac{\partial \alpha}{\partial x} (\alpha - \beta)^2(2\alpha + \beta) \frac{\partial^2 u_1}{\partial t \partial x} - 30 (\alpha - \beta)^2 (\alpha + \beta)^2 \frac{\partial \beta}{\partial x}\frac{\partial^2 u_1}{\partial t \partial x}$$

$$+ 2 (\alpha - \beta)^3 (8\alpha^2 + 9\alpha\beta + 3\beta^2)\frac{\partial^3 u_1}{\partial t \partial x^2} =$$

$$60(\alpha - \beta)(\alpha + \beta)\frac{\partial \beta}{\partial x}g - 60(\alpha - \beta)^2(\alpha + \beta)\frac{\partial \beta}{\partial x}\frac{\partial {u_0}^2}{\partial x}$$

$$+ 5(\alpha - \beta)^3 (5\alpha + 3\beta) \frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2} + 60 (\alpha - \beta)(\alpha + \beta) \frac{\partial u_0}{\partial x}(u_0 + \alpha u_1)$$

$$+ 60\frac{\partial \alpha}{\partial x}(\alpha - \beta)(\alpha + \beta) u_1 (u_0 + \alpha u_1)$$

$$- \left(15\frac{\partial^2 \alpha}{\partial x^2}\frac{\partial u_0}{\partial x} + 15\frac{\partial \alpha}{\partial x}\frac{\partial^2 \alpha}{\partial x^2} u_1 + 5\frac{\partial^3 \alpha}{\partial x^3}(u_0 + \alpha u_1)\right).$$

$$(\alpha - \beta)^2(4(2\alpha + \beta)u_0 + 3(\alpha + \beta)^2 u_1)$$

$$- 60 \frac{\partial \alpha}{\partial x} (\alpha - \beta) (\alpha + \beta) \frac{\partial \beta}{\partial x}\frac{\partial u_0}{\partial x} (- u_0 + (\alpha - 2\beta)u_1)$$

$$+ 60(\alpha - \beta)^2(\alpha + \beta)\frac{\partial \beta}{\partial x}\frac{\partial^2 u_0}{\partial x^2}(u_0 + \beta u_1) + 60 \frac{\partial \alpha^2}{\partial x} (\alpha + \beta) (\alpha - \beta)\frac{\partial \beta}{\partial x}u_1 (u_0 + \beta u_1)$$

$$+ 60\frac{\partial^2 \alpha}{\partial x^2}(\alpha - \beta)(\alpha + \beta)\frac{\partial \beta}{\partial x}(u_0 + \alpha u_1)(u_0 + \beta u_1)$$

$$- 5\frac{\partial \alpha}{\partial x}(\alpha - \beta)^2\frac{\partial^2 u_0}{\partial x^2}(8(2\alpha + \beta)u_0 + (\alpha^2 + 14\alpha\beta + 9\beta^2)u_1)$$

$$- (\alpha - \beta)^3 \frac{\partial^3 u_0}{\partial x^3}(5(5\alpha + 3\beta)u_0 + (7\alpha^2 + 21\alpha\beta + 12\beta^2)u_1)$$

$$- 30(\alpha - \beta)^2 (\alpha + \beta) (\alpha + 3\beta) \frac{\partial\beta}{\partial x} \frac{\partial u_0}{\partial x} \frac{\partial u_1}{\partial x} + 2(\alpha - \beta)^3 (8\alpha^2 + 9\alpha\beta + 3\beta^2) \frac{\partial^2 u_0}{\partial x^2} \frac{\partial u_1}{\partial x}$$

$$- 15 \frac{\partial\alpha^2}{\partial x} (\alpha - \beta)^2 (4 (2\alpha + \beta) u_0 + 3(\alpha + \beta)^2 u_1) \frac{\partial u_1}{\partial x}$$

$$- 15\alpha \frac{\partial^2\alpha}{\partial x^2} (\alpha - \beta)^2 (4 (2\alpha + \beta) u_0 + 3(\alpha + \beta)^2 u_1) \frac{\partial u_1}{\partial x}$$

$$+ 60 \frac{\partial\alpha}{\partial x} (\alpha - \beta) (\alpha + \beta) \frac{\partial\beta}{\partial x} ((2\alpha - \beta) u_0 + \alpha\beta u_1) \frac{\partial u_1}{\partial x}$$

$$+ 5 (\alpha-\beta) (8 (\alpha^2+\alpha\beta+\beta^2) u_0+3 (\alpha+\beta)(3 \alpha^2+\beta^2) u_1) \frac{\partial u_1}{\partial x}$$

