SPECIAL REPORT 91-21

CRREL

An Analysis of the Stress Wave in Solids (SWIS) Finite Element Code

Karen J.L. Faran

November 1991

For conversion of SI metric units to U.S./British customary units of measurement consult ASTM Standard E380, Metric Practice Guide, *published by the American Society for Testing and Materials, 1916 Race St., Philadelphia, Pa. 19103.*

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U.S. Army Corps of Engineers
Cold Regions Research & Engineering Laboratory

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November 1991

Prepared for
OFFICE OF THE CHIEF OF ENGINEERS

PREFACE

This report was prepared by Captain Karen J.L. Faran, Research and Development Coordinator, Applied Research Branch, Experimental Engineering Division, U.S. Army Cold Regions Research and Engineering Laboratory.This work was funded by the Directorate of Research and Development, Office of the Chief of Engineers, Project 4A161102AT24, Task FS, Work Unit 017, *Winter Seismic/ Acoustic Energy Interactions*.

Mark Burnett, Institute of Geophysics and Planetary Physics, Scripps Institute of Oceanography, La Jolla, California, provided a copy of the SWIS code and some helpful notes on its use. Dr. Donald Albert and Randy McGilvary of CRREL provided technical review of this report.

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An Analysis of the Stress Wave in Solids (SWIS) Finite Element Code

KAREN J.L. FARAN

INTRODUCTION

The ability to analyze wave propagation for geometrically complex circumstances is important in calculating ground motion caused by earthquakes, explosions or other sources of seismic waves. Analytical models derived using separation of variables methods are limited in this area because they can only solve problems with simple geometry. For more complex situations, it is necessary to use finite element or finite difference schemes.

In 1973, Frazier (1974) developed the finite element code Stress Waves In Solids, or SWIS. It has been used to solve several challenging problems because it includes a variety of seismic propagation modes, including body waves, interface waves and diffraction. SWIS is able to simulate a number of seismic phenomena. Some examples are:

1. Explosions in geologically complex formations.
2. Spontaneous earthquake ruptures and near-field ground motions.
3. Disturbances in laterally varying earth models.
4. Wave propagation through buried and surface structures.

SWIS is a versatile code in that it can solve problems in one, two or three spatial dimensions in either Cartesian or cylindrical coordinates. Although the code assumes linear elasticity and isotropic materials, it is possible to solve problems in regions containing up to nine material types. The grid generator has a feature in which the grid size may be progressively expanded at 10% per zone to simulate a non-reflecting boundary. Finally, SWIS can solve either static, diffusion or wave propagation problems.

This report describes how to use the SWIS code, which was upgraded at the Center for Seismic Studies in 1985. (The upgrade was annotated in the code.) First, it describes how to create the input file. A discussion of the output files follow. Finally, examples of how SWIS was used to solve three wave propagation problems are discussed.

PROBLEM INITIALIZATION

The numerical algorithm in the SWIS code contains features from both finite element and finite difference methods. The continuum is divided, using spatial interpolation functions and a virtual work principle, but the sequence is modeled after Langrangian finite difference shock codes. Also, the SWIS code directly computes strain rate, stress and restoring forces instead of developing the conventional finite element stiffness matrix.

To define a stress wave problem for the SWIS code, the following quantities are required (Frazier 1974, pp 11–12.):

1. Coordinate system designation:
    a. Number of spatial dimensions to appear in the grid.
    b. Orthogonal curvilinear coordinate system to be employed in the calculations.
2. Grid configuration: Although most grids can be produced using the grid generator in the code, it is possible to supersede the generator in local regions. Grid configuration is described by:
    a. Spatial location of the node points.
    b. Node map to associate nodes with elements
3. Boundary conditions and applied forces: Each directional component of each node point is assigned one of the following constraint conditions:
    a. Unconstrained, with applied body force or surface traction to form an array of nodal forces.
    b. Constrained, with nodal displacement components constrained to follow a specified time history.
4. Material properties, described by:
    a. Density.
    b. Constitutive properties (P-wave and S-wave velocities).
    c. Dimensionless coefficient to regulate the damping of spurious high frequency numerical oscillations.
5. Time stepping data:
    a. Start and finish times.
    b. Time step, Δt.
6. Starting conditions:
    a. Velocity and displacement with respect to some reference frame.
    b. Stress at the centroid of each element.
7. Presentation of results:
    a. Element and node numbers for which results are to be printed at designated time intervals.
    b. Printer plots for displaying results at designated time intervals.
    c. Time histories of individual node points.
    d. Plot files producing graphical displays of the computed results.

FILES USED BY SWIS

For both input and output files, SWIS uses a two part code for its file names. The first half is the letter "u" followed by a one or two digit code for the Fortran unit number used in SWIS. The second half consists of a two or three letter description of the contents. Thus, file "u15in" is designated as unit 15 in SWIS, and is used as the input file, and file "u8hn" is the name of unit 8 and contains the time history for selected nodes.

Input file

To run SWIS, create an ASCII file, for unit 15 titled u15in. The format of this file and variable definitions are given in Appendix A.

Output files

SWIS produces eight ASCII files that present computed displacement and velocity results in different formats. By setting variables in input file u15in to appropriate values, it is possible to either suppress printing or set the time intervals for recording.

Each file can be divided into several blocks of information. A descriptive summary and format outline for each of the output files is given below. Format A indicates a character string, I indicates an integer and E represents exponential format.

1. *u8hn* provides displacement time histories for specified nodes at selected time intervals.

Block 1: Problem description (A).
Block 2: Grid generation description (A).
Block 3: a. Number of time steps (I6).
b. Number of degrees of freedom (I6).
c. Number of nodes with recorded histories (I6).
d. Time step (E12.4).
Block 4: Node numbers for plot history (11I7).
Block 5: Displacements for each listed node, for each time interval (8E12.4).

2. *u9he* contains time histories of element stress and displacement. Block 5 is printed for each *n*th iteration (set in input file u15in).

Block 1: Problem description (A).
Block 2: Grid generation description (A).
Block 3: a. Number of time steps (I6).
b. Number of degrees of freedom and stress components (I6).
c. Number of time history elements (I6).
d. Time step (E12.4).
Block 4: Element numbers for time histories (11I7).
Block 5: Displacements and stress components for each element (8E12.4).

3. *u10g* contains information about the deformed grid. Block 6 is printed only if a force greater than 0.0001 N is applied to the node. If time history nodes are identified, both blocks 7 and 8 are printed; if no nodes are identified, only block 8 is printed. Blocks 9–12 are printed every *n*th iteration (set in input file u15in).

Block 1: Problem description (A).
Block 2: Grid generation description (A).
Block 3: a. Number of spatial dimensions [ndimt] (I7).
b. 2^{ndimt} (I7).
c. Total number of elements (I7).
d. Number of different material types (I7).
Block 4: Coordinates to be used in grid generation mapping (8E12.4).
Block 5: a. *P*-wave velocity (E12.4).
b. *S*-wave velocity (E12.4).
c. Density (E12.4).
Block 6: a. Digit used to separate data (I6).
b. Node coordinates (8E12.4).
Block 7: a. Digit used to separate data (=10) (I6).
b. Node coordinates of nodes with time histories (8E12.4).
Block 8: a. Digit used to separate data (=999) (I6).
b. Node coordinates of node 1 (8E12.4).
Block 9: Time (E12.4).
Block 10: a. 2^{ndimt} (I7).
b. Material number (I6).
Block 11: Node coordinates of lowest node number in elements (8E12.4).
Block 12: (Displacement)+(velocity)*(damping) of lowest node number in elements (8E12.4).

4. *ul1vn* supplies data for plotting node vectors. Block 5 is printed for every *n*th iteration.

Block 1: Problem description (A).
Block 2: Grid generation description (A).
Block 3: a. Number of spatial dimensions (I7).
b. $2^{\text{number degrees of freedom}}$ (I7).
c. Total number of nodes (I7).
Block 4: a. Integer code used for specifying nodal constraints (I6).
b. Node coordinates (3E12.4).
Block 5: a. Time advance (E12.4).
b. Displacements and velocities (8E12.4).

5. *ul2ve* is supposed to provide data for plotting element vectors. Currently, no information is sent to this file.

6. *ul3ln* provides displacement and velocity information for specified lines of nodes. Block 4 is repeated for each line of nodes. Block 5 is printed for every *n*th iteration of the program (set in file u15in). In Block 5, the items b, c and d are printed for each line of nodes. Furthermore, displacement and velocities (item d) are printed for each node in the line.

Block 1: Problem description (A).
Block 2: Grid generation description (A).
Block 3: a. Number of dimensions (I7).
b. (Number of degrees of freedom)*2 (I7).
c. Number of node lines (I7).
Block 4: a. Node line number (I7).
b. Number of nodes (I7).
c. Node positions (8E12.4).
Block 5: a. Time advance (E12.4).
b. Node line number.
c. Number of nodes in line.
d. Displacements and velocities for each node (8E12.4).

7. *ul4div* provides the divergence and curl information of the nodes specified in file u13ln. Information is sent to u14div only if information is requested for lines of nodes, i.e., if data are sent to file u13ln. The output file has only one output format block, which is printed for each *n*th iteration and for each specified line of nodes. Currently, u14div is only printed for problems with two spatial dimensions and with a rectangular mesh.

Block 1: a. Node line number (I7).
b. Number of nodes in line (I7).
c. Divergence and curl for each node in the line (8E12.4).

8. *ul6out* summarizes analysis description, provides summary of control parameters, grid definition, material definition, node constraints and output specifications. If so desired, u16out also contains the computed results for specified time intervals. The organization of this file is self-evident. An example follows.

