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Special Report 102

ON MEASURING DISPERSED POPULATIONS

Robert W. Waterhouse

November 1968

U.S. ARMY MATERIEL COMMAND
TERRESTRIAL SCIENCES CENTER
COLD REGIONS RESEARCH & ENGINEERING LABORATORY
HANOVER, NEW HAMPSHIRE

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PREFACE

This paper was prepared by Robert W. Waterhouse, Research Civil Engineer, of the Construction Engineering Branch (E.F. Lobacz, Chief), of the Experimental Engineering Division (K.A. Linell, Chief), Cold Regions Research and Engineering Laboratory, U.S. Army Terrestrial Sciences Center (USA TSC).

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ABSTRACT

Accuracy and speed of property estimating of randomly dispersed populations can be improved by identifying and accounting for the influence of shape of both the element and the viewing stage. Perimeter intercept count of high density non-overlapping dispersions provides a rapid method of density evaluation, provided the influence of shape on location and orientation of elements and on the random definition of the intercepts is recognized. The efficiency of the circular stage is seen to be due to its randomizing of intercepts. Models demonstrate these influences and comparative accuracy of three systems of estimating.

ON MEASURING DISPERSED POPULATIONS

by

Robert W. Waterhouse

Introduction

The uniform* dispersion of a population permits the utilization of simple estimating relationships between the dimensions of the elements and the stage bounding the population, in order to solve for properties of the population which would be very time-consuming to determine rigorously.

In the study of thin sections of snow and ice it becomes evident that to assess the effects of metamorphism in a quantitative way, numbers must be assigned to the time change in the size, shape and arrangement of particles (grains or crystals).

Fuchs (1960) and others, using rigorous methods of counting, have provided valuable intrinsic information on the structure of snow and data with which to test new concepts. Statistical methods employed for such work provide accuracy commensurate with the amount of time spent in the particular process but we do not have both rapid and accurate methods for determining the properties from thin sections.

In such work there appears to be a need to represent the shape of an object numerically. The shape of the stage or counting area as it relates to the shape of the objects being studied is also of a significance that apparently has been overlooked.

In an attempt to understand the problem more thoroughly and possibly improve on present techniques the following brief study has been made using models and three counting techniques. Populations of squares, triangles and circles of pasteboard have been photographed with various dispersions arranged by hand under a square or triangular mask which constitutes the stage. These laminae model the 2-dimensional projections of solids found in practice.

The laminae were 1 inch on a side or 1 inch in diameter; the stages 6 inches on a side. The area of solids on the stage is to be determined. Results of the model studies are shown in Table I.

Counting techniques and results

The true areas were determined by planimeter (Table I, col. 6).

In column 5, the number of intercepts identified with 100% solidity ratio is a theoretical mean which presumes that the shapes can physically fill all voids.

The first statistical method adds lengths of the intercept of the laminae with the perimeter of the stage and divides the sum by the stage perimeter P_s, using a relationship demonstrated by Chalkley *et al.* (1949):

$$\frac{\Sigma \text{ segments}}{P_s} = \frac{\text{area of solids}}{\text{area of stage}} . \qquad (1)$$

---

* Short for "homogeneously random."

Table I. Solidity ratios of dispersed model sets.

1	2	3	4	5	6	7	8(4x5)	9
	Shape			Number of intercepts N	Solidity ratio Z%			Z by count / Z by planimeter %
Set	Lamina	Stage	$\frac{25}{[FF_s]^{\frac{1}{2}}}$ $\left[\frac{A}{A_s}\right]^{\frac{1}{2}}$		By planimeter	By length of intercept	By number of intercepts	
1	S	S	3.27	18	64.0	64.6	58.9	92
2				22	71.4	66.6	72.0	101
3				23	77.7	76.0	75.2	97
*				30.65	100.0			
4	S	T	4.36	11	48.9	51.6	48.0	98
5				15	63.2	50.3	65.4	103
6				20	68.7	74.9	87.2	127
*				22.9	100.0		99.5 Fig. 2	
7	C	S	3.32	17	56.7	44.0	56.4	99
8				20	64.5	66.0	66.4	103
9				25	80.3	84.4	83.0	103
*				30.1	100.0			
10	C	T	4.36	13	54.4	60.0	56.7	104
11				17	71.2	68.2	74.2	104
12				20	84.5	79.3	87.2	103
*				22.9	100.0			
13	T	S	1.89	19	31.85	28.2	35.9	112
14				21	36.25	42.7	39.7	109
15				23	41.05	45.4	43.5	106
*				53.0	100.0			
16	T	T	2.51	13	30.9	32.85	32.6	106
17				15	40.6	42.0	37.6	93
18				18	53.2	59.0	45.2	85
*				39.8	100.0			
								103

S = Square; T = Equilateral triangle; C = Circle

Average Error +3%

* Theoretical Mean.

