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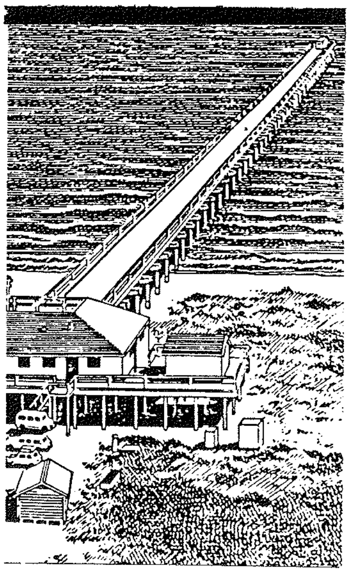

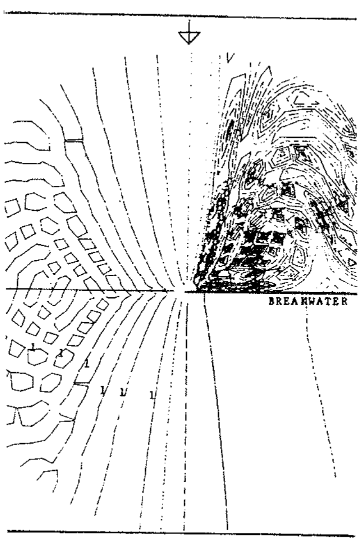

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COMBINED REFLECTION AND DIFFRACTION BY A VERTICAL WEDGE

by

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This report presents an analytical solution for the combined wave reflection and diffraction by a vertical wedge, of arbitrary wedge angle, subject to the excitation of a plane simple harmonic wave train coming from a far field. Results of two special cases are calculated: one a thin semi-infinite breakwater and the other a wedge of 90-deg angle. The results are presented in amplification factor diagrams. A subroutine WEDGE is also documented in the appendix. The subroutine can be used to calculate the case of arbitrary wedge angle.

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PREFACE

The work in this report was authorized by the Office, Chief of Engineers (OCE), Coastal Engineering Functional Area of Civil Works Research and Development, under Waves at Entrances Work Unit 31673, Harbor Entrances and Coastal Channels Program, at the Coastal Engineering Research Center (CERC) of the US Army Engineer Waterways Experiment Station (WES). Messrs. John H. Lockhart, Jr., and John G. Housley were OCE Technical Monitors. Dr. Charles L. Vincent is CERC Program Manager.

This report was prepared by Dr. H. S. Chen, Coastal Oceanography Branch (CR-O), Research Division (CR). Work was performed under direct supervision of Dr. Edward F. Thompson, Chief, CR-O, and Mr. H. Lee Butler, Chief, CR; and under general supervision of Dr. James R. Houston and Mr. Charles C. Calhoun, Jr., Chief and Assistant Chief, CERC, respectively.

This study was initiated after a discussion by Dr. Vincent and the author on the possibility of implementing a scheme to redistribute wave energy behind islands in numerical models. The author acknowledges and appreciates the review and comments provided by Drs. Edward F. Thompson and Norman W. Scheffner. This report was edited by Ms. Shirley A. J. Hanshaw, Information Products Division, Information Technology Laboratory, WES.

Commander and Director of WES is COL Dwayne G. Lee, CE, and Technical Director is Dr. Robert W. Whalin.

CONTENTS

COMBINED REFLECTION AND DIFFRACTION BY A VERTICAL WEDGE

PART I: INTRODUCTION

1. The boundary value problem of linear wave reflection and diffraction by a vertical wedge of arbitrary wedge angle has been well formulated and presented by Stoker (1957) among many other investigators. The technique to obtain an analytical solution for the problem is also depicted in the cited book. However, analytical solutions are not available for the problem, except for the special case of wave diffraction by a thin semi-infinite breakwater, that is, a wedge with wedge angle equal to zero.

2. The solution of the thin semi-infinite breakwater was presented in the dimensionless diffraction diagrams by Wiegel (1962). The diagrams have been especially useful in preliminary engineering design and have been included in the Shore Protection Manual (SPM) (1984). Although equally useful, the combined reflection and diffraction diagrams are not available, perhaps because of the complexity of the diagrams which makes them difficult to create without using modern high-speed computers for computation and graphing.

3. The objectives of the present study are (a) to obtain an analytical solution for the combined wave reflection and diffraction by a vertical wedge of arbitrary wedge angle subject to excitation of a plane simple harmonic wave train coming from infinity and (b) to provide the combined reflection and diffraction diagrams. The diagrams included in this report have two cases: one for a thin semi-infinite breakwater and the other for a 90-deg vertical wedge. Subroutine WEDGE for computing the combined reflection and diffraction by a vertical wedge of arbitrary wedge angle is also documented in the report (Appendix A).

PART II: BOUNDARY VALUE PROBLEM

Mathematical Formulation

4. In this study our primary interest is the wave reflection and diffraction by a vertical wedge of arbitrary wedge angle in a constant water depth h * subject to the excitation of monochromatic incident waves of infinitesimal amplitude coming from infinity. Let (r,θ,z) be cylindrical coordinates, with $z = 0$ representing the undisturbed water free surface and upward direction representing the positive z-axis. The tip of the wedge is chosen to be the origin of the coordinates and two rigid walls of the wedge to coincide with $\theta = 0$ and $\theta = \theta_o$, respectively, as illustrated in Figure 1. Cartesian coordinates (x,y,z), corresponding to the cylindrical coordinates, are also occasionally used and shown in the same figure. Therefore, the wedge

