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DENSITY OF ICE AS A FUNCTION OF TEMPERATURE AND STRESS

by

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SUMMARY

The equations for calculating the density of ice of moderate porosity (density > 0.8) as a function of temperature and stress condition are developed, and the values of parameters are calculated from the best available experimental data.

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If a crystal does not change modification when subjected to a change in temperature and pressure, the path followed in the change does not affect the result. We can change temperature isopiezically, and then pressure isothermally, or vice-versa, or change both temperature and pressure simultaneously. The nature of the data available on ice makes it convenient to take the following path:

    I. change temperature isopiezically
    II. change pressure isothermally.

The three following statements are not strictly true, but the error resulting from assumption of validity is here considered to be negligible.

    1. Ice aggregates are isotropic with respect to thermal expansion and elastic reaction to stress.

    2. Ice of low porosity (very dense snow or bubbly ice) has a coefficient of thermal expansion equal to that of bubble free ice.

    3. The magnitude of naturally occurring bubble air pressure has no effect on thermal expansion nor on elastic behavior.

I. Isopiezic density of ice as a function of temperature

The coefficient (α) of linear thermal expansion of ice at atmospheric pressure was measured by Butkovich (1957) between very close to 0°C and -30°C. Anisotropism was found to be very small and can be neglected in calculating the volume coefficient.

$$\alpha = \frac{1}{L} \frac{\partial L}{\partial t} \quad \text{at constant pressure.} \tag{1}$$

where L is any length of a piece of ice, and t is the temperature.

$$\int_{L_1}^{L_2} \frac{dL}{L} = \int_{t_1}^{t_2} \alpha \, dt \tag{2}$$

where L_1 and L_2 are lengths at t_1 and t_2 respectively.

$$\ln \frac{L_2}{L_1} = \int_{t_1}^{t_2} \alpha \, dt. \tag{3}$$

If γ is density, and V is volume, then

$$\ln \frac{\gamma_2}{\gamma_1} = \ln \frac{V_1}{V_2} = \ln \frac{L_1^3}{L_2^3} = -3 \ln \frac{L_2}{L_1}$$

$$\gamma_2 = \gamma_1 \exp \left[-3 \int_{t_1}^{t_2} \alpha \, dt \right]. \tag{4}$$

The exponent is sufficiently small to permit use of the approximation $e^x \cong 1 + x$

$$\gamma_2 \cong \gamma_1 \left(1 - 3 \int_{t_1}^{t_2} \alpha \, dt \right). \tag{5}$$

Butkovich determined the parameters of

$$\alpha = a + bt + ct^2 + dt^3 \qquad \text{for } 0 > t > -30°C$$

$$a = 52.5186 \times 10^{-6}$$
$$b = 0.18525 \times 10^{-6}$$
$$c = 0.008854 \times 10^{-6}$$
$$d = 0.00023712 \times 10^{-6}.$$

Butkovich gives $\underline{b}$ and $\underline{d}$ as negative, but it is clear that he inserts $\underline{t}$ as a positive number; we enter it as a negative Celsius temperature.

$$\int_{t_1}^{t_2} \alpha \, dt = a(t_2 - t_1) + \frac{b}{2}(t_2^2 - t_1^2) + \frac{c}{3}(t_2^3 - t_1^3) + \frac{d}{4}(t_2^4 - t_1^4). \tag{6}$$

Substituting eq 6 in eq 5

$$\gamma_2 = \gamma_1 \left\{ 1 + 3(t_1 - t_2)\left[a + \frac{b}{2}(t_1 + t_2) + \frac{c}{3}(t_1^2 + t_2^2 + t_1 t_2) + \frac{d}{4}(t_1 + t_2)(t_1^2 + t_2^2) \right] \right\} \tag{7}$$

$$0 > t > -30°C.$$

Butkovich (1953) also accurately measured ice density. We now recalculate Butkovich's values for the isopiezic density of bubble-free polycrystalline commercial ice. Accurate density values for polycrystalline clear glacier ice are unavailable.