Example of u16out

```
1. ANALYSIS DESCRIPTION
   --------------------
 u15in.1d.2, one-dim prob, dt=0.01


2. CONTROL PARAMETERS
   ------------------
   - Spatial Representation:
     Number of Space Dimensions used ............     1
     Number of Degrees of Freedom per Node .......    1
     Number of Stress Components ................     1
     Solution Coordinate Designation ............     0
     Order of Fourier Interpolation in Azimuth ...    0

   - Time Control:
     Number of Time Deriviatives ................     2
     Time Step ..................................     0.0100
     Starting Time ..............................     0.0000
     Ultimate Time ..............................     2.0000


3. GRID DEFINITION
   ---------------
   - Grid Generation, Designator MAPYZ = 2
 Regular grid, each element 0.05 meter long
     grid size:  NEI  100      NEJ     1     NEK     1
     producing:          100 elements and       101 nodes
     grid growth to element: IS    0  JS    0  KS    0
     grid growth begins at:  IG    0  JG    0  KG    0
     corner nodes of the grid exterior:
     0.00      10.00


4. MATERIAL DEFINITION
   -------------------
   - Number of Different Constituents     1
      MAT      DENS     P-VEL      S-VEL      POIS      DAMP
        1    2.7000    6.3000     3.1000    0.3403    0.0000
   - Material Numbers Assigned to Individual Elements
     Lines of Data     0


5. NODE CONSTRAINTS
   ----------------
   - Lines of Constraint Data     2
      NODE     IDNODE  SPECIFIED CONSTRAINTS
        1          1    0.0000    0.0000    0.0000          0          0
      101          0    1.0000    0.0000    0.0000          0          0

6. OUTPUT SPECIFICATIONS
   ---------------------
   - Print Results at Interval .........   0
   - Plot Deformed Grid at Interval ....   0
   - Plot Node Vectors at Interval .....   0
   - Plot Element Vectors at Interval ..   0
   - Plot (  0) Node Lines at Interval ..   0
   - Plot Time Histories of (  5) Nodes:
             21        41        61        81        101
   - Plot Time Histories of (  0) Elements:


NODE     F T P BOUND                 INITIALIZATION SUMMARY                     R P T M
 ...     O Y R  TYPE                 ----------------------                     A R Y A
 ...     R P I    1   SPECIFIED CONSTRAINTS   NODE COORDINATES         MASS   N I P T
 ...     C E N   1 2                                                           G N E
 ...     E   T 1 2 3    S1        S2        S3        Y1        Y2        Y3          E T

      1 1 0 0 0 0 1  0.000   0.000   0.000     0.00     0.00     0.00     0.13 1 0 0 1
    101 2 0 2 0 0 0  1.000   0.000   0.000    10.00     0.00     0.00     0.14


            MOTION AT TIME =      0.0100 (time step =      1)

ODE BND ...........DISPLACEMENT............ .......VELOCITY COMPONENTS........  ...
 21   0  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00  0.
 41   0  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00  0.
 61   0  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00  0.
 81   0  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00  0.
101   0  0.7407E-03  0.0000E+00  0.0000E+00  0.7407E-01  0.0000E+00  0.0000E+00  0.
```

SELECTED EXAMPLES

To test the SWIS code, stress waves were calculated for three wave propagation problems: one-dimensional longitudinal displacement subjected to impulse loading; a cantilever beam with an impulse load applied along the axis, at the unsupported end; and two-dimensional wave propagation with a vertical impulse force (Lamb's problem). The input files and results for these test calculations follow.

Example 1: One-dimensional longitudinal displacement

Analytical solution

The first problem considered was that of one-dimensional stress longitudinal displacement, i.e., only displacements in the x-direction were allowed. This situation describes the wave propagation in the middle of a large piece of material, rigidly constrained at one face and with a uniform pressure applied impulsively at the other (see Fig. 1). The material is allowed to move only in the direction of the applied force, and as a result, all other displacements vanish. The equations of motion, initial conditions and boundary conditions reduce to the following one-dimensional problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c_1^2}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

initial conditions: $$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0$$

boundary conditions: $$u(0,t) = 0$$

$$P(\ell,t) = P\,\delta(t)$$

where u = displacement
t = time
x = position along beam
c_1 = $[(\lambda+\mu)/\rho]^{1/2}$, the longitudinal or P-wave velocity
P = magnitude of the constant pressure
$\delta(t)$ = delta function
ℓ = length of the beam.

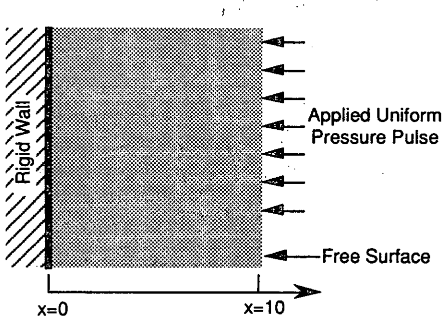


Figure 1. Geometry for example 1, one-dimensional stress, longitudinal displacement.

ρ = material density.

The solution to eq 1 can be found by either a separation of variables or by using transforms. The latter technique gives the solution as (Graff 1975, pp. 91–94)

$$u(x,t) = \frac{P}{\rho c_1}\left[\left\{H < t - \frac{(\ell - x)}{c_1} > - H < t - \frac{(\ell + x)}{c_1} >\right\} - \right.$$

$$\left\{H < t - \frac{(3\ell - x)}{c_1} > - H < t - \frac{(3\ell + x)}{c_1} >\right\} +$$

$$\left.\left\{H < t - \frac{(5\ell - x)}{c_1} > - H < t - \frac{(5\ell + x)}{c_1} >\right\} - \ldots\right] \tag{2}$$

where $H < t - a >$ is the Heaviside function, defined such that

$$H < t - a > = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

Equation 2 defines a square wave propagating between the two ends of the material, with wave speed equal to the longitudinal wave speed.

Input file

The mesh created for this example was a string of 201 nodes, lined in the x-direction, which created 200 line elements (Fig. 2). Since displacement is restricted to only the x-direction, it is unnecessary to create a three-dimensional mesh. If the material is aluminum, values for element length, time step, material properties, magnitude of the impulse and dimensions of the region are as follows:

time step (Δt):	DT = 0.005 (ms)
density (ρ):	DENS(1) = 2.70 (Mg/m^3)
P-wave velocity (c_1):	VP(1) = 6.30 (km/s)
S-wave velocity (c_t):	VS(1) = 3.10 (km/s)
damping:	DAMP(1) = 0.0
impulse force (P):	VSPEC(2,1) = 1.0 (N)
length of region (ℓ):	YGRID(1,2)–YGRID(1,1) = 10.0 (m).

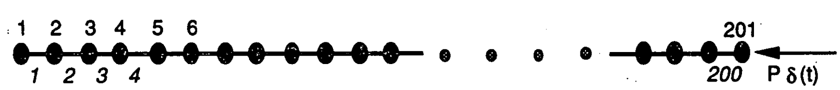
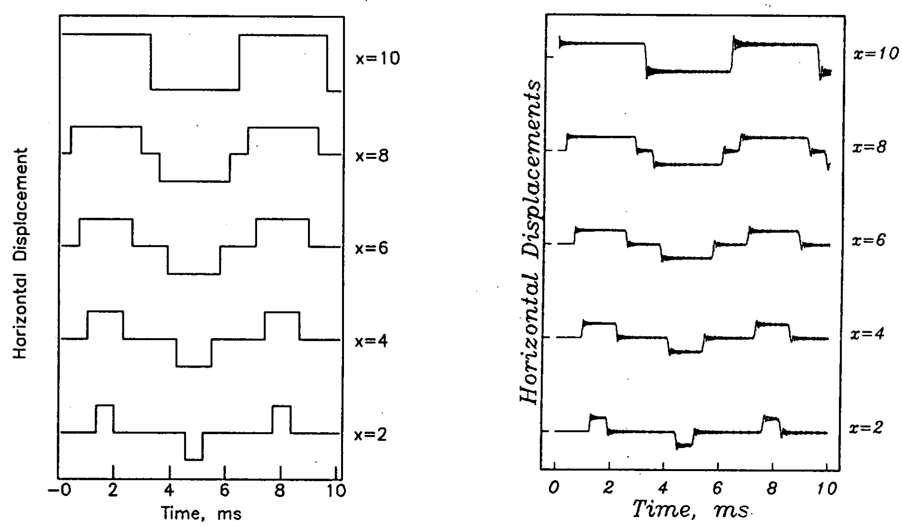


Figure 2. Finite element mesh for one-dimensional stress problem (200 elements, 201 nodes).

For this problem, node 1 was assigned zero displacement to meet the fixed end condition (line 13 of the following file). A unit impulsive force was applied to the free end of the beam, node 201, at time $t = 0$ (line 14). Finally, records of the displacements were made for five nodes along the beam: 41, 81, 121, 161 and 201 (line 17). The input file for this example follows (entries correspond to Appendix A).

Entry	Line	
A	1	Test input, one-dim prob, *dt*=0.005 ms
B	2	1 1 0 0
C	3	2 0.005 0 10
D	4	Regular grid, each element 0.05 m long
E	5	200 0 0 2 2 0 0 0 0 0 0
F	6	0.0 10

Entry	Line							
G	7	0						
I	8	0						
K	9	1						
L	10	1	2.70	6.30	3.10	0.0		
M	11	0						
O	12	2						
P	13	1	1	0	0	0	0	0
	14	201	0	1.0	0	0	0	0
R	15	1	0	0	0			
S	16	0	0					
U	17	5	41	81	121	161	201	
V	18	0	0					



a. Analytical solution. *b. SWIS* (dx = *0.05 m;* dt = *0.005 s).*

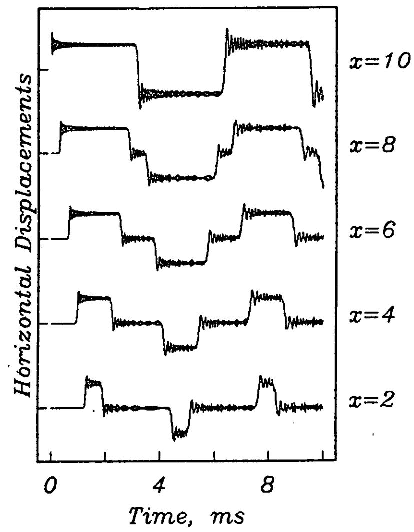
Figure 3. Comparison of analytical and SWIS waveforms calculated for example 1.