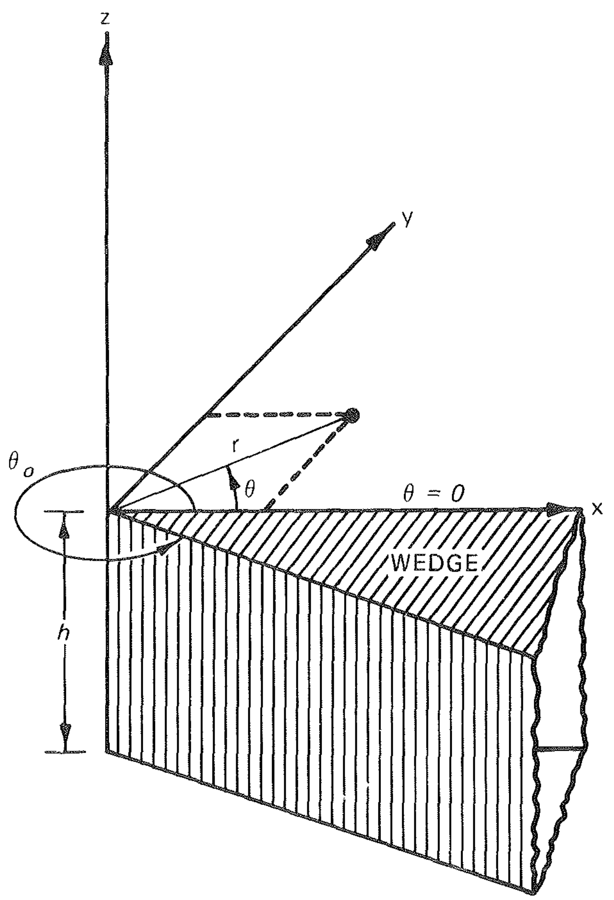

Figure 1. A vertical wedge of arbitrary wedge angle

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* For convenience, symbols and abbreviations are listed in the notation (Appendix B).

angle is $2\pi - \theta_o$, and the water region is defined by $\theta_o \geq \theta \geq 0$ and $0 \geq z \geq -h$.

5. The velocity field for the wave reflection and diffraction in an ideal fluid can be represented by the velocity potential function $\Phi(r,\theta,z,t)$ which must satisfy the Laplace equation, where t is the temporal coordinate. We assume that the waves are sinusoidal in time with radian frequency ω. Water depth is constant, and the bottom is rigid and impermeable. Therefore, the vertical and temporal components of the velocity potential function, which follow from separation of variables, can be factored out and the velocity potential written as

$$\Phi(r,\theta,z,t) = A_o \frac{\cosh k(z + h)}{\cosh kh} \phi(r,\theta)e^{i\omega t} \tag{1}$$

where

$A_o = -iga_o/\omega$
$i = \sqrt{-1}$
g = gravitational acceleration
a_o = incident wave amplitude
k = wave number
ϕ = horizontal component of the velocity potential function

6. Substituting Equation 1 into the Laplace equation and using both the kinematic and dynamic boundary conditions at the free surface, the Laplace equation is then reduced to the Helmholtz equation which is written in polar coordinates as follows:

$$r^2 \frac{\partial^2\phi}{\partial r^2} + r \frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial\theta^2} + k^2r^2\phi = 0 \tag{2}$$

where k must be a real number and satisfy the dispersion relationship

$$\omega^2 = gk \tanh kh \tag{3}$$

7. The free surface displacement η from the mean water level $z = 0$ can be obtained from linear wave theory and is represented as

$$\eta(r,\theta,t) = \frac{1}{g} \frac{\partial\Phi}{\partial t} = a_o\phi(r,\theta)e^{i\omega t} \tag{4}$$

8. Thus only the horizontal part of the velocity potential function ϕ is needed to be determined as a solution of Equation 2 in the water region $\theta_o \geq \theta \geq 0$, with the following boundary conditions at the rigid and impermeable walls of the wedge:

$$\frac{\partial \phi}{\partial \theta} = 0 \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta_o \tag{5}$$

9. A condition at infinity is also required to ensure a unique solution. The classic approach is to use the Sommerfeld radiation condition at infinity which states that the scattered wave ϕ_s must behave like a cylindrical outgoing progressing wave at infinity such that '

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial \phi_s}{\partial r} + ik\phi_s \right) = 0 \tag{6}$$

The total wave represented by ϕ is the linear superposition of an incident wave ϕ_i, a reflected wave from the the $\theta = 0$ wall of the wedge ϕ_r, and the scattered wave ϕ_s from the tip of the wedge.

$$\phi = \phi_i + \phi_r + \phi_s \tag{7}$$

Equation 6 can be satisfied if

$$\phi_s \sim \frac{e^{-ikr}}{\sqrt{kr}} \quad \text{at} \quad r \to \infty \tag{8}$$

10. The incident wave coming from a large distance from the tip of the wedge is assumed to be a plane progressive wave of amplitude a_o and incident angle α to the x-axis as given by

$$\phi_i = e^{ikr \cos (\theta - \alpha)} \tag{9}$$

Consequently, the perfectly reflected wave from the $y = 0$ wall of the wedge is

$$\phi_r = e^{ikr \cos (\theta + \alpha)} \tag{10}$$

Thus the boundary value problem (in which the governing equation is Equation 2, the boundary condition is Equation 5, and the radiation condition is Equation 6) is completely formulated.