	t	γ_{it}	γ_{io} (at 0°C)	$\bar{\gamma}_{io}$
Specimen 21a	-4.18	0.917122	0.916522 ⎫	
	-4.10	0.917111	0.916521 ⎬	0.916514
	-4.17	0.917099	0.916499 ⎭	
Specimen 21b	-4.17	0.917082	0.916482 ⎫	
	-4.06	0.917059	0.916474 ⎬	0.916477
	-3.99	0.917050	0.916475 ⎭	

average 0.916496

Until new measurements suggest otherwise, we can accept 0.91650 as the density of bubble free ice at 0°C and one atmospheric pressure.

The density at any temperature $0 > t > -30°C$ is given by

$$\gamma_{it} = 0.91650 \left[1 - 10^{-6}t(157.556 + 0.2779t + 0.008854t^2 + 0.0001778t^3) \right]$$

t is entered as a negative Celsius temperature.

Table I gives the factor by which the density at 0°C and one atmospheric pressure is multiplied to obtain that at another temperature at the same pressure, and also the density for $\gamma_{io} = 0.9165$.

Table I.

-t°C	$\dfrac{\gamma_{it}}{\gamma_{io}}$	γ_{it}	-t°C	$\dfrac{\gamma_{it}}{\gamma_{io}}$	γ_{it}
0	1.000000	0.91650	16	1.002474	0.91877
1	1.000157	0.91664	17	1.002627	0.91891
2	1.000314	0.91679	18	1.002779	0.91905
3	1.000470	0.91693	19	1.002931	0.91919
4	1.000626	0.91707	20	1.003082	0.91932
5	1.000782	0.91722	21	1.003234	0.91946
6	1.000937	0.91736	22	1.003384	0.91960
7	1.001092	0.91750	23	1.003535	0.91974
8	1.001246	0.91764	24	1.003685	0.91988
9	1.001401	0.91778	25	1.003834	0.92001
10	1.001555	0.91793	26	1.003983	0.92015
11	1.001709	0.91807	27	1.004131	0.92029
12	1.001862	0.91821	28	1.004279	0.92042
13	1.002016	0.91835	29	1.004426	0.92056
14	1.002169	0.91849	30	1.004572	0.92069
15	1.002322	0.91863			

II. Isothermal density of ice as a function of stress

The ice aggregate is considered to be elastically isotropic. Let $\partial\sigma_x$, $\partial\sigma_y$, $\partial\sigma_z$ be the changes of the principal stresses, $\underline{E}$ Young's modulus, and μ Poisson's ratio. Both $\underline{E}$ and μ are assumed to be independent of stress. The specific change in volume is obtained by using the additive theorem for stresses and strains.

$$\frac{\partial V}{V} = -\frac{\partial\sigma_x + \partial\sigma_y + \partial\sigma_z}{E}(1 - 2\mu). \tag{8}$$

A useful application will be to calculate in situ density of glacier ice from drill-core density measured in the laboratory. For this purpose we consider two likely stress conditions.

1. Ice is vertically compressed under lateral constraint. σ_z is the vertical ice load pressure.

$$\partial\sigma_x = \partial\sigma_y = \partial\sigma_z \frac{\mu}{1-\mu} \tag{9}$$

$$\frac{1}{\gamma}\frac{\partial\gamma}{\partial\sigma} = -\frac{1}{V}\frac{\partial V}{\partial\sigma} = \frac{1}{E}\frac{1-\mu-2\mu^2}{1-\mu}. \tag{10}$$

2. Ice is compressed hydrostatically

$$\frac{1}{\gamma}\frac{\partial\gamma}{\partial\sigma} = \frac{3}{E}(1-2\mu). \tag{11}$$

Equations 10 and 11 have the same form:

$$\frac{1}{\gamma}\frac{\partial\gamma}{\partial\sigma} = \frac{n}{E} \tag{12}$$

where the value of $\underline{n}$ depends on the stress condition and Poisson's ratio.