$$- 30(\alpha - \beta)^2(\alpha + \beta)^2 \beta \frac{\partial\beta}{\partial x} \frac{\partial u_1}{\partial x}^2 + (\alpha - \beta)^3(7\alpha^2 + 21\alpha\beta + 12\beta^2) \frac{\partial u_0}{\partial x} \frac{\partial^2 u_1}{\partial x^2}$$

$$- 15 \frac{\partial\alpha}{\partial x} (\alpha - \beta)^2 ((7\alpha^2 + 2\alpha\beta - \beta^2) u_0 + 2\alpha (\alpha + \beta)^2 u_1) \frac{\partial^2 u_1}{\partial x^2}$$

$$+ 30(\alpha - \beta)^2(\alpha + \beta)^2 \frac{\partial\beta}{\partial x} (u_0 + \beta u_1) \frac{\partial^2 u_1}{\partial x^2} + 5(\alpha - \beta)^3(\alpha + \beta)^3 \frac{\partial u_1}{\partial x} \frac{\partial^2 u_1}{\partial x^2}$$

$$- (\alpha - \beta)^3 (2(8\alpha^2 + 9\alpha\beta + 3\beta^2) u_0 + 5(\alpha + \beta)^3 u_1) \frac{\partial^3 u_1}{\partial x^3}$$

PART IV: SOLUTION ALGORITHM

Integration of evolution equations, such as Equations 81 - 83 is now considered. These equations are very complex from an algebraic standpoint, but in fact may be integrated with little difficulty. Together they form a system of three coupled, partial differential equations that are first-order in time and third-order in space. Further, they generally have boundary conditions at both ends of the domain (two-point boundary conditions). The key to an efficient scheme for their solution lies in the fact that the highest-order mixed derivatives are only first order in time and second-order in space. The scheme presented below has been developed to handle similar problems and has been used extensively to integrate evolution equations for deep-water waves (Webster and Kim 1990). There is no specific limit to three governing equations; in fact, any number is allowable, so long as there are sufficient boundary conditions.

Ertekin (1984) devised a scheme for solving similar equations and boundary conditions corresponding to Theory I. His analysis was simplified since there were only two governing equations, and these were not coupled in the time derivatives of the variables. The equations are identical to Equation 50. The first equation could be integrated directly. The second equation, implicit in the spatial coordinate, was expressed as a tri-diagonal system of linear equations and solved using the Thomas algorithm.

The Theory II equations, Equations 81 - (83, are considerably more complex than those of Theory I. Moreover, the last two equations, Equations 82 and 83, have the disadvantage of being coupled in the time derivatives of the variables. However, it is still possible to use essentially the same efficient scheme for their solution. The current procedure results from a hybridization of Ertekin's method and the scheme devised by Newman (1968) for coupled ordinary differential equations.

In the interest of future work with the theory, this algorithm will herein be described in general terms. In particular, consideration will not be limited to the three equations at hand, but instead a more general system of equations will be considered which would be applicable to a higher order approximation of the theory.

A system of K coupled, quasi-linear partial differential equations will be considered in the K-dependent variables. The variables are expressed as a K-dimensioned vector, $\xi(x, t)$, and the equations have the special form:

$$\boldsymbol{A}\,\dot{\xi} + \boldsymbol{B}\,\dot{\xi}_{,x} + \boldsymbol{C}\,\dot{\xi}_{,xx} = \boldsymbol{g} \tag{92}$$

where **A**, **B**, and **C** are K x K matrices, **g** is a K-dimensioned vector. **A**, **B**, **C**, and **g** are perhaps functions of x and ξ and its spatial derivatives, although this dependence will not be shown in the interest of simplicity. The dot over ξ signifies a derivative with respect to time. It is assumed that the problem is posed as a two-point boundary-value problem in x and an initial-value problem in t.

The domain of x over which a solution to the equations is desired is assumed to be a uniform grid of x's spaced a distance Δx apart. The i-th point on the grid will be denoted by $x_i = i\,\Delta x$, i = 1,ns. Time is also assumed to be discretized with intervals Δt, with $t_j = j\,\Delta t$. The value of the solution vector $\xi(x_i,t_j)$ will be denoted by $\xi^{(i,j)}$, and similar superscripts will be used for the other vectors and matrices. The spatial derivatives will be approximated by central differences

$$\dot{\xi}_{,x}^{(i,j)} = \left(\dot{\xi}^{(i+1,j)} - \dot{\xi}^{(i-1,j)}\right) /2\ \Delta x$$
$$\dot{\xi}_{,xx}^{(i,j)} = \left(\dot{\xi}^{(i+1,j)} - 2\ \xi^{(i,j)} + \xi^{(i-1,j)}\right) /\Delta x^2 \tag{93}$$