Analytical Solution

11. Analytical solution to the problem formulated in the preceding section is obtained by following the solution technique by Stoker (1957). To obtain the solution, the water region is divided into three subregions--I, II, and III--by the incident wave ray passing through the tip of the wedge and the reflected wave ray reflected away from the tip of the wedge, as shown in Figure 2. Obviously, the total wave in subregion I is the sum of the incident, reflected, and scattered waves; the total wave in subregion II, where the reflected wave does not exist, is the sum of the incident and scattered waves; and the total wave in subregion III, where the incident and reflected waves have been shaded out, is only the scattered wave. For certain combinations of

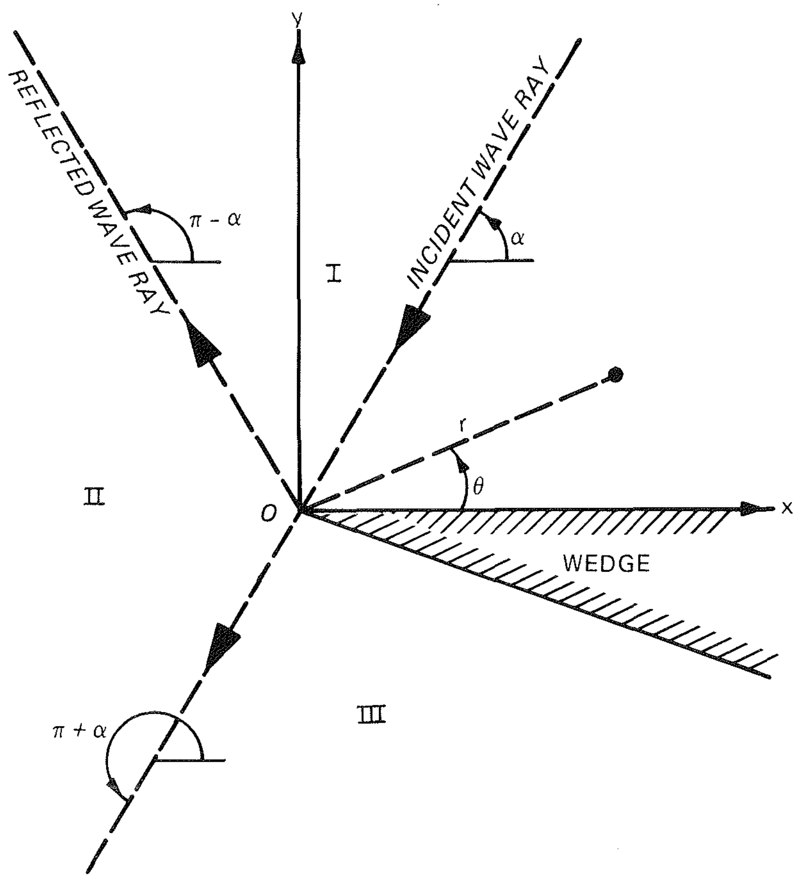

Figure 2. Three subregions and the wedge

the wedge angle and incident wave angle, subregions II and III may not exist at all. In general, the solution function can be written as

$$\phi = \phi_o(r,\theta) + \phi_s(r,\theta) \tag{11}$$

where

$$\phi_o(r,\theta) = \begin{cases} \phi_i + \phi_r & \pi - \alpha > \theta > 0 \\ \phi_i & \pi + \alpha > \theta > \pi - \alpha \\ 0 & \theta_o > \theta > \pi + \alpha \end{cases} \tag{12}$$

The equation reveals that ϕ_o is the sum of the incident and reflected waves ϕ_i and ϕ_r and is a known function. The scattered wave ϕ_s is the only unknown function to be determined in the problem. Nevertheless, the total wave ϕ instead of the scattered wave ϕ_s is the desired solution to be obtained in this study.

12. The solution for the total wave ϕ is pursued. The finite cosine transform of ϕ, denoted by $\overline{\phi}$, is introduced by the formula

$$\overline{\phi}(kr,n) = \int_0^{\nu\pi} \phi(kr,\theta) \cos \frac{n\theta}{\nu} \, d\theta \tag{13}$$

where $n = 0,1,2,\ldots$ are integers, and ν is related to the wedge angle as defined by

$$\theta_o = \nu\pi \tag{14}$$

Applying the finite cosine transform and using the boundary condition in Equation 5, Equation 2 becomes

$$r^2 \frac{\partial^2 \overline{\phi}}{\partial r^2} + r \frac{\partial \overline{\phi}}{\partial r} + \left[(kr)^2 - \left(\frac{n}{\nu}\right)^2 \right] \overline{\phi} = 0 \tag{15}$$

Equation 15 is a form of the Bessel equation for which general solutions are the Bessel functions of the first and second kinds, $J_{n/\nu}(kr)$ and $Y_{n/\nu}(kr)$, respectively. Since $Y_{n/\nu}(kr)$ are singular at the origin, the solution is chosen to be

$$\overline{\phi}(kr,n) = a_n J_{n/\nu}(kr) \tag{16}$$

where a_n are constants to be determined.

13. Taking the finite cosine transform of Equation 11 and using Equation 16, we have

$$\int_0^{\nu\pi} \phi_s \cos\frac{n\theta}{\nu}\, d\theta = a_n J_{n/\nu}(kr) - \int_0^{\nu\pi} \phi_o \cos\frac{n\theta}{\nu}\, d\theta \tag{17}$$

or

$$\overline{\phi}_s = a_n J_{n/\nu}(kr) - \overline{\phi}_o \tag{18}$$

Then applying the operation $\lim_{r\to\infty} \sqrt{r}(\partial/\partial r + ik)$ to both sides of Equation 18, and using the Sommerfeld radiation condition (Equation 6) we have

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} + ik\right)\left[a_n J_{n/\nu}(kr) - \int_0^{\nu\pi} \phi_o \cos\frac{n\theta}{\nu}\, d\theta\right] = 0 \tag{19}$$