Poisson's ratio is density-dependent.   From seismic data by Bentley et al. (1957) the relation is approximately

$$\mu \cong 0.220 + 0.122\,\gamma \tag{13}$$

and we calculate $\underline{n}$ in eq 12:

γ	$n_1 = \dfrac{1-\mu-2\mu^2}{1-\mu}$	$n_2 = 3(1-2\mu)$
0.80	0.761	1.094
0.90	0.675	1.021
0.93	0.667	1.000

$n_2 \cong 1$ and n_1 is a weak function of γ.

Nakaya (1959) determined $\overline{E}$ as a function of density and temperature on vibrating bars.   His results indicate the approximate validity of the following two functions.

$$E_{t_1} = a_1 + b_1\,\gamma \quad \text{at temperature } t_1 \tag{14}$$

$$\frac{1}{E_0}\,\frac{\partial E}{\partial t} = p + q\,\gamma \tag{15}$$

where E_0 is the modulus at 0°C by extrapolation.

We differentiate eq 14 with respect to $\underline{t}$, holding γ constant, noting that $\underline{a}$ and $\underline{b}$ are functions of $\underline{t}$ only.

$$\frac{\partial E}{\partial t} = \frac{da}{dt} + \gamma\,\frac{db}{dt}\,. \tag{16}$$

From eq 15 and 16 we obtain

$$\frac{da}{dt} = pE_0 \quad \text{and} \quad \frac{db}{dt} = qE_0\,, \tag{17}$$

Integration gives

$$a_t - a_1 = pE_0(t-t_1) \quad \text{and} \quad b_t - b_1 = qE_0(t-t_1) \tag{18}$$

also

$$a_1 - a_0 = pE_0 t_1 \quad \text{and} \quad b_1 - b_0 = qE_0 t_1 \tag{19}$$

We also write eq 14 as

$$E_0 = a_0 + b_0\,\gamma \tag{20}$$

and can now get an expression for E_t, the modulus as a function of temperature

$$E_t = a_t + b_t\,\gamma \tag{21}$$

where

$$a_t = \frac{a_1(1 + q\gamma t_1 + pt) + b_1\,\gamma p(t-t_1)}{1 + pt_1 + q\gamma t_1} \tag{22}$$

$$b_t = \frac{b_1(1 + pt_1 + q\gamma t) + a_1 q(t-t_1)}{1 + pt_1 + q\gamma t_1}\,. \tag{23}$$

The change in density produced by isothermal compression is small enough for $\underline{E}$ and $\underline{n}$ to be constant in integrating eq 12

$$\gamma_2 \cong \gamma_1 \exp \frac{n(\sigma_2 - \sigma_1)}{E}. \tag{24}$$

For a pressure difference as large as 300 kg/cm², the value of the exponent is of the order of 10^{-2}, so eq 24 simplifies to

$$\gamma_2 \cong \gamma_1 \left[1 + \frac{n(\sigma_2 - \sigma_1)}{E} \right] \text{ at constant temperature.} \tag{25}$$

In using eq 25, σ and $\underline{E}$ must, of course, be given in the same units. If $\underline{E}$ is in dynes/cm², then σ in $\overline{g^*}$/cm² must be multiplied by 981.
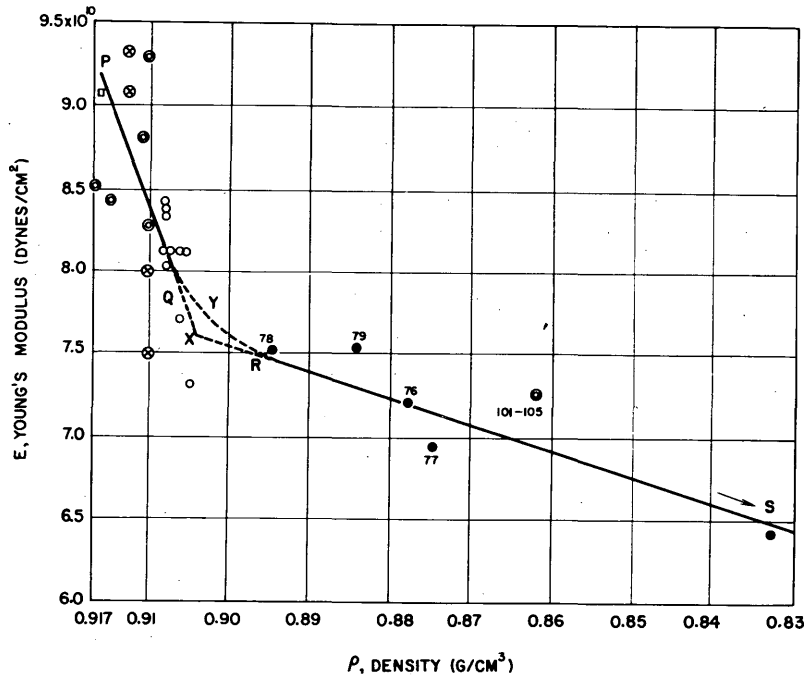