Comparison of output to theory

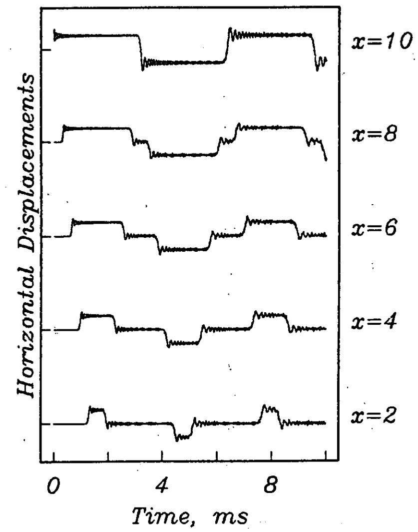
The disturbance for the given parameters should be a square wave reflecting between the two ends of the material, at the longitudinal velocity of 6.3 km/s. Figure 3 shows that the expected and calculated waveforms match.

One of the shortcomings of the SWIS solution is the large, unrealistic amount of ringing in the results. Much of this oscillation has been eliminated from previous runs by decreasing both the element size and time step (Fig. 4). It is expected that the solution could be further refined by additional reductions in the spatial and time increments.

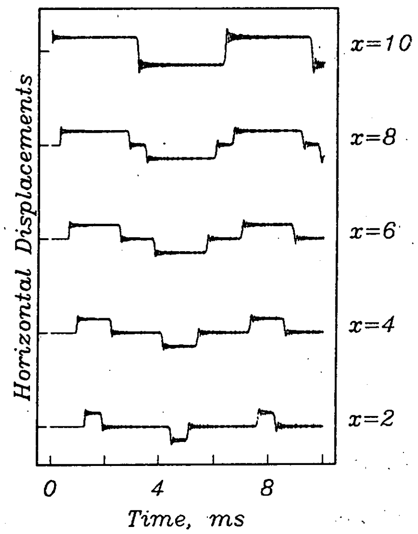
Another possible way of reducing the oscillations and removing the high frequency noise in the figures would be to introduce a damping factor with the material parameters. Figure 5 shows that a damping factor of 0.2 significantly removes the oscillations in Figure 3b, and the solution using this damping factor closely resembles the analytical solution. This method may have adverse effects on the solution, however, in that the higher damping factors change the form of the calculated waves. As seen in Figure 5, the solutions obtained using non-zero damping factors have slightly rounded corners and finite rise times. However, the damping factors considered did not seem to affect the amplitude of the wave, nor did they change the velocities at which the disturbances travel.



a. dx = *0.10 m*, dt = *0.01 s.*



b. dx = *0.10 m*, dt = *0.005 s.*



c. dx = *0.05 m*, dt = *0.005 s.*

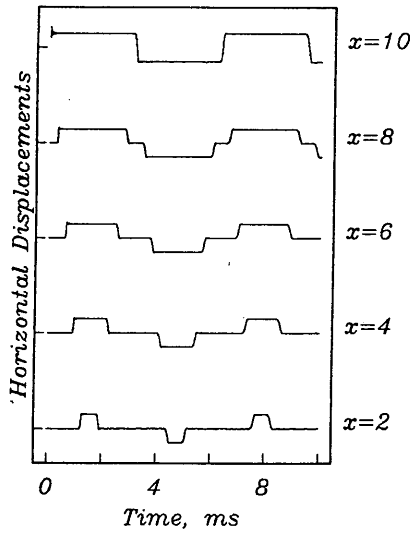
Figure 4. Comparison of different space and time steps for example 1. The above plots use the same vertical scale.

Example 2: Cantilever beam

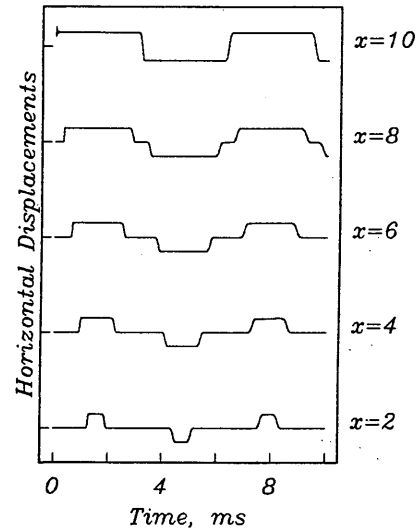
Analytical solution

The second example considered was that of a wave propagating along a long and very thin rod, or one-dimensional stress where the longitudinal normal stress σ_x is a function of position along the rod and time only (Fig. 6). All other stresses vanish, and elements are allowed to deform in the transverse direction. The equations of motion reduce to

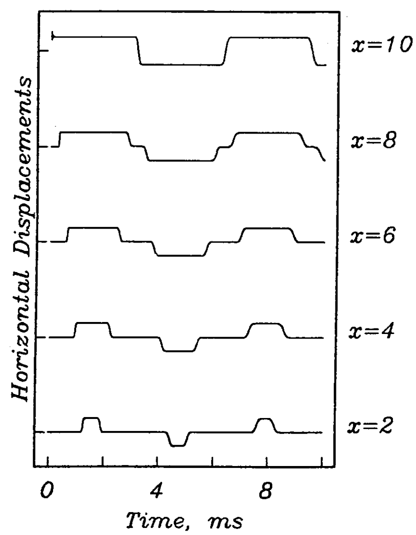
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c_b^2}\frac{\partial^2 u}{\partial t} \tag{3}$$



a. Damping = 0.2.



b. Damping = 0.4.



c. Damping = 0.6.

*Figure 5. Damping effects on one-dimensional model (example 1). The above plots use the same vertical scale (*dx *= 0.05;* dt *= 0.005 s).*

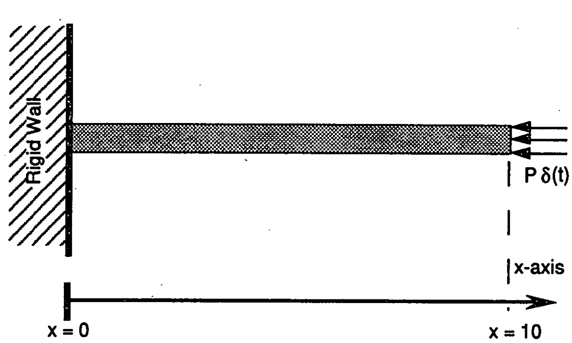


Figure 6. Geometry of wave propagation for example 2, a cantilevered beam.

where $c_b^2 = \dfrac{E}{\rho}$, the beam velocity

$E = \mu[(3\lambda+2\mu)/(\lambda+\mu)]$, Young's modulus of elasticity

ρ = material density.

With no initial displacements nor velocities along the beam, and boundary conditions of $u(0,t) = 0$ and $P(\ell,t) = P\delta(t)$, the solution to this problem is almost identical to the previous problem. The only difference between the two solutions is the velocity at which the wave propagates through the beam ($c_b < c_l$). Manipulation of the relations between the material constants yield the following relation for c_b in terms of the longitudinal and transverse velocities

$$c_b = 2c_t^2\left[\left(1.5\, c_l^2 - 2\, c_t^2\right) / \left(c_l^2 - c_t^2\right)\right]. \tag{4}$$

Laplace transform techniques (Graff 1975, pp. 91–94) give the solution to the problem as

$$\begin{aligned} u(x,t) = \frac{P}{\rho c_b}\Bigg[&\left\{H < t - \frac{(\ell - x)}{c_b} > - H < t - \frac{(\ell + x)}{c_b} >\right\} - \\ &\left\{H < t - \frac{(3\ell - x)}{c_b} > - H < t - \frac{(3\ell + x)}{c_b} >\right\} + \\ &\left\{H < t - \frac{(5\ell - x)}{c_b} > - H < t - \frac{(5\ell + x)}{c_b} >\right\} - \dots\Bigg] \end{aligned} \tag{5}$$

where u = displacement

P = magnitude of the load

ρ = material density

c_b = beam velocity

ℓ = length of the beam

t = time

x = position along beam

$H<t-a>$ = Heaviside step function, defined in the previous example.

The solution to eq 3 is a square wave propagating between the ends of the material at the beam velocity. Because the beam velocity c_b is less than the longitudinal velocity c_l, this wave travels slower than the wave in example 1. The amplitude of the resulting wave, however, is larger than that of the previous example. A plot of displacement versus time, for five points on the beam, is given in Figure 8a.

Input file

This example differs from the longitudinal displacement problem because the nodes must be allowed to move in the transverse directions (because of the Poisson effect). A one-dimensional mesh is not capable of handling these displacements, and so either a two- or three-dimensional grid must be used. To reduce computation time, a two-dimensional mesh was created (Fig. 7) to model an aluminum beam. The parameters for this example follow.

time step (Δt):	DT = 0.005 (ms)
density (ρ):	DENS(1) = 2.70 (Mg/m^3)
P-wave velocity (c_l):	VP(1) = 6.30 (km/s)

S-wave velocity (c_t):	VS(1) = 3.10 (km/s)
beam velocity (c_b):	$c_b = 5.1$ (km/s)
damping:	DAMP(1) = 0.0
impulse force (P):	VSPEC(2,1) = 1.0 (N)
length of beam (ℓ):	YGRID(1,2) – YGRID(1,1) = 10.0 (m).