14. Equation 19 can be asymptotically evaluated to determine a_n. Firstly, the first term involving the Bessel function is evaluated. The function $J_{n/\nu}(kr)$ at $r\to\infty$ behaves asymptotically (Abramowitz and Stegun 1964) as follows:

$$J_{n/\nu}(kr) \sim \sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{n\pi}{2\nu} - \frac{\pi}{4}\right) \tag{20}$$

Hence, we have

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} + ik\right) J_{n/\nu}(kr) \sim \sqrt{\frac{2k}{\pi}}\, e^{i(kr - n\pi/2\nu + \pi/4)} \tag{21}$$

Secondly, the second term involving the integral of ϕ_o is evaluated. The asymptotic behavior of the integral over $\theta = (0, \nu\pi)$ and at large distance $r \to \infty$ can be found by the method of stationary phase. The integral, after substituting ϕ_o from Equations 9, 10, and 12, can be written as

$$\int_0^{\nu\pi} \phi_o \cos\frac{n\theta}{\nu}\, d\theta = \int_0^{\pi-\alpha} \left[e^{ikr\cos(\theta-\alpha)} + e^{ikr\cos(\theta+\alpha)}\right] \cos\frac{n\theta}{\nu}\, d\theta$$

$$+ \int_{\pi-\alpha}^{\pi+\alpha} e^{ikr\cos(\theta-\alpha)} \cos\frac{n\theta}{\nu}\, d\theta \qquad (22)$$

In the integrals, there are three points of stationary phase at $\theta = \alpha$ and $\theta = \pi \pm \alpha$. If the same argument as that of Stoker (1957) is followed, of the three contributions only the first one $\theta = \alpha$ furnishes a nonvanishing contribution for $r \to \infty$ when the operator $\sqrt{r}(\partial/\partial r + ik)$ is applied to it. The physical significance of this statement is that only the incident wave is effective in determining the cosine coefficients of the solution. Therefore,

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} + ik\right) \int_0^{\nu\pi} \phi_o \cos\frac{n\theta}{\nu}\, d\theta \sim 2\sqrt{2\pi k}\cos\frac{n\alpha}{\nu}\, e^{i(kr+\pi/4)} \qquad (23)$$

Substituting Equations 21 and 23 into Equation 19, we obtain the unknown coefficients a_n :

$$a_n = 2\pi\cos\frac{n\alpha}{\nu}\, e^{in\pi/2\nu} \qquad (24)$$

15. Since the solution ϕ in the cosine series expression is

$$\phi(r,\theta) = \frac{1}{\nu\pi}\,\bar{\phi}(r,0) + \frac{2}{\nu\pi}\sum_{n=1}^{\infty} \bar{\phi}(r,n)\cos\frac{n\theta}{\nu} \qquad (25)$$

the solution is obtained by substituting Equations 16 and 24 into Equation 25 as follows:

$$\phi(r,\theta) = \frac{2}{\nu}\left[J_0(kr) + 2\sum_{n=1}^{\infty} e^{in\pi/2\nu} J_{n/\nu}(kr)\cos\frac{n\alpha}{\nu}\cos\frac{n\theta}{\nu}\right] \qquad (26)$$

Equation 26 is the solution for the combined wave reflection and diffraction by a vertical wedge of arbitrary wedge angle and is considered to be extended from the solution by Stoker (1957) who only solved the problem of a thin semi-infinite breakwater. The solution in Equation 26 and the one by Stoker are not only in nonclosed form but also in terms of Bessel functions. It seems that the calculations of the solutions are very difficult without using a modern high-speed computer. This is probably the reason why Stoker arrived at his solution expressed in the same cosine series but did not use it to calculate the result. Instead, he further transformed the expression into a very complex integral form for further approximation in calculating the result.

16. Notably, the solution at the origin point is obtained by simply substituting $r = 0$ into Equation 26 to arrive at

$$\phi(0,\theta) = \frac{2}{\nu} \qquad (27)$$

Therefore, wave response at the origin point depends only on the wedge angle and does not depend on the incident wave angle.

Two Special Cases

17. The solutions for two special cases are used to verify Equation 26: one for the case of a thin semi-infinite breakwater and the other for the case of an infinite wall extending from $x = -\infty$ to ∞ .

18. The vertical wedge should reduce to a thin semi-infinite breakwater as the wedge angle reduces to 0 deg. Therefore, solution of the combined wave reflection and diffraction by a thin semi-infinite breakwater is obtained by substituting $\nu = 2$ (that is, $\theta_o = 2\pi$) into Equation 26 which then becomes

$$\phi(r,\theta) = J_0(kr) + 2 \sum_{n=1}^{\infty} e^{in\pi/4} J_{n/2}(kr) \cos \frac{n\alpha}{2} \cos \frac{n\theta}{2} \tag{28}$$

Equation 28 is precisely the same one obtained by Stoker (1957).

19. The vertical wedge should also become an infinite wall extending from $x = -\infty$ to ∞ with the water occupying only the half plane of $y \geq 0$ as the wedge angle increases to 180 deg. In this situation the scattered wave is absent from the solution, and the total wave is only the sum of the incident and reflected waves as follows:

$$\phi(r,\theta) = e^{ikr \cos(\theta-\alpha)} + e^{ikr \cos(\theta+\alpha)} \tag{29}$$

After expansion of the exponential functions in terms of Bessel functions (Abramowitz and Stegun 1964), Equation 29 becomes

$$\phi(r,\theta) = 2\left[J_0(kr) + 2 \sum_{n=1}^{\infty} i^n J_n(kr) \cos n\alpha \cos n\theta\right] \tag{30}$$

Equation 30 is the same equation reduced from Equation 26 by substituting $\nu = 1$ into it.