Figure 1.   Young's modulus vs density, tunnel ice (PQ) and core samples (RS). (From Nakaya, 1959).

Figure 1, reproduced from Nakaya (1959), shows that there is a great deal of scatter, perhaps attributable mainly to the sensitivity of resonance vibration frequency to internal and external shape of the specimen. Air bubble size and shape, for instance, appreciably influence the modulus obtained from resonance frequency. But it would be surprising if this were not less so with respect to the bulk modulus determined by hydrostatic tests.

Jellinek and Brill (1956) found, by uniaxial static loading to some 2×10^6 dynes/cm², a value of $E \cong 5\frac{1}{2} \times 10^{10}$ dynes/cm² at -5°C for bubbly ice, $\gamma = 0.886$, made from soaked snow. From our equations we calculate $E = 7.3 \times 10^{10}$. The difference could be attributed to scatter or to the different methods of obtaining $\underline{E}$. Confidence in calculating the effect of stress on density must remain relatively low until new data from static testing become available.

In the meantime we can accept Nakaya's two linear segments of Figure 1.

The parameters of equations 14, 22, and 23 are

$$\left.\begin{array}{l} t_1 = -9°C \\ a_1 = -7.2 \times 10^{10} \text{ dynes/cm} \\ b_1 = +16.4 \times 10^{10} \text{ dynes/cm} \end{array}\right\} \quad \text{for } 0.905 > \gamma > 0.60$$

$$\left.\begin{array}{l} t_1 = -5°C \\ a_1 = -111.7 \times 10^{10} \text{ dynes/cm} \\ b_1 = +132.0 \times 10^{10} \text{ dynes/cm} \end{array}\right\} \quad \text{for } 0.917 > \gamma > 0.905$$

Lacking data for $\gamma > 0.917$, we assume continued validity of the latter set of parameters.

The parameters of eq 15 are
$$p = -0.089$$
$$q = +0.095.$$

The use of eq 15 must have a limit, since it states that the temperature effect vanishes at $\gamma = 0.937$, which is unlikely to be true.

Table II gives examples of the influence of hydrostatic pressure on density, calculated from eq 25 for $n = 1$ and $t = -5°C$.

Table II. $\dfrac{\gamma + \Delta\gamma}{\gamma}$ for different values of $\Delta\sigma$ in $kg*/cm^2$.

γ	$\Delta\sigma = 10$	50	100	200	300
0.80000	1.000166	1.000829	1.00166	1.00332	1.00497
0.85000	1.000145	1.000726	1.00145	1.00290	1.00436
0.90000	1.000129	1.000647	1.00129	1.00259	1.00388
0.92000	1.000101	1.000505	1.00101	1.00202	1.00303
		$\gamma + \Delta\gamma$			
0.80000	0.80013	0.80066	0.80133	0.80266	0.80398
0.85000	0.85012	0.85062	0.85123	0.85247	0.85371
0.90000	0.90012	0.90058	0.90116	0.90233	0.90349
0.92000	0.92009	0.92046	0.92093	0.92186	0.92279

It is evident that even moderate pressure affects the value of the fourth decimal place.

References

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