With these approximations, Equation 92 can be written as

$$\tilde{A}^{(i,j)}\ \dot{\xi}^{(i-1,j)} + \tilde{B}^{(i,j)}\ \dot{\xi}^{(i,j)} + \tilde{D}^{(i,j)}\ \dot{\xi}^{(i+1,j)} = g^{(i,j)} \tag{94}$$

where

$$\tilde{A}^{(i,j)} = \left(\frac{C^{(i,j)}}{\Delta x^2} - \frac{B^{(i,j)}}{2\Delta x} \right)$$

$$\tilde{B}^{(i,j)} = \left(-2\ \frac{C^{(i,j)}}{\Delta x^2} + A^{(i,j)} \right) \tag{95}$$

$$\tilde{D}^{(i,j)} = \left(\frac{C^{(i,j)}}{\Delta x^2} + \frac{B^{(i,j)}}{2\Delta x} \right)$$

Suppose that the solution $\dot{\xi}^{(0,j)} = \dot{\xi}_0^{(0)}$ is known as the result of a boundary condition at this point. Then the solution at $i = 1$ can be readily found from Equation 93 as

$$\dot{\xi}^{1,k} = \left[\tilde{B}^{(1,j)}\right]^{-1} \left\{\left[g^{(1,j)} - \tilde{A}^{(1,j)}\ \dot{\xi}_0^{(0)}\right] - \tilde{D}^{(1,j)}\ \dot{\xi}^{(2,j)}\right\}$$
$$= \dot{\xi}_0^{(1)} + \tilde{E}^{(1,j)}\ \dot{\xi}^{(2,j)} \tag{96}$$

where

$$\dot{\xi}_0^{(1)} = [\tilde{B}^{(1,j)}]^{-1}\ [g^{(1,j)} - \tilde{A}^{(1,j)}\ \dot{\xi}_0^{(0)}]\ , \quad and$$
$$\tilde{E}^{(1,j)} = -\ [B^{(1,j)}]^{-1}\ D^{(1,j)} \tag{97}$$

This process is continued throughout the domain. That is,

$$\dot{\xi}^{(i,j)} = \dot{\xi}_0^{(i)} + \tilde{E}^{(i,j)}\, \dot{\xi}^{(i+1,j)}$$

$$\dot{\xi}_0^{(i)} = \left[\tilde{B}^{(i,j)}\right]^{-1} \left[g^{(i,j)} - \tilde{A}^{(i,j)}\, \dot{\xi}_0^{(i-1)}\right] \tag{98}$$

$$\tilde{E}^{(i,j)} = -\left[\tilde{B}^{(i,j)}\right]^{-1} \tilde{D}^{(i,j)}$$

At the boundary i = ns it is assumed that $\dot{\xi}^{(ns+1,j)}$ is known and given as a boundary condition at this point. Then

$$\dot{\xi}^{(ns,j)} = \left[\tilde{B}^{(ns,j)} + \tilde{A}^{(ns,j)}\, \tilde{E}^{(ns-1,j)}\right]^{-1} \left\{g^{(ns,j)} - \tilde{A}^{(ns,j)}\, \dot{\xi}_0^{(ns-1)} - \tilde{D}^{(i,j)}\, \dot{\xi}^{(ns+1)}\right\} \tag{99}$$

With the value of $\dot{\xi}^{(ns,j)}$ thus determined, the other values of $\dot{\xi}^{(i,j)}$ can be determined by back substitution

$$\dot{\xi}^{(i-1,j)} = \dot{\xi}_0^{(i-1,j)} + \tilde{E}^{(i-1,j)}\, \dot{\xi}^{(i,j)} \tag{100}$$

The values of $\dot{\xi}^{(i,j)}$ are then used to estimate the values of $\xi^{(i,j+1)}$ (i.e. at the next time step) by

$$\xi^{(i,j+1)} = \xi^{(i,j)} + \dot{\xi}^{(i,j)}\, \Delta t \tag{101}$$

This estimate is only first-order accurate and would be unsatisfactory. However, we can obtain an estimate for $\dot{\xi}^{(i,j+1)}$ by using these new values and reapplying the procedure at t = (j+1) Δt. A new estimate for $\xi^{(i,j+1)}$ can be formed by

$$\xi^{(i,j+1)} = \xi^{(i,j)} + \left(\dot{\xi}^{(i,j)} + \dot{\xi}^{(i,j+1)}\right) \frac{\Delta t}{2} \tag{102}$$

This new estimate is now second-order accurate in both Δt and Δx.