PART III: CALCULATION AND RESULTS

20. Results of the combined reflection and diffraction by a wedge of arbitrary wedge angle can be calculated from Equation 26. Since the solution is not only in terms of Bessel functions but also in a nonclosed form, the computer program WEDGE is therefore written to calculate the solution.

21. In the program the subroutine BESJ for calculating Bessel function of fractional or integer order was used. The subroutine was originally written by Amos, Daniel, and Weston in 1975 (Morris 1984) and is collected in the Naval Surface Weapons Center Library of Mathematics Subroutines (Morris 1984).

22. In the calculation the summation of the infinite terms in Equation 26 was carried out to the term which is preceded by eight successive terms of the absolute value of the Bessel function, all equal to or less than 10^{-8}. The solution has a truncation error less than 10^{-8}, and it is of the order of one.

23. In this study, results of the combined wave reflection and diffraction for the wedge are calculated for two cases: one for a vertical wedge of 0-deg wedge angle and the other for a vertical wedge of 90-deg wedge angle.

Vertical Wedge of 0-Deg Wedge Angle

24. When the wedge angle is equal to zero, the wedge is actually a thin semi-infinite breakwater extending from $x = 0$ to ∞. Figure 3 shows the thin semi-infinite breakwater along with the polar coordinates. In this case the diffraction results for various incident wave angles in the water region from $\theta = \pi$ to 2π and $r/\lambda \leq 10$, where λ is the incident wave length, have already been presented by Wiegel (1962) and are shown in the SPM, Volume I (1984). The present results combine reflection and diffraction effects and cover the water region from $\theta = 0$ to 2π and $r/\lambda \leq 10$. Therefore, the present results for this particular case can be considered to be a complementary and extended version to the ones in the SPM.

25. In this study wave response was calculated at 1,460 grid points intersected by $r/\lambda = 0.5,(0.5),10$, which means that the values of r/λ are from 0.5 to 10.0 with each value increment being 0.5. Hereafter, all similar expressions are to be interpreted in the same way (e.g., $\theta = 0,(\pi/36),2\pi$ for

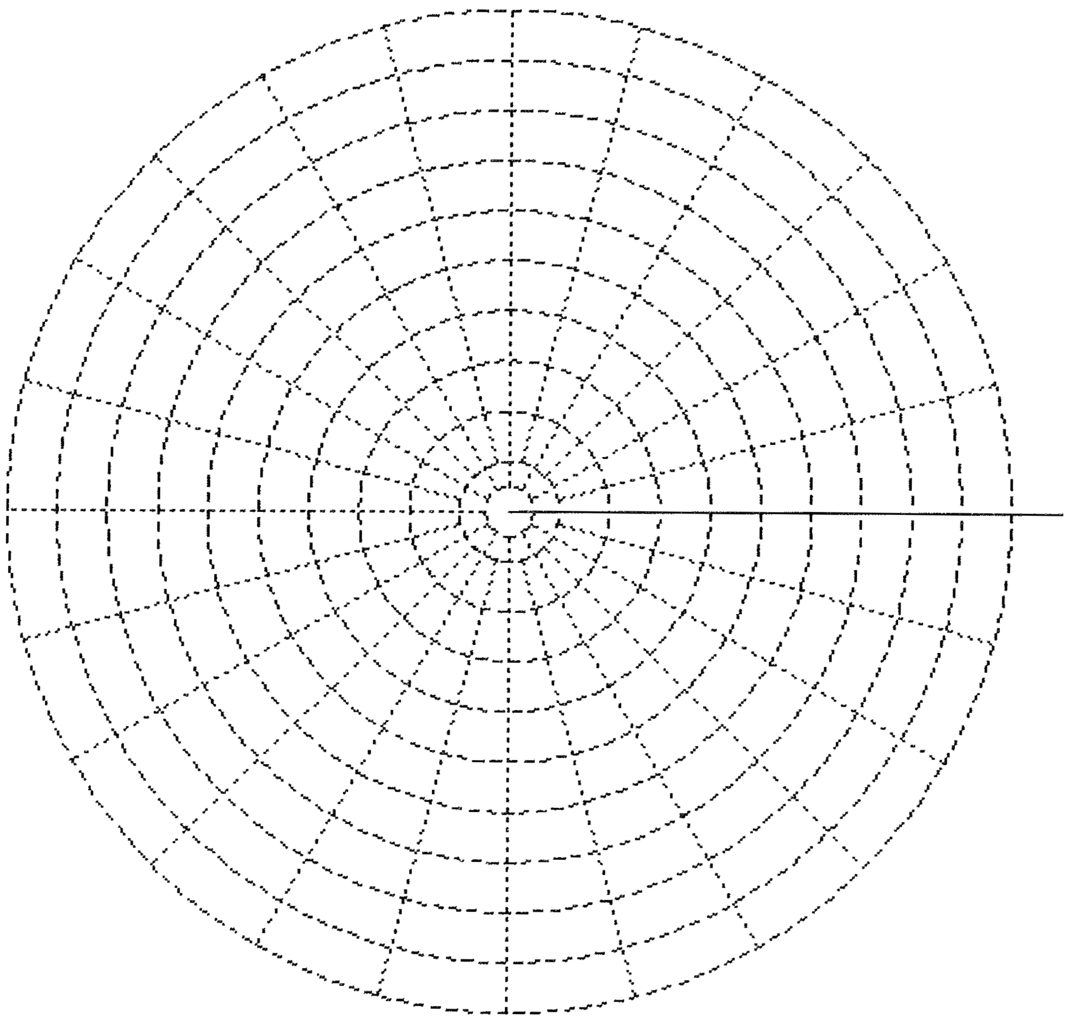

Figure 3. Thin semi-infinite breakwater and polar coordinates

the incident wave angle $\alpha = 0,(\pi/12),\pi)$. The wave response at the origin point is obtained substituting $\nu = 2$ into Equation 27, as follows:

$$\phi(0,\theta) = 1 \tag{31}$$

Those calculated values were used to interpret the value for each non-overlapping pixel of size $0.1r/\lambda$ by $0.1r/\lambda$ in the area within the $10r/\lambda$ radius from the origin. A diagram was then constructed by patching those pixels over the entire area. The wave response diagrams for each incident wave angle are shown in Figures 4 through 15. Notably, the values in the diagrams constitute the amplification factor which is defined as the ratio of the total wave height to the incident wave height. Therefore, in subregions II