This ratio will be called the solidity ratio, Z. The results are shown in Table I (col. 7). Time is saved with this method but accuracy is forfeited. Validity of this method is based on a statement by Crofton (1885). Crofton's essential statement is, "The average of all possible intercepts of straight lines across any shaped (non re-entrant) lamina $= \frac{\pi A}{P}$, where A is the area of the lamina and P is its perimeter." If the mean segment, S, is $\frac{\pi A}{P}$, s is the individual segment, and N is the number of segments intercepted then

$$S = \frac{\Sigma s}{N} = \frac{\pi A}{P} \qquad \text{(cutting many equal laminae by one line, say, the perimeter of a counting area).} \tag{2}$$

From eq 1 and 2

$$Z = \frac{\Sigma s}{P_s} = \frac{\pi A N}{P P_s}. \tag{3}$$

Let the shape of a lamina be represented by a shape factor, F:

$$F = \frac{P^2}{4\pi A} \tag{4}$$

which uniquely characterizes shape for all regular polygons and is non-dimensional, independent of size and normalized on circles. Accordingly, $F = 1$ for a circle, $F > 1$ for all other shapes. Eliminating A in eq 3 by substitution from eq 4

$$Z = \left(\frac{P}{4 P_s F}\right) N. \tag{5}$$

Since $\frac{P}{4 P_s F}$ is constant for a population of equal laminae on a given stage, Z can be found by intercept count, N, which is a more rapid process than summing the intercepts.

In the event that the area of particle A is more readily available than P the following can be used:

$$\text{Since } F = \frac{P^2}{4\pi A},$$

$$P \propto A^{1/2} F^{1/2} \tag{6}$$

and

$$P_s \propto A_s^{1/2} F_s^{1/2} \tag{7}$$

where A_s = area of stage.

Substitute eq 6 and 7 in eq 5

$$Z = \frac{25N}{(F F_s)^{1/2}} \left(\frac{A}{A_s}\right)^{1/2} \% \tag{8}$$

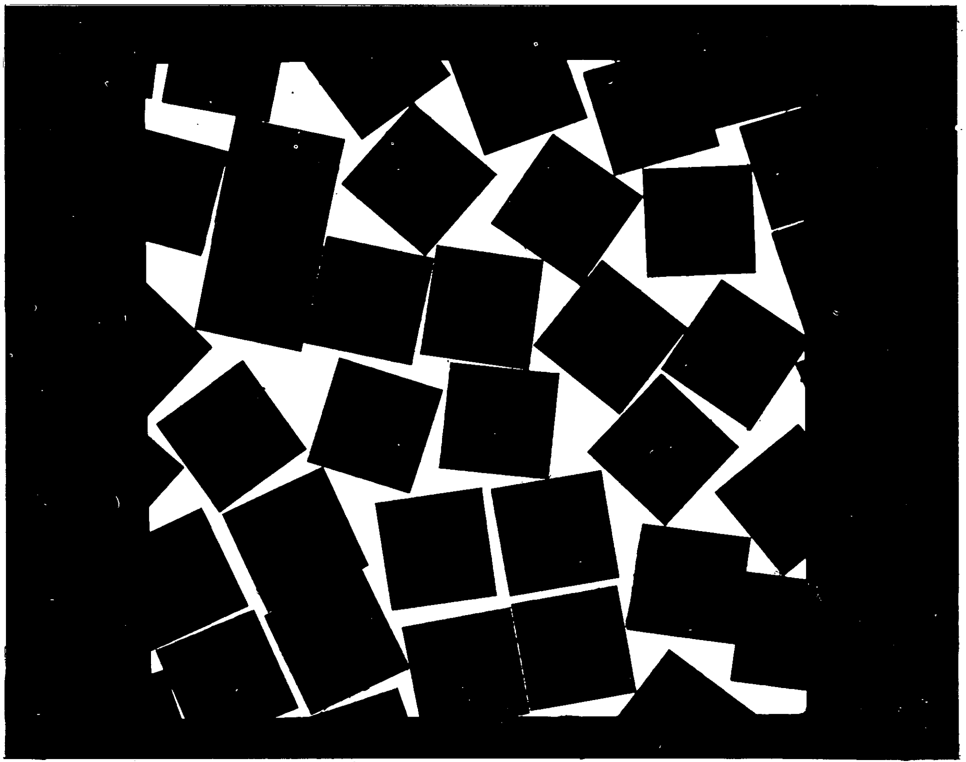

Figure 1. Uniform dispersion of squares on a square stage $\frac{1}{6}$ linear size ratio. Set no. 3.

which more clearly shows the influence of the stage shape. Table I (col. 8) gives solidity ratios computed from expression 5 or 8.

These systems are believed to be most useful when ratios of particle-stage area $(A/A_S)^{1/2}$ are from 1/10 to 1/100. For ratios larger than 1/10, accuracy is very sensitive to a small non-uniformity of dispersion. Where the area of solids or voids can be conveniently determined by photometry, eq 8 can be used to determine the mean shape factor or to indicate the lack of uniform dispersion. This will be most useful where odd-shaped particles are under study.

The applicability of this work will be tested further on current problems and the reasoning extended to populations of distributed size. With the ability to describe the mean shape as well as the mean size and mean distribution, the breadth of application will be further extended.