PART V: NUMERICAL IMPLEMENTATION

A separate report in this series will present a detailed description of the Fortran program based on the shallow-water Level II Green-Naghdi theory developed herein. This program was developed primarily for the use of Corps of Engineers projects associated with military and Civil Works and is far superior in performance and in programming style to the programs used by Shields (1986) and Shields and Webster (1988) for the previous shallow-water work. The program was specifically written to be generally applicable to two-dimensional solutions of all sorts of GN theory water wave problems.

The flow chart included herein (Figure 1) describes the general modular structure of this numerical model. The part of the program which is unique to the particular level of the theory (and type of GN theory for deep- or shallow-water) is the subroutine called **coeff**(j,neq). The coeff subroutine in the program corresponds to Equations 89 - 91 of this report (Level II shallow-water evolution equations); this subroutine is complicated and critical. The program was checked for accuracy, and preliminary computations using it have been performed.

The program consists of a main routine which provides for the input, output and the flow of the information. The principal subroutines are **solve**(neq), a coding of the Thomas algorithm presented above, **coeff**(j,neq) which includes the GN equations, **invmat**(a,b,neq,n), a standard linear equation inversion routine using Gauss-Jordan elimination, and **filter**, a digital smoothing filter to remove the spurious ripples near the wavemaker. A routine **bottom** determines the various spatial derivatives of the bottom topography required in the theory. Several small routines to perform standard vector and matrix operations are included to make the program self-sufficient.

For the shallow-water wave study, it is assumed that the time history of the waves is known at the left-hand boundary of the domain. The waves are input not only as a local wave height history, $\beta(t)$ but also as a history of the corresponding values of the other variables in this Level II theory [$u_0(t)$ and $u_1(t)$]. These variables are obtained from the solution to the steady flow equations (linearized for small wave amplitude) and only roughly represent the flow which corresponds to steep waves. These linear solutions are (for waves proceeding from left to right with a celerity c):

$$\begin{aligned} \beta(t) &= \beta_0 cos[kx - \omega t] = \beta_0 cos[k(x - ct)] \\ u_0 &= \beta_0 \frac{12g[20 + 7(kh)^2]}{c[240 + 104(kh)^2 + 3(kh)^4]} cos[\omega t] \\ u_1 &= \beta_0 \frac{120g(kh)^2}{ch[240 + 104(kh)^2 + 3(kh)^4]} cos[\omega t] \\ c &= \sqrt{\frac{24gh[(kh)^2 + 10]}{[240 + 104(kh)^2 + 3(kh)^4]}} \end{aligned} \tag{103}$$

and where β_0 is the wave amplitude and h is the water depth. In that these linear, small-amplitude waves are not exact boundary conditions for finite amplitude waves, there are some small oscillations near the wave maker as the solution "finds itself." This is similar to the flow near a flap-type wave maker in a physical wave tank, since the boundary condition at the flap is only roughly like a free water wave. These numerical ripples cause problems if allowed to propagate and are thus filtered out near the wave maker. Free waves elsewhere are not filtered. The historical wavemaker input values are placed sequentially in the first three elements of the variable arrays, so that spatial derivatives can be formed. An Orlansky free boundary condition is placed at the other end of the domain.

The equations are integrated and the results are dumped into several files as they become available. Appropriate output for use at CERC may be tailored to the desired user needs in a given application. An important feature of this program is that it has re-start capability and the program is arranged to be able to continue using previous computations. This feature will be particularly good if one is using a microcomputer.

PART VI: CONCLUDING REMARKS

This report details the development and some of the philosophy behind the Green-Naghdi theory of fluid sheets. The fundamental principals of the Level I and Level II Green-Naghdi theories are described for completeness. Derivation of both the Level I and Level II theories have been presented in a systematic manner due to complexities of these theories and high level of mathematics. The powerful symbolic manipulator Mathematica™ was used for the formulation and derivation of equations. A modular Fortran program has been developed for shallow-water Level I and II theories. The 2-D numerical model developed in this research is applicable to solutions of all Green-Naghdi theory water wave problems. The program allows a general description of the bottom contours and of boundary conditions at the right-hand side of the domain (either open or reflective).

It is recommended that the Level III Green-Naghdi theory be developed in the next phase of this research for coastal applications. The Level I and II theories presented herein can be easily extended to 3-D flows, although the equations will be considerably more complex. The 3-D Green-Naghdi equations may for all practical purposes be beyond human capabilities, but these are easily manageable with the use of mathematical and symbolic manipulators such as Mathematica™.