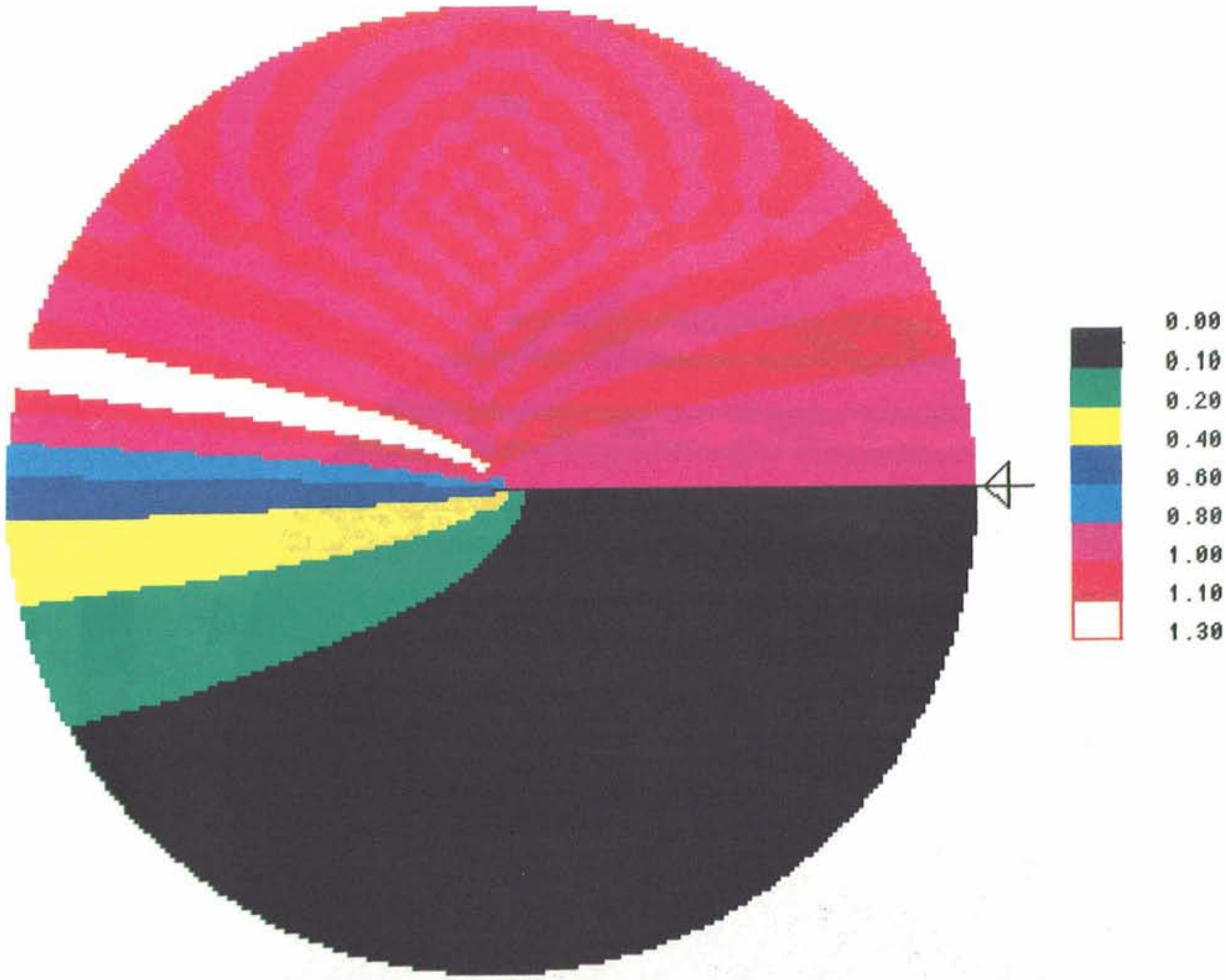

Figure 4. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 0 deg

and III (as defined in Figure 2) where the reflected wave is absent, the amplification factor is essentially the diffraction coefficient as defined in the SPM.

26. Figures 4 through 15 reveal that the amplification factors in subregion I change very rapidly between 0 and 2.35 over the subregion, and the diagram patterns become very complex because of the interesting superposition of the incident, reflected, and scattered waves. (In the legend of Figures 4 through 15, the width of the pixel is one incident wave length, and the values are amplification factors.) Such patterns would be very difficult to construct without using a high-speed computer and computer graphics. In subregions II and III, the amplification factors change smoothly from 1.15 roughly along the reflected wave ray reflected from the origin point to nearly