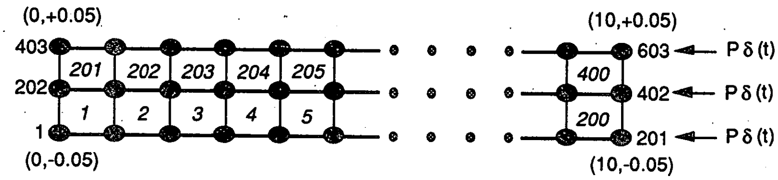


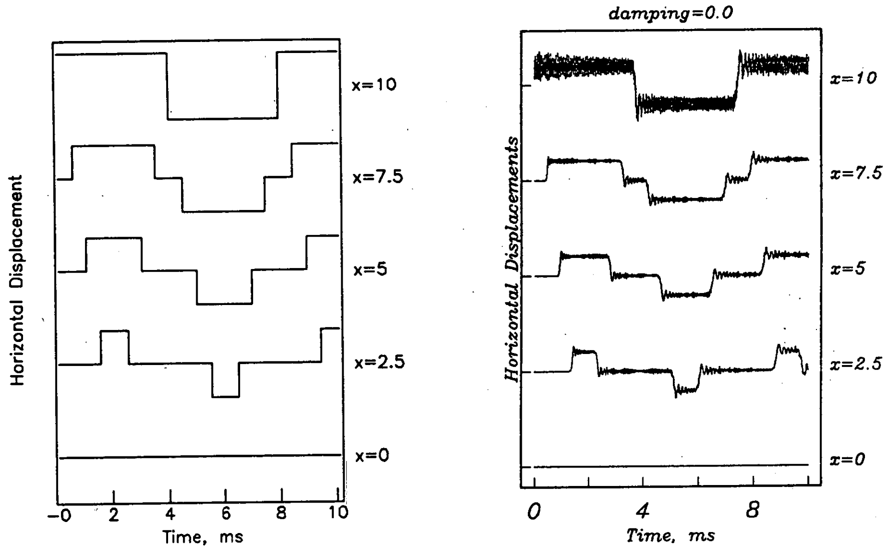
Figure 7. Finite element mesh of beam problem (400 elements, 603 nodes). Impulse force applied at nodes 201, 402 and 603; nodes 1, 202 and 403

For this example, the displacements for the nodes at $x = 0$, nodes 1, 202 and 403, were set identically equal to zero (lines 13, 15 and 17 of the following input file). At the free end of the beam, a unit impulse was applied in the direction of the beam axis (lines 14, 16 and 18 of the following input file). The longitudinal displacements were recorded for five nodes located on the center fiber of the beam (line 22) (entries correspond to Appendix A).

Entry	Line											
A	1	2-d model of beam, dt=0.005 ms										
B	2	2	2	0	0							
C	3	2	0.005	0	10							
D	4	Regular elements, 0.05 × 0.05 m long										
E	5	200	2	0	2	4	0	0	0	0	0	0
F	6	0.0	–0.05		10.0	–0.05	0	0.05	10.0	0.05		
G	7		0									
I	8		0									
K	9		1									
L	10		1	2.70	6.30	3.10	0.0					
M	11		0									
O	12		6									
P	13		1	11	0	0	0	0	1			
	14		201	00	1.0	0	0	0	1			
	15		202	11	0	0	0	0	1			
	16		402	00	1.0	0	0	0	1			
	17		403	11	0	0	0	0	1			
	18		603	00	1.0	0	0	0	1			
R	19		0	0	0	0						
S	20		0	0								
U	21		5	202	252	302	352	402				
V	22		0	0								

Comparison of output to theory

A plot of displacement versus time, as calculated by SWIS, for the above input file is shown in Figure 8b. SWIS calculates a waveform with a shape and velocity close to that of the analytical solution. As in the case of the one-dimensional strain example, it is expected that refining the input mesh and decreasing the time step could further improve the results.



a. Analytical solution. *b. SWIS (dx = 0.05 m; dt = 0.005 s).*

Figure 8. Comparison of analytical and SWIS waveforms calculated for example 2.

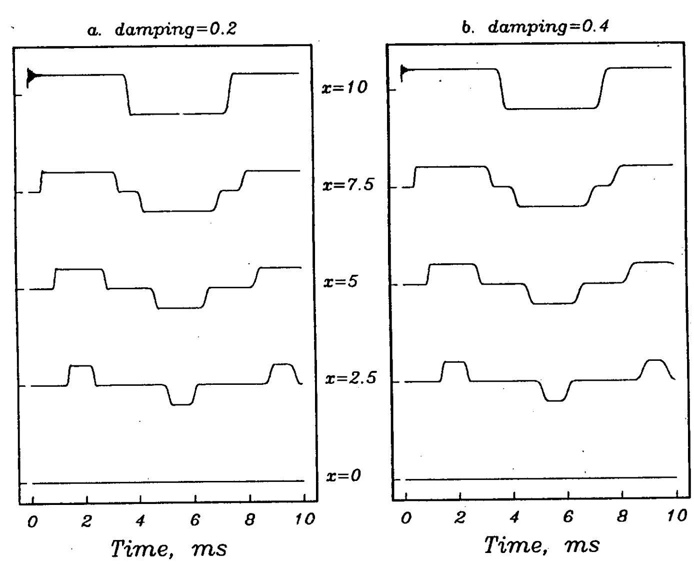


Figure 9. Damping effects on beam. Plots use the same vertical scale (dx = 0.05 m; dt = 0.005 s).

Applying a damping factor again removes much of the oscillations (Fig. 9). A factor of 0.2, however, already seems to modify the solution in that the waveform is no longer a square wave. Increasing the damping factor from 0.2 to 0.4 removes more of the high frequency components, but results in only a small change in the solution. For these damping factors, little or no reduction is noticed in the amplitudes or the wave velocities.

Example 3: Lamb's problem in two-dimensional Cartesian coordinates

The third example treated the two-dimensional Lamb's problem, a vertical point load applied impulsively in the plane of the grid (Fig. 10). The results from this example were compared to the waveforms generated by other computing schemes.

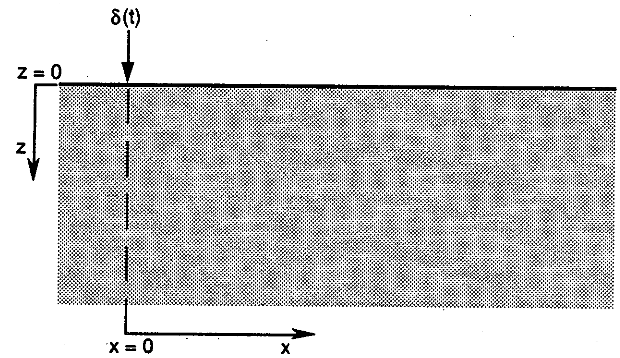


Figure 10. Example 3, geometry of Lamb's problem.

Input file

No-displacement constraints were applied to the vertical sides of this mesh so that the waves would reflect off of the sides. To compare the solution from SWIS to other models (Kuhn 1985, p. 1112), the following parameters were used:

time step (Δt):	DT = 2 (ms)
ending time:	TMAX = 400 (ms)
density (ρ):	DENS = 1.0 (Mg/m^3)
P-wave velocity (c_1):	VP = 1.00 (km/s)
S-wave velocity (c_t):	VS(1) = 0.60 (km/s)
impulse force (*P*):	VSPEC(2,1) = 1.0 (N).

For this problem, SWIS was run with several input files to observe the effect of the damping factor and to get information for different types of plots. For all of the input files, however, the mesh used was a two-dimensional grid with rectangular elements, each 3 by 3 m. The mesh had 70 elements in each direction, and had a total of 5041 nodes. A vertical force was applied at the left upper corner of the mesh (node 4971) and the vertical sides of the grid were constrained so that these nodes had no horizontal movement (Fig. 11).

The file shown below was run to obtain information for a contour plot. For every 10 time steps (20 ms), data were recorded for 15 strings of nodes (line 17 of the input file), each string containing 15 nodes (lines 18–32). The damping factor in this run is 0.2 (last entry in line 10) (entries correspond to Appendix A).

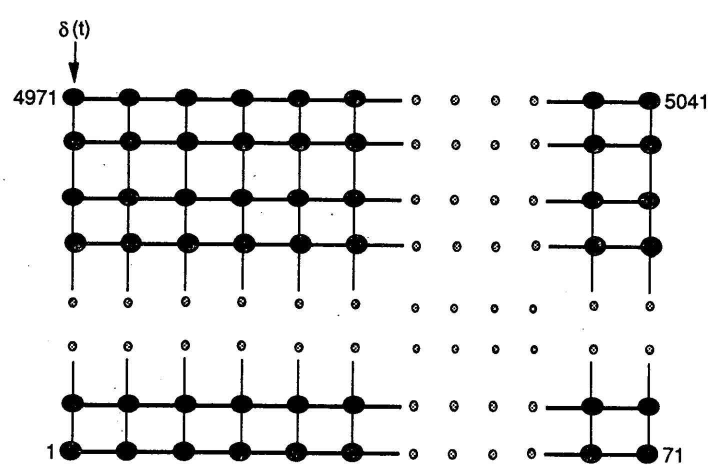


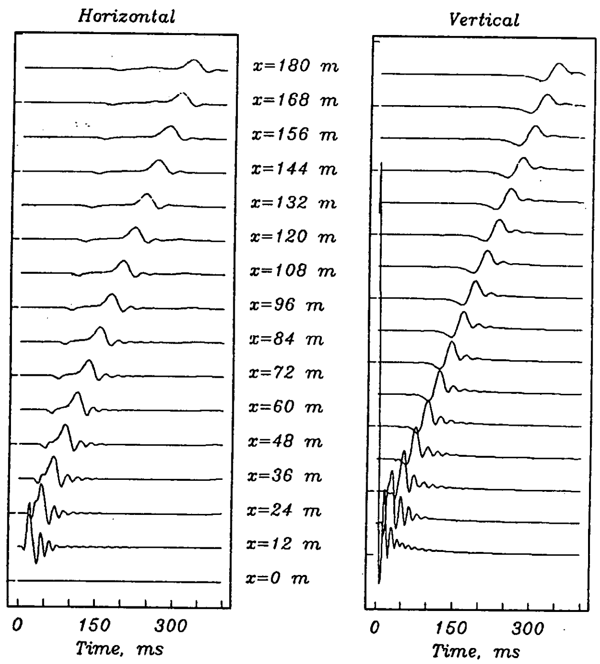
Figure 11. Finite element mesh used for Lamb's problem.