A 3-D Green-Naghdi theory will be better suited for military applications, including the Logistics-Over-The-Shore operations and coastal wave problems, since it can precisely describe effects of amplitude nonlinearities, frequency dispersion, refraction, shoaling, reflection, and diffraction of waves propagating over arbitrary water depths, seabed topography, and non-uniform boundaries. The problems of wave breaking and re-formation may realistically be represented only by a 3-D time-dependent Green-Naghdi theory. The solution algorithm for 3-D dimensional flows will have to be redeveloped to accommodate the new domain and several design problems in coastal applications, both for military and civil works.

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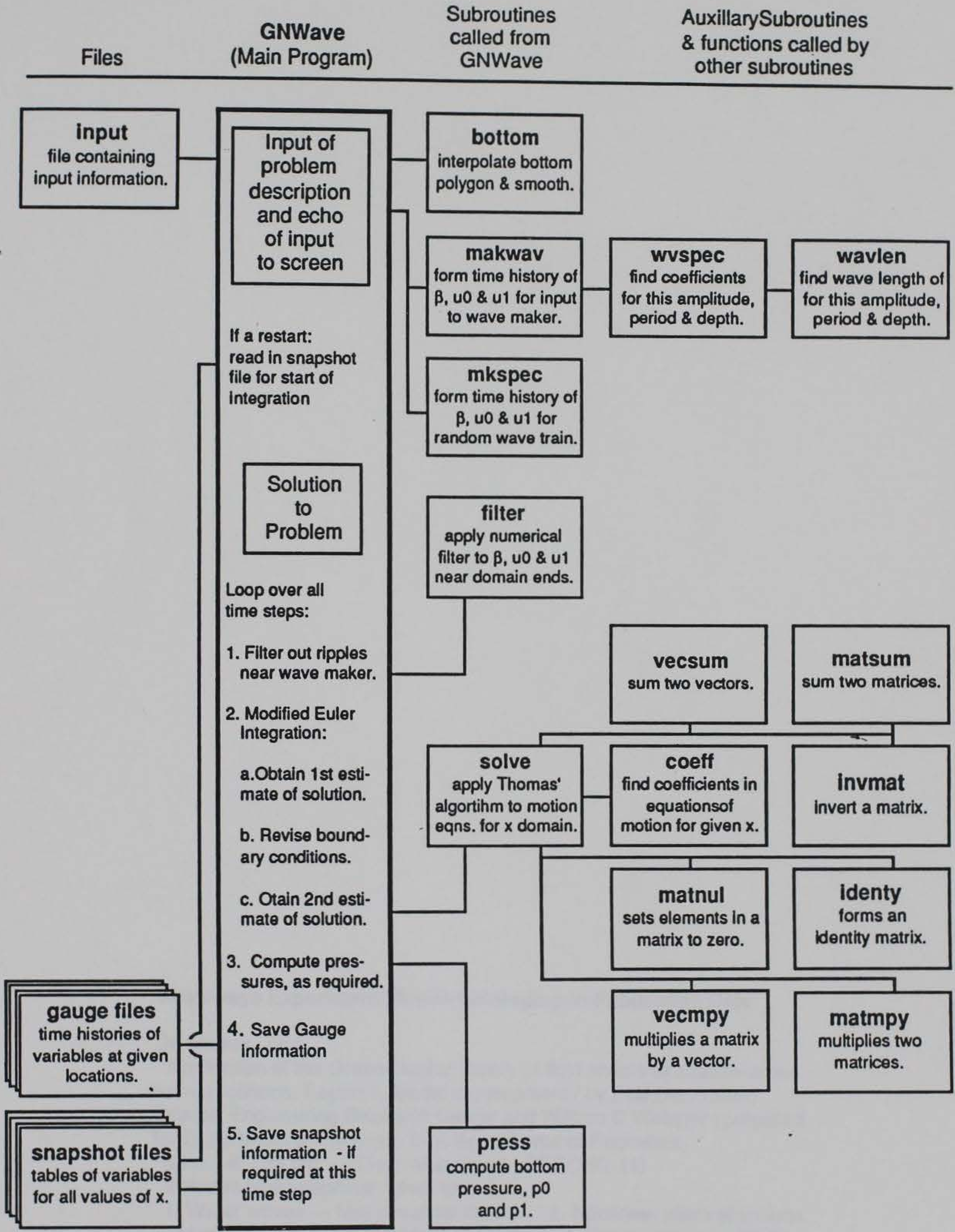

Figure 1. Program level flow chart for GNWAVE