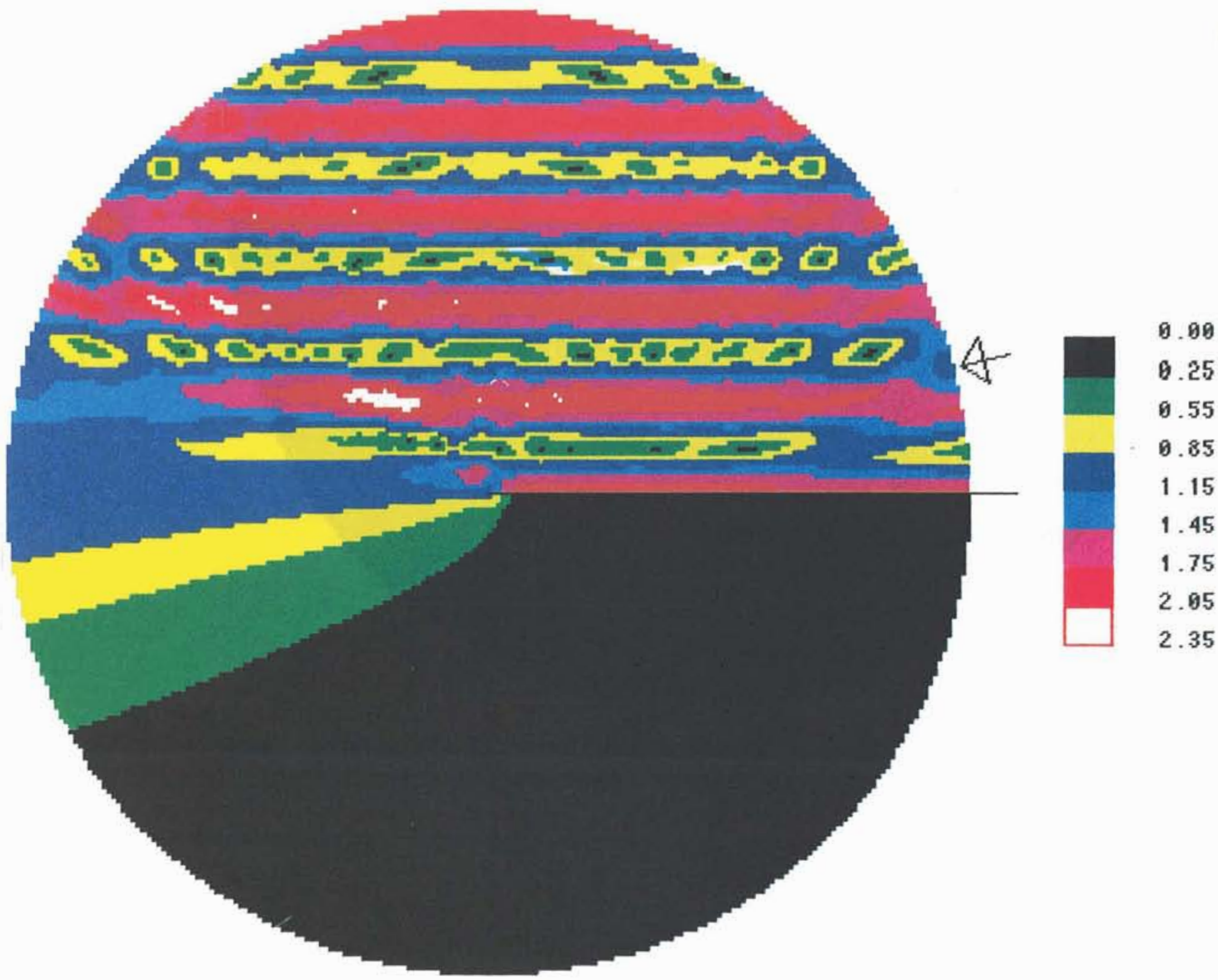

Figure 5. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 15 deg

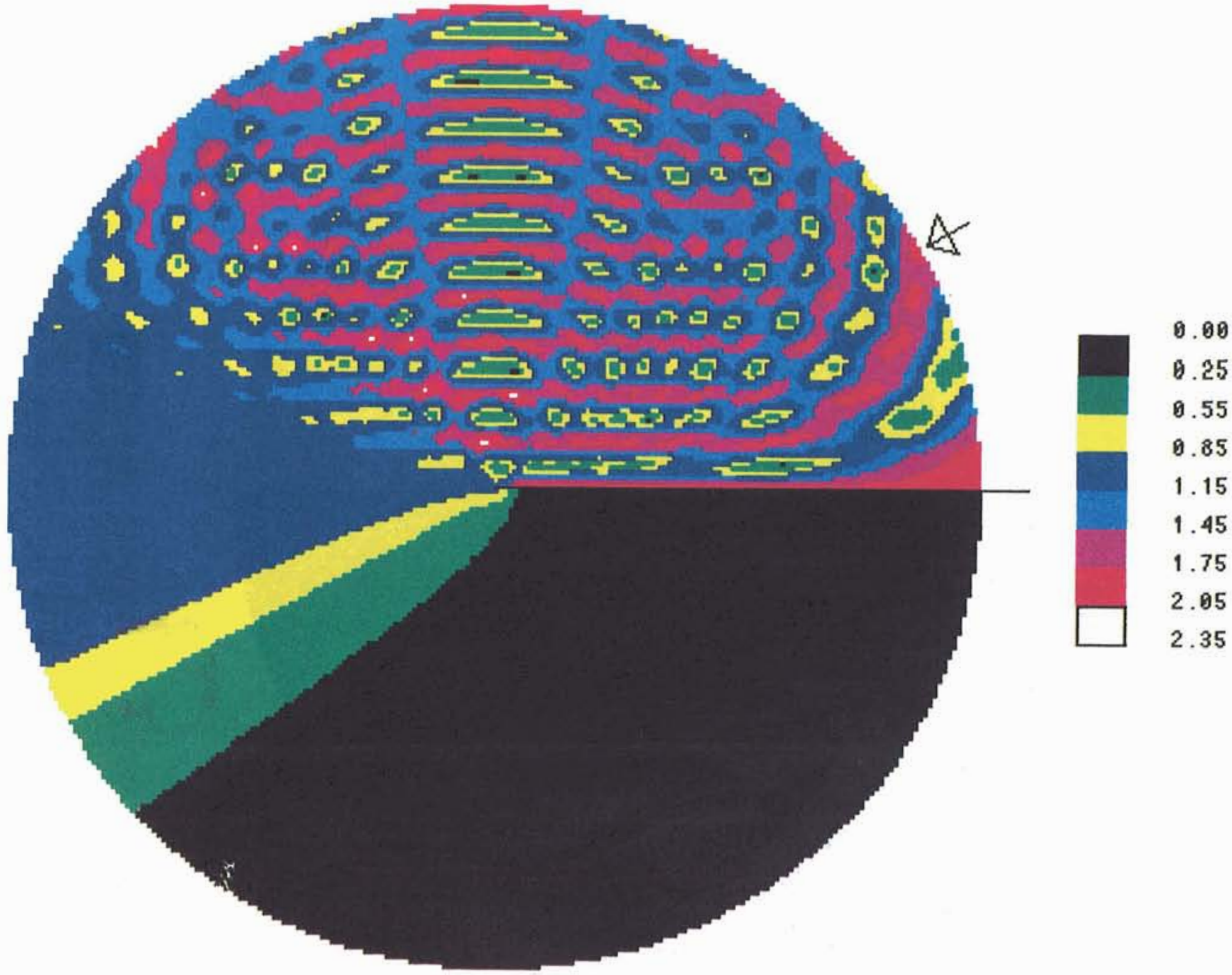