Figure 1 shows a typical model arrangement of squares on a square stage.

Summary

1. In each family of laminae (i.e., conics, polygons, etc.) there may exist one of the same shape factor as a member of another family. For the infinity of shapes within a family, this lack of uniqueness is not very significant in a practical sense and it does not prevent the factor from representing the influence of shape on other properties.

2. The technique of counting requires a minimum of judgement. *All* intercepts are counted. Should an intercept be doubtful, the stage is simply moved into an unquestioned position.

3. The range of the system is demonstrated well with a limit set of close packed squares on a triangular stage (Fig. 2). Twenty-three intercepts represent 100% solidity ratio: the theoretical number of intercepts is 22.9 as shown in column 5, Table I.

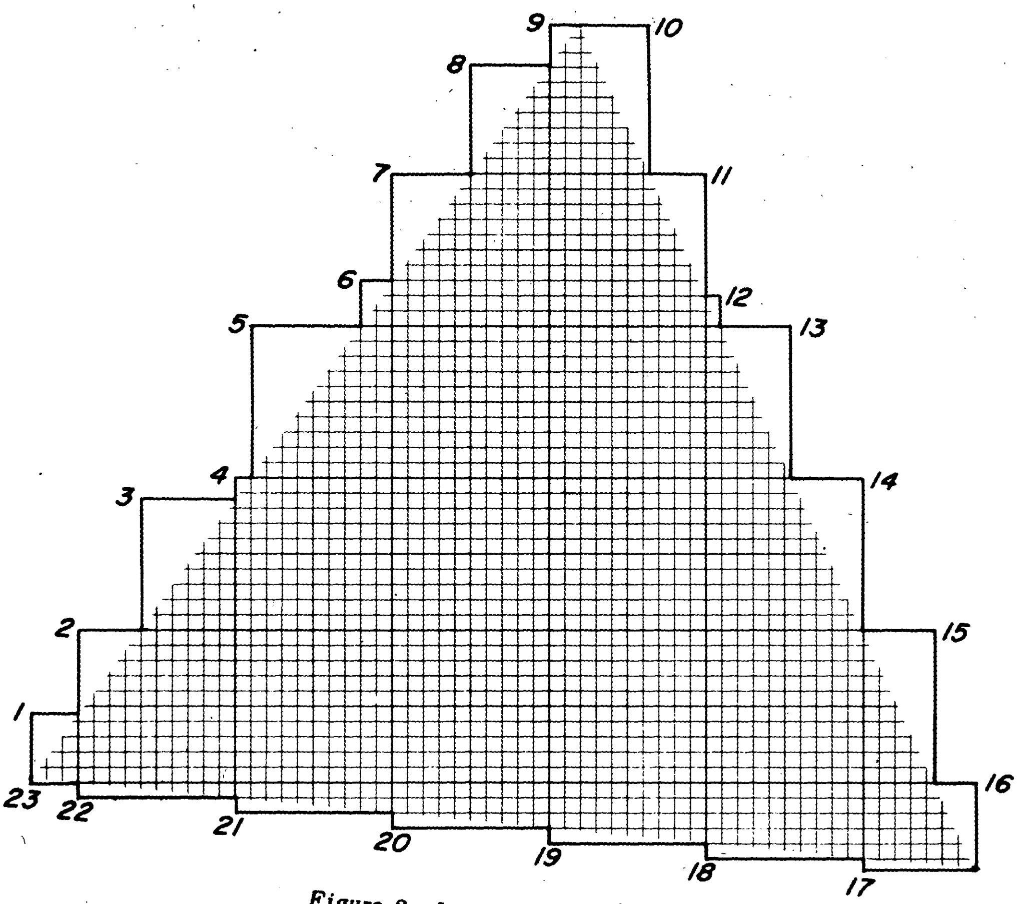

Figure 2. Limit to sets 4, 5, 6.

4. The results indicate that in 10 of the 18 sets the *counting* of intercepts is more rapid and more accurate than the summing of lengths of intercepts.

5. From these results, it appears that, for a given set, the largest possible error depends on the arrangement of the particles and also on the difference in shapes of particles and stage. Non-uniformity of dispersion is the greatest cause of error with the systems studied.

6. The accuracy of the counting system is improved with an increase in stage/particle size ratio. With 50 intercepts or more, the theoretical error would be reduced to less than 1%.

Voids cannot be measured in the same manner as solids in *this* study because of the complexity of contiguous-void shape-factors. However, porosity of a dispersion $= n = \frac{\text{Area of voids}}{A_s} = 1 - \frac{A}{A_s} = 1 - Z$.

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Accuracy and speed of property estimating of randomly dispersed populations can be improved by identifying and accounting for the influence of shape of both the element and the viewing stage. Perimeter intercept count of high density non-overlapping dispersions provides a rapid method of density evaluation, provided the influence of shape on location and orientation of elements and on the random definition of the intercepts is recognized. The efficiency of the circular stage is seen to be due to its randomizing of intercepts. Models demonstrate these influences and comparative accuracy of three systems of estimating.

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