Entry	Line											
A	1	Lamb's problem, *dt*=2 ms, *t*max=400 ms										
B	2	2	2	0	0							
C	3	2	2	0	400							
D	4	each element 3 × 3 m, total grid 210 × 210 m										
E	5	70	70	0	2	4	0	0	0	0	0	0
F	6	0	−210	210	−210	0	0	210	0			
G	7	0										
I	8	0										
K	9	1										
L	10	1	1.00	1.0	0.6	0.2						
M	11	0										
O	12	3										
P	13	1	10	0.0	0.0	0.0	69	71				
	14	71	10	0.0	0.0	0.0	70	71				
	15	4971	10	0.0	1.0	0.0	0	1				
R	16	0	0	0	0							
S	17	15	10									
T	18	4971	14	5								
	19	4616	14	5								
	20	4261	14	5								
	21	3906	14	5								
	22	3551	14	5								
	23	3196	14	5								
	24	2841	14	5								
	25	2486	14	5								
	26	2131	14	5								
	27	1776	14	5								
	28	1421	14	5								
	29	1066	14	5								
	30	711	14	5								
	31	356	14	5								
	32	1	14	5								
U	33	0										
V	34	0	0									

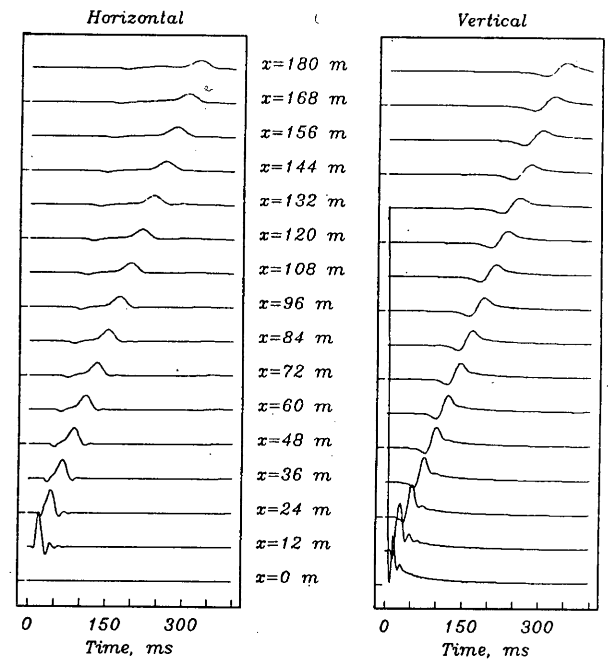
Discussion of output

To evaluate the results, horizontal and vertical displacements and velocities were plotted against time (Fig. 12 and 13) and contour plots of the displacements (Fig. 14) were produced. The range and depth scales for Figures 12 through 14 were chosen to match those of Kuhn's (1985) figures. It is important to note that the plots in Figure 14 may contain some artifacts attributable to the automatic contouring algorithm. For example, the contour plot of the horizontal displacement at 80 ms (Fig. 14a1) indicates zero displacement at about 108 m. This particular contour line is not part of the wave front, but a result of the automatic smoothing in the contouring algorithm. Despite the artifacts, however, it is relatively easy to identify the wavefronts in the contour plots. The contour lines of interest are grouped closely to each other, and compose the "steep" portions of the mapping.

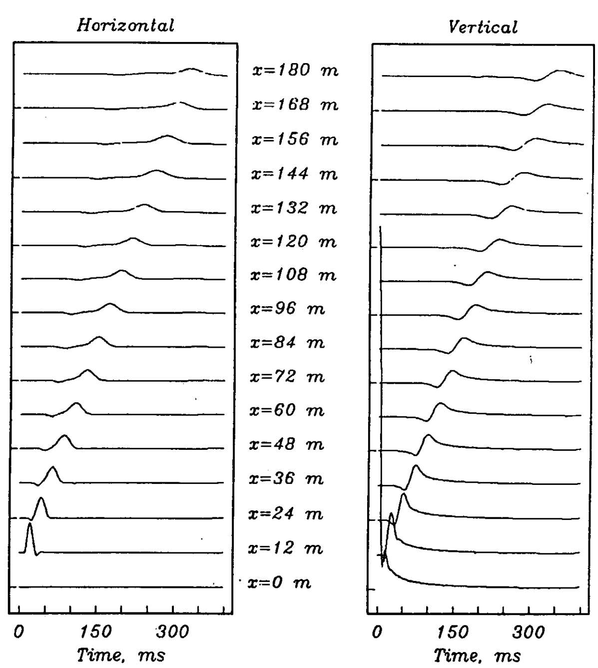
The plots of horizontal and vertical displacement in Figure 12 show the disturbance propagating through the material. Since the wave velocities for the material are known, it is possible to determine the arrival of each wave front. For example, on the 180-m trace of horizontal displacements in Figure



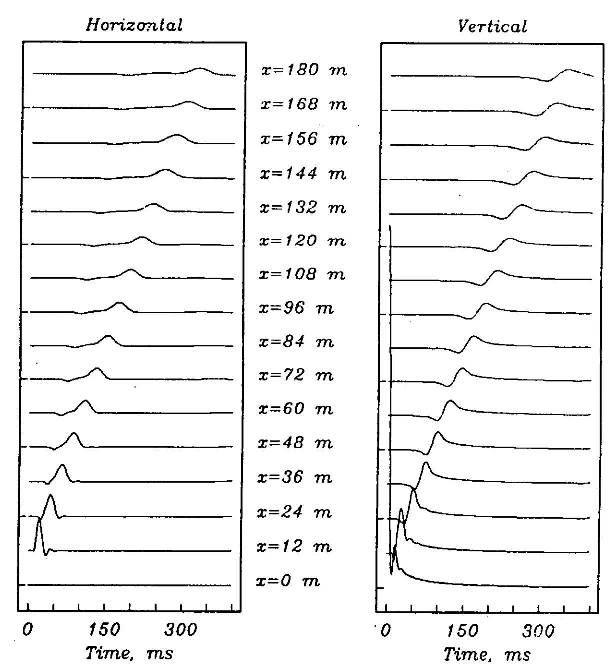
a. *Damping = 0.2.*



b. Damping = 0.4.



c. Damping = 0.6.



d. Damping = 0.8.

Figure 12. Horizontal and vertical surface displacements vs time for example 3. All plots use the same vertical scale (dx = 3 m; dt = 2 ms).

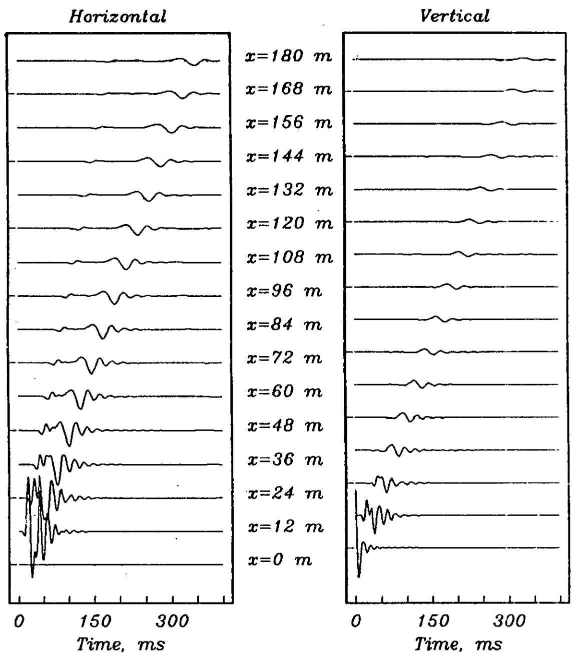


*Figure 13. Horizontal and vertical surface velocities vs time for example 3 (*dx *= 3 m;* dt *= 2 ms; damping = 0.2).*

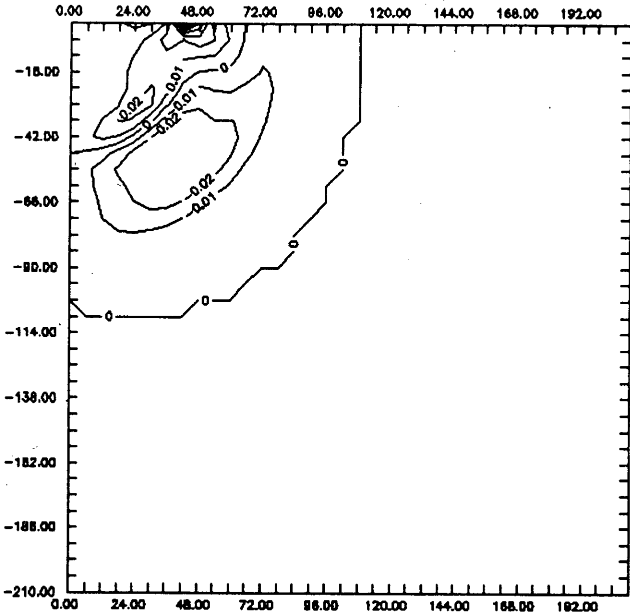
12a, a disturbance arrives at approximately 180 ms. This corresponds to a velocity of 1 km/s, and implies that the disturbance is a pressure wave. A second wave front reaches the 180-m range at approximately 300 ms, has a velocity of about 0.6 km/s, and could be either the shear or Rayleigh wave. Figure 13, a plot of horizontal and vertical surface velocities against time, also shows the propagation of the three waves. Finally, notice that the waves are non-dispersive. This agrees with theory, since the example models a non-layered half-space.

The Nyquist frequency, or the highest frequency that can be monitored owing to the sampling time step, is $f_N = 1/(2\Delta t) = 1/(0.004\ \mathrm{s}) = 250$ Hz. From Figure 12, the period of the Rayleigh wave is approximately 45 ms, and corresponds to a dominant frequency of 22 Hz. This is an order of magnitude smaller than the Nyquist frequency, and so it is reasonable to expect that the Rayleigh wave is well represented in the plot.

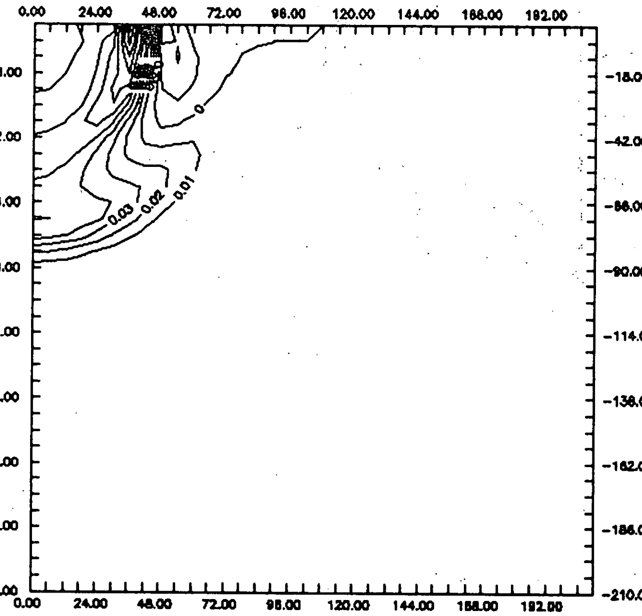
The displacement contours in Figure 14 yield results consistent with theory. First, there are no horizontal displacements directly beneath the source (x = 0-m axis), a constraint set in the input file. Disturbances at the depths of 80 and 160 m are observed on the $x = 0$ axis of the vertical displacement contour plots at 80 and 160 ms respectively. These disturbances traveled at a rate of 1 km/s, and probably correspond to the pressure wave. The second disturbance, the combined effect of the shear and Rayleigh waves, is observed near the range of 48 m on the 80-ms plot and at about 96 m on the 160-ms plot. Finally, the displacement magnitudes, especially in the horizontal displacement contour plots, fall away to zero with increase in depth and indicate the presence of a Rayleigh wave.

Since the compressional energy and shear energy are proportional to the squares of the divergence and curl of displacement, respectively (Dougherty and Stephen 1987, p. 242), contour plots of the divergence and curl were created to better observe the arrival of the various wavefronts at $t = 80$ and 160 ms. The equations used to find divergence and curl are given in Appendix B.

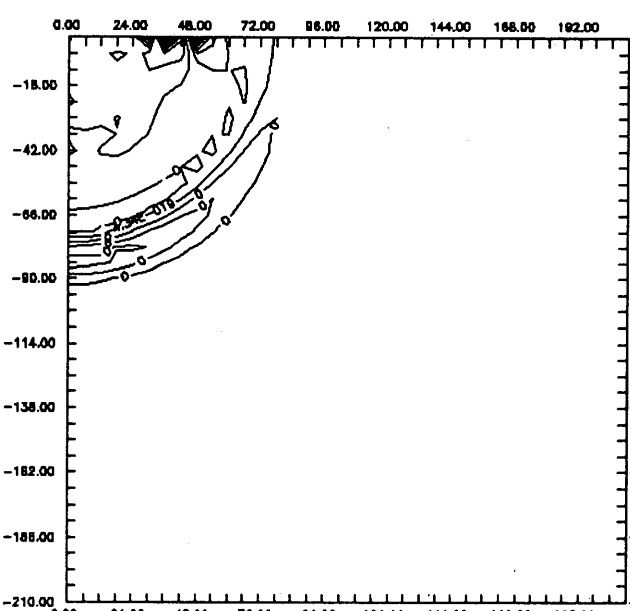
The contour plots of the divergence and curl facilitate observation of the wave fronts. The divergence contour plots show the pressure wave front as being almost spherical. Disturbances from shear waves and surface waves are present on the curl contour plots, but it is difficult to distinguish



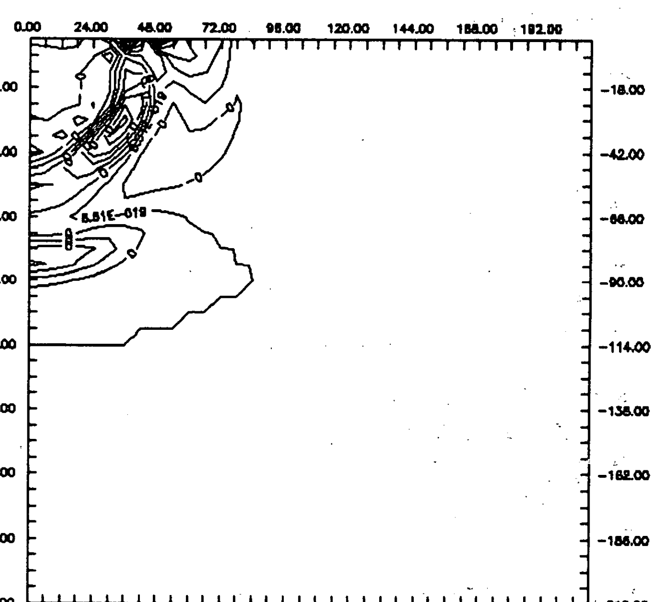
1. Horizontal displacement.



2. Vertical displacement.



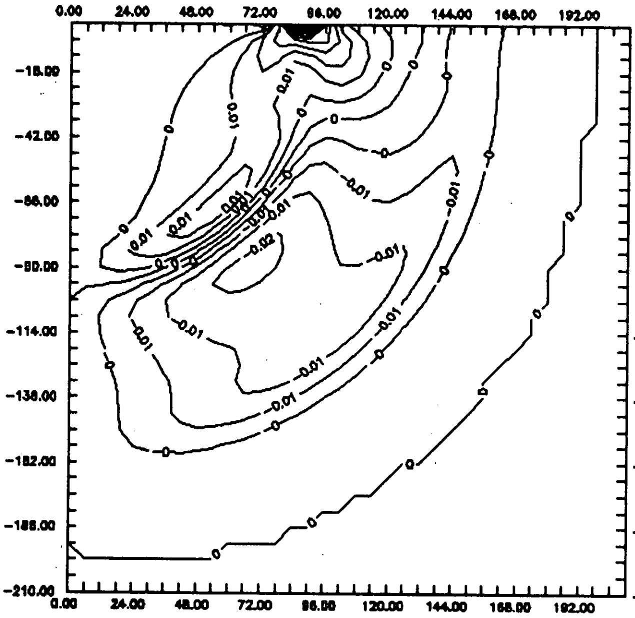
3. Divergence.



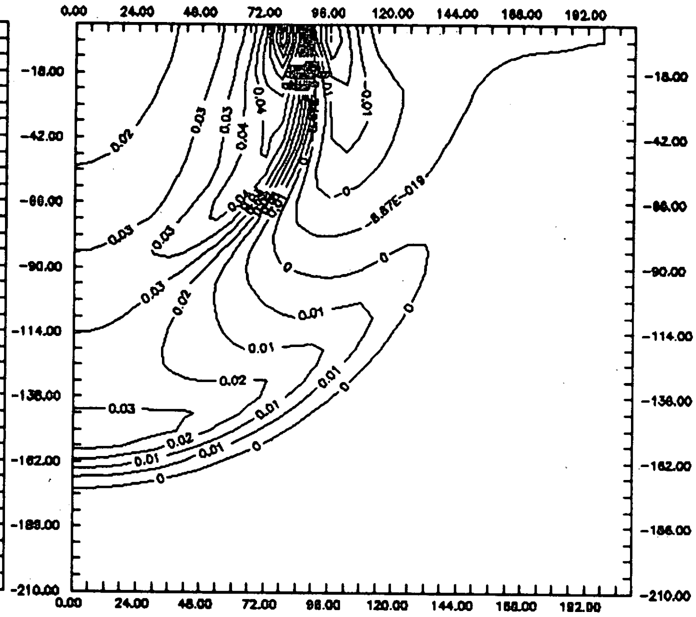
4. Curl.

a. At time t = *80 ms. Wave fronts are at the following locations:* P*-wave at 80 m; S-wave at 48 m; Rayleigh wave at 44 m.*

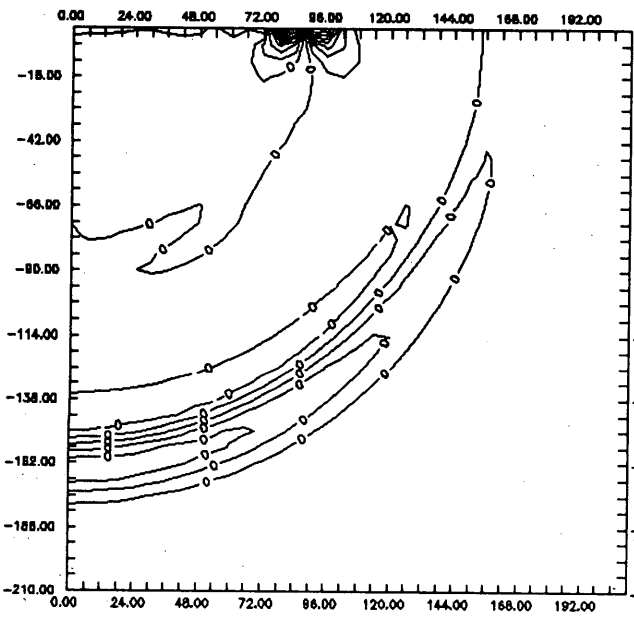
Figure 14. Contour plots of displacements and divergence and curl of displacements for example 3.



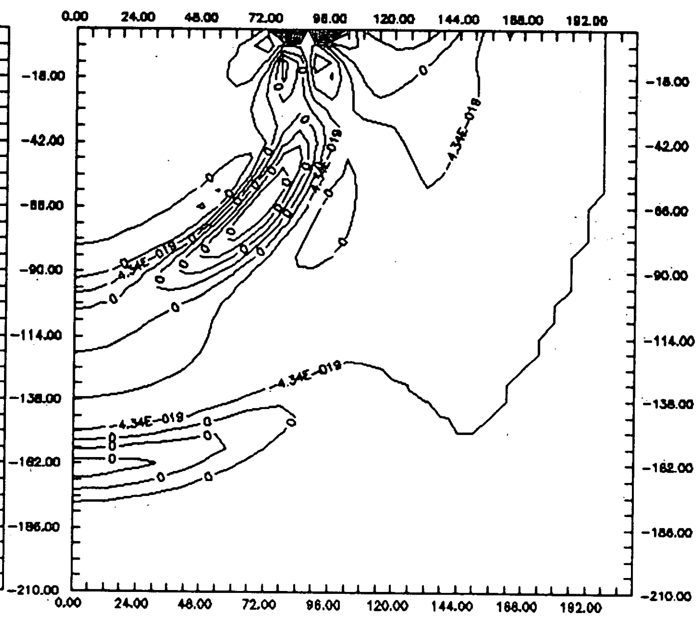
1. Horizontal displacement.



2. Vertical displacement.



3. Divergence.



4. Curl.

b. At time t = *160 ms. Wave fronts are at the following locations:* P-*wave at 160 m;* S-*wave at 108 m; Rayleigh wave at 88 m.*

Figure 14 (cont'd).

between the two waves at the surface of the material since the wave speeds are almost equal. At a depth greater than 18 m, however, the Rayleigh wave displacements fall away, and only the shear wave remains.

Computation time

On the ILLIAC computer, Frazier (1974, p. 65) estimated that the calculations were processed that the rate of 0.4 ms per two-dimensional element per numerical time step. With 200 time steps, and 4900 elements (5041 nodes), each of the runs took approximately 30 minutes of real time on a Masscomp 5550, a 32-bit computer running at 20 MHz. At this rate, the computer processes at approximately 1.8 ms per element per numerical time step.

Damping factor

Frazier, when using SWIS, used different damping factors for the longitudinal and transverse waves. It is not apparent, however, how he specified the two factors in the input file as our version of the code does not allow this option. At this point, the magnitude required to reduce only the high frequency noise resulting from numerical dispersion has not yet been determined. A value of 0.2 does not seem sufficient because the source wave oscillates much more than what has been observed in both field work and other mathematical models. Damping factors set to 0.4 and 0.6 reduced the amount of oscillation, but also damped the results. Finally, a value of 0.8 caused the disturbance to die out almost immediately.

Comparison with other models

Kuhn (1985) also conducted a study of Lamb's problem in two dimensions. He used the same material parameters and numerically integrated the analytical solution. In his work, however, Kuhn

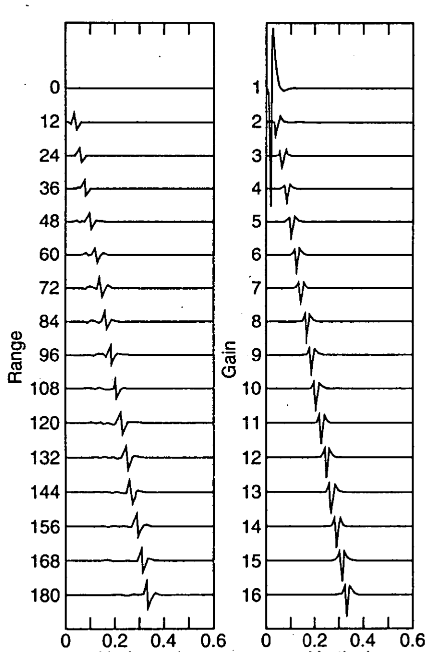


Figure 15. Horizontal and vertical velocities (m/s) calculated by Kuhn (after Kuhn 1985, p. 1114, his Fig. 6a).

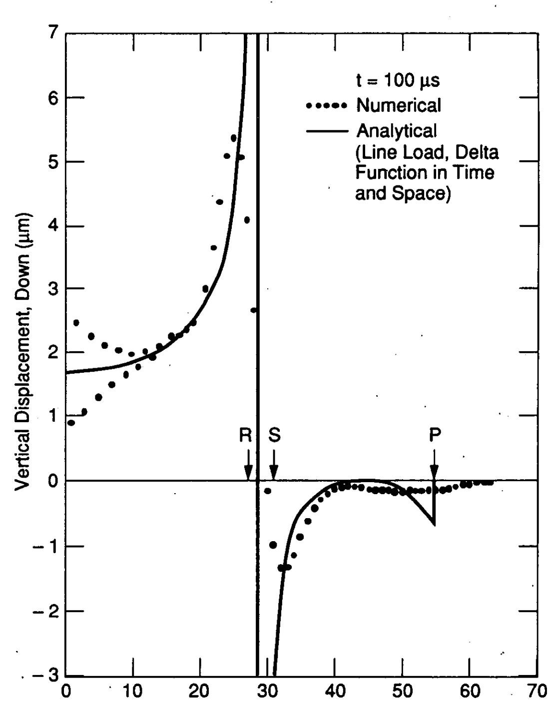


Figure 16. Vertical displacement time history calculated by Frazier (after Frazier 1974, p. 67, his Fig. 4.2), damping = 0.8.

used a different approximation for the impulse force, solved the problem in cylindrical coordinates and used a mildly viscoelastic material for his half-space. These results are shown in Figure 15. Kuhn's figure shows surface velocities, and contains two records of 16 traces each, with ranges varying from 0–180 m. The middle column of numbers represents gain, which is constant along each trace, and allows the comparison of absolute amplitudes between different offset traces.

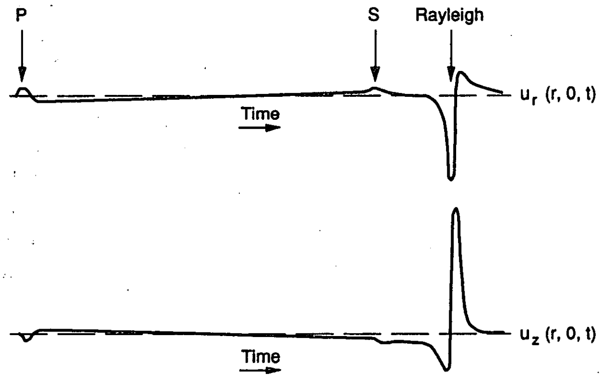


Figure 17. Horizontal and vertical displacements calculated by Lamb (after Graff 1975, p. 369, his Fig. 6.21).

In all of his calculations, Kuhn used only one source function, which had a dominant frequency of about 20 Hz (Kuhn 1985, p. 1108). Because he solves Lamb's problem by numerically computing the analytic integral solution, his waveforms are much cleaner and it is easier to distinguish between the different waves. It is difficult to see the similarities between our solution (Fig. 13) and Kuhn's (Fig. 15) because the finite element results contain much noise, resulting from numerical dispersion. However, the waves arrive at approximately the same time, and the initial forms of the waves are similar.

Frazier (1974, pp. 65–74) used Lamb's problem in a two-dimensional Cartesian coordinate system to evaluate the SWIS code written for the ILLIAC computer. As mentioned in the section above, Frazier was able to specify different damping factors for the various waves. He also investigated the effectiveness of transmitting boundary conditions, an option that is not available on our version of SWIS. Finally, Frazier used different parameters for his calculations, including a different material, smaller time and space steps, and a different force. Since the parameters are so different from those in our model, our comparison is limited to the form of the displacements (Fig. 16).

As a final comparison, we considered the calculations of Lamb (Graff 1985, p. 369). In his analysis of the half-space problem, he used a line loading with a time variation of

$$Z(t) = \frac{\tau}{t^2 + \tau^2}$$

where τ is a constant. If τ is small, $Z(t)$ describes a sharp impulse. Lamb's results for the horizontal and vertical surface displacements from the above loading are shown in Figure 17. The time and amplitude scales are not included in this figure, but the first disturbance shows the arrival of a P-wave, the second corresponds to the S-wave, and the major response is ascribable to the arrival of the Rayleigh wave.

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APPENDIX A: FORMAT FOR INPUT FILE u15in

Listing

A. HED
B. NDIMT NDFNT MAPXY NFOUR
C. NDBYDT DT TMIN TMAX
D. GRIDH
E. NEI NEJ NEK MAPYZ NBNODES IS JS KS IG JG KG
F. ((YGRID(NDIM,NBN)NDIM=1,NDIMT),NBN=1,NBNODES)
G. NNCRDC (if NNCRDC=0, go to *I*)
H. for NC=1 to NNCRDC:
  NODEC(NC) (Y(I,NC),I=1,3) (DELY(I,NC),I=1,3) NANCRD(NC) IANCRD(NC)
I. NENNC (if NENNC=0, go to *K*)
J. for NC=1,NENNC:
  NELN(NC) (NODEE(N,NC),N=1,NNET) NAEL(NC) IAEL(NC) IANE(NC)
K. NMAT
L. for N=1,NMAT:
  MAT DENS(MAT) VP(MAT) VS(MAT) DAMP(MAT)
M. NEMATC (if NEMATC=0, go to *O*)
N. for NC=1,NEMATC:
  NELM(NC) NEMAT(NC) NAEMAT(NC) IAEMAT(NC)
O. NNBCC (if NNBCC=0, go to *Q*)
P. for NC=1, NNBCC (if NODEB(NC)>0, go to *R*)
  NODEB(NC) NBTYPE(NC) (VSPEC(I,NC),I=1,3) NANBC(NC) IANBC(NC)
Q.(BCDIR(NCOMP,NAXIS,NC),NCOMP=1,3) NAXIS=1,2)
R. INTPRT INTPG INTPNV INTPEV
S. NPLTNL INTPNL (if either=0, go to *U*)
T. for NL=1,NPLTNL:
  NDLN(NL) NANLN(NL) IANLN(NL)
U. NTHPTS, (NNPRT(I),I=1,NTHPTS)
V. NTHELM NEPRT(I),I=1,NTHELM

Definition of entries

Entry formats are noted in parentheses (A = character string; I = integer; and E = exponential format).

A. HED

(A) A character string used to describe the problem; to be used as a heading on output. An example is:

Input file for uniform material-2D with point source at surface.

Be sure to leave a space as the first entry so the first letter doesn't get read as a carriage control character.

B. NDIMT NDFNT MAPXY NFOUR

NDIMT: (I5) the number of spatial dimensions (1, 2 or 3).

NDFNT: (I5) the number of degrees of freedom per node.

MAPXY: (I5) designates the type of spatial operator; choices are as follows:

For uniform, rectilinear grid in Cartesian coordinates, MAPXY = 0.

For non-uniform, skewed grid in Cartesian coordinates, MAPXY = 1.

For non-uniform, skewed grid in Cartesian coordinates, and to store stresses for non-linear constitutive, MAPXY = 2.

For cylindrical coordinates (r,z) with harmonic interpolation in azimuth, MAPXY = 5.

NFOUR: (I5) Fourier azimuthal order in cylindrical coordinates.

If MAPXY=5, NFOUR = 0.

C. NDBYDT DT TMIN TMAX

NDBYDT: (I5) the number of time derivatives in the partial differential equation:

For static problem, NDBYDT=0.

For diffusion, NDBYDT=1.

For wave propagation, NDBYDT=2.

DT: (F10.4) grid size in time. DT should be less than the space grid size divided by the longitudinal wave (P-wave) velocity.

TMIN: (F10.4) starting time.

TMAX: (F10.4) ending time (number of time increments = TMAX/DT).

D. GRIDH

(A) A description of the grid generation. As with HED, leave a space for the carriage control character. An example for an entry is:

Regular grid, each element 10 m × 10 m, 7 km vertical by 10 km horizontal.

E. NEI NEJ NEK MAPYZ NBNODES IS JS KS IG JG KG

NEI: (I5) number of elements along the I-direction of a block of elements.

NEJ: (I5) number of elements along the J-direction of a block of elements.

NEK: (I5) number of elements along the K-direction of a block of elements.

MAPYZ: (I5) designates the mapping from the curvilinear problem.

For identity mapping, MAPYZ = 0.*

For bi-quadratic mapping, MAPYZ = 2.

For cylindrical coordinate mapping, MAPYZ = 3.

For spherical coordinate mapping, MAPYZ = 4.

NBNODES: (I5) number of nodes that are specified along exterior corners of the grid (NBNODES = 4 is a typical entry).

IS, JS, KS: (I5) starting numbers for expanding the grid size at 10% per zone. Grid elements less than IS, JS and KS are progressively expanded.

IG, JG, KG: (I5) starting numbers for expanding the grid size at 10% per zone. Grid elements greater than IG, JG and KG are progressively expanded.

F. ((YGRID(NDIM,NBN)NDIM=1,NDIMT),NBN=1,NBNODES)

YGRID(NDIM,NBN): (F10.4) coordinates of the nodes at the exterior corners of the grid, specified in the order:

For I_{min}, J_{min}, K_{min}, NBN = 1.

For I_{max}, J_{min}, K_{min}, NBN = 2.

For I_{min}, J_{max}, K_{min}, NBN = 3.

For I_{max}, J_{max}, K_{min}, NBN = 4.

For I_{min}, J_{min}, K_{max}, NBN = 5.

* MAPYZ = 0 is not operational; use MAPYZ = 2 for identity mapping.

For I_{max}, J_{min}, K_{max}, NBN = 6.
For I_{min}, J_{max}, K_{max}, NBN = 7.
For I_{max}, J_{max}, K_{max}, NBN = 8.

Supply NBN's depending on the dimensionality of the problem:

For a one-dimensional problem, NBN = 1–2.
For a two-dimensional problem, NBN = 1–4.
For a three-dimensional problem, NBN = 1–8.

For a two-dimensional problem, 5000 units along the top and 6000 m deep, an entry could be (the numbering used for generating the grid need not align with the coordinate axes, Y1,Y2,Y3):

0.0	–6000.0	5000.0	–6000.0	0.0	0.0	5000.0	0.0

For cylindrical coordinates, enter the radius first, and then the angle in radians. To specify a full circle (2Π radians) with radius of 10, the entry would be:

0	0	10	0	0	6.2832	10	6.2832

G. NNCRDC

(I5) number of lines (sequences) of data used to supersede node coordinates. If NNCRD = 0, skip to entry *I*.

H. NODEC(NC) Y(I,NC) (DELY(I,NC), I = 1,3) NANCRD(NC) IANCRD(NC)

OPTIONAL. Specify node sequence *only* if NNCRDC>0!

Complete for NC = 1 to NNCRDC:

NODEC(NC): (I5) first node number of sequence on line NC.

Y(I,NC),I = 1,3): (3F10.4) coordinates of node number NODEC(NC).

(DELY(I,NC),I = 1,3): (3F10.4) increment to be added to the node coordinates for generating additional nodes in the sequence.

NANCRD(NC): (I5) number of additional nodes in sequence NC.

IANCRD(NC): (I5) increment to be added to the node numbers to identify subsequent nodes in sequence NC.

I. NENNC

(I5) number of sequences (lines) of data used to supersede node numbers associated with individual elements. SET NENNC = 0 and go to entry *K*!

J. NELN(NC) NODEE(N,NC) NAEL(NC) IAEL(NC) IANE(NC)

If NENNC = 0, do not enter values. Currently, the code only reads, and does not process these variables.

K. NMAT

(I5) number of materials being specified. $1 \leq$ NMAT ≤ 9.

L. MAT DENS(MAT) VP(MAT) VS(MAT) DAMP(MAT)

Specify properties for each material, N = 1 to NMAT.

MAT: (I5) material number, $1 \leq$ MAT ≤ 9.

DENS(MAT): (F10.4) mass density for material number MAT.

VP(MAT): (F10.4) *P*-wave velocity for material number MAT.

VS(MAT): (F10.4) *S*-wave velocity for material number MAT.

DAMP(MAT): (F10.8) dimensionless damping coefficient to suppress high-frequency amplitudes from numerical dispersion.

M. NEMATC

(I5) number of assignment sequences; assigns material numbers to elements. NEMATC=0 for a uniform material. If NEMATC = 0, skip to entry *O*.

N. NELM(NC) NEMAT(NC) NAEMAT(NC) IAEMAT(NC)

Do not enter values if NEMATC = 0 (uniform material). Enter values for NC = 1 to NEMATC.

NELM(NC): (I10) first element in sequence NC.

NEMAT(NC): (I10) material number for sequence NC.

NAEMAT(NC): (I10) number of additional elements in sequence NC.

IAEMAT(NC): (I10) increment in element number for identifying subsequent elements in the sequence.

O. NNBCC

(I5) number of sequences used to constrain nodes. The number of different "boundary conditions," such as applied forces or displacements, or both. If NNBCC = 0, go to entry *Q*.

P. NODEB(NC) NBTYPE(NC) VSPEC(I,NC),I=1,3) NANBC(NC) IANBC(NC)

Used to specify constraints; enter values for NC=1 to NNBCC.

NODEB(NC): (I10) first node in sequence NC. To apply a rotation to a node, enter the negative of the node number.

NBTYPE(NC): (I10) multi-digit constraint code for interpreting components of the values specified by VSPEC(I,NC). The ones digit of NBTYPE pertains to I = NDFNT; the tens digit pertains to I = NDFNT–1, etc. The individual digits are interpreted as follows:

0: VSPEC is an applied force.

1: VSPEC is an applied displacement.

Thus, for NDFNT = 2, NBTYPE = 00010 indicates:

VSPEC(1,NC) = displacement assigned to component #1.

VSPEC(2,NC) = force applied to component #2.

Whereas, for NDFNT = 3, NBTYPE = 00010 indicates:

VSPEC(1,NC) = force applied to component #1.

VSPEC(2,NC) = displacement assigned to component #2.

VSPEC(3,NC) = force applied to component #3.

(VSPEC(I,NC),I = 1,3): (3F10.4) the value for the *I*th component of the force or displacement (as specified by NBTYPE).

IANBC(NC): (I10) increment in node number for the subsequent nodes.

If NODEB > 0, go to entry *R*.

Q. (BCDIR(NCOMP,NAXIS,NC), NCOMP=1,3), NAXIS=1,2)

(F10.4) used to specify rotations, but not fully operational; vectors to specify rotated directions for degree of freedom NCOMP with respect to axis NAXIS.

R. INTPRT INTPG INTPNV INTPEV

Used to specify print control. Set the value = 0 to suppress the plot.

INTPRT: (I5) interval between time steps for printing computed results to unit 16, file 'u16out'. Set INTPRT < 0 to plot intermediate values.

INTPG: (I5) interval between time steps for plotting deformed grid to unit 10, file 'u10g'. Set INTPG < 0 to plot only the undeformed grid.

INTPNV: (I5) interval between time steps for plotting node vectors to unit 11, file 'u11vn'.

INTPEV: (I5) interval between time steps for plotting element vectors to unit 12, file 'u12ve'.

S. NPLTNL   INTPNL

Plot along specified lines of nodes (sends output to unit 13, file 'u13ln' and unit 14, file 'u14div').

NPLTNL: (I5) number of node lines (set NPLTNL = 0 to suppress plots).

INTPNL: (I5) interval between time steps (set INTPNL = 0 to suppress plots).

If either NPLTNL or INTPNL = 0, go to entry *U*.

T. NDLN(NL)   NANLN(NL)   IANLN(NL)

Specify lines of nodes to plot displacement; enter values for NL = 1,NPLTNL. Do not enter values if NPLTNL = 0.

NDLN(NL): (I10) first node number in the line NL.

NANLN(NL): (I10) number of additional nodes in the line.

IANLN(NL): (I10) increment in node number along the line.

U. NTHPTS, (NNPRT(I), I = 1,NTHPTS)

Used to plot time histories of node displacement to unit 8, file 'u8hn'.

NTHPTS: (I10) number of nodes for which time histories are to be plotted.

(NNPRT(I),I = 1,NTHPTS): (I10) node number for plot history. No entries are needed if NTHPTS = 0.

V. NTHELM   (NEPRT(I), I = 1,NTHELM)

Used to plot time histories of element stress and displacement to unit 9, file 'u9he'.

NTHELM: (I10) number of elements for which time histories are to be plotted.

(NEPRT(I),I = 1,NTHELM): (I10) element number for plot history. No entries are needed if NTHELM = 0.

APPENDIX B: CALCULATION OF DIVERGENCE AND CURL

SWIS was modified so that the divergence and curl would be calculated for a two-dimensional problem in rectangular coordinates. The following discussion applies to this specific case only.

For the two-dimensional case, divergence and curl are defined by:

$$\text{divergence } (x) = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}$$

$$\text{curl } (x) = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}$$

where x = position
u_1 = displacement in x-direction
u_2 = displacement in y-direction.

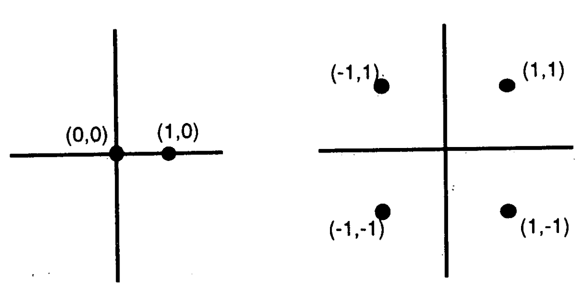
The divergence and curl were calculated using finite differences. The values for the corner nodes were calculated using forward differences for both directions; edge node values resulted from a forward difference for the direction perpendicular to the