Figure 6. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 30 deg

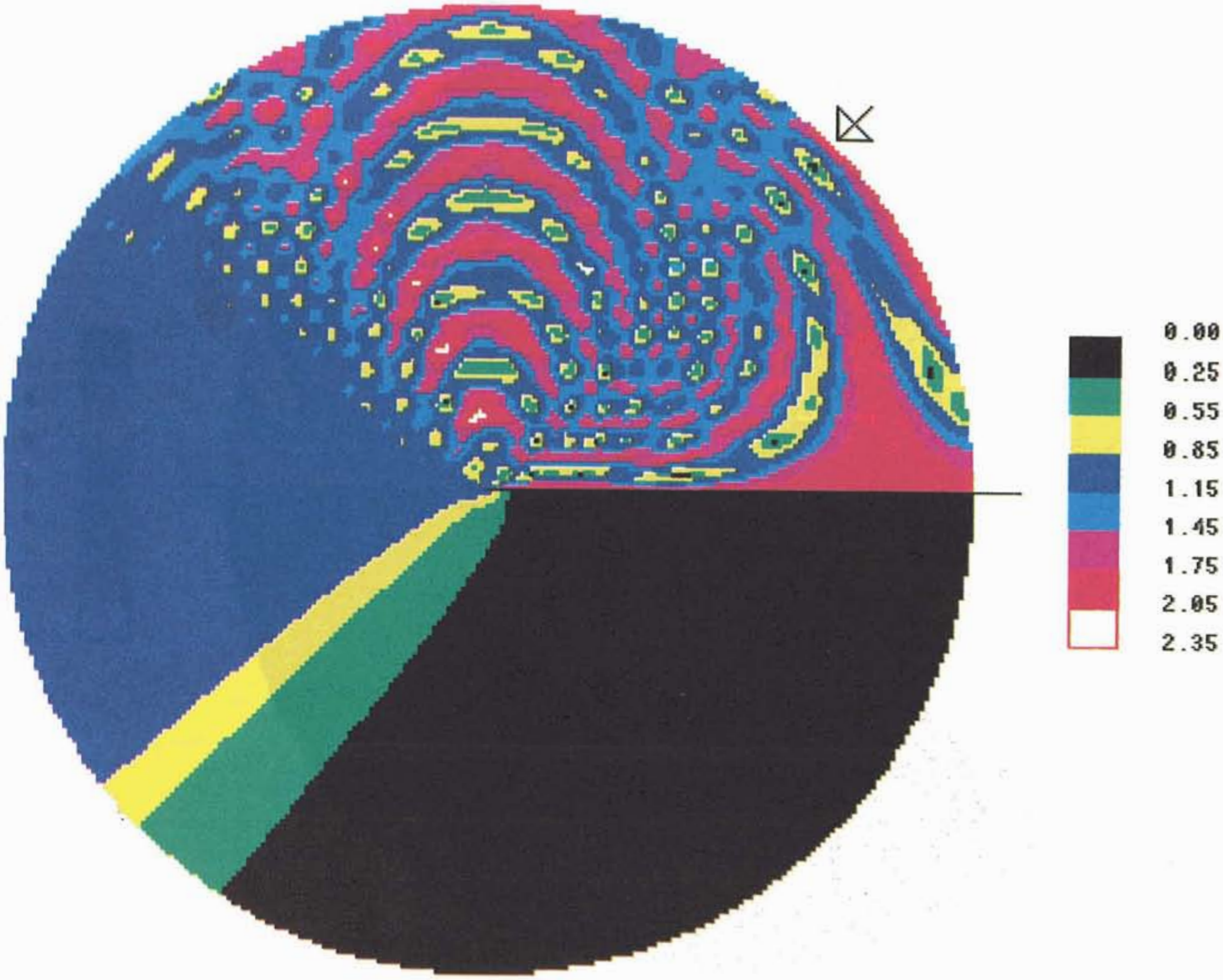

Figure 7. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 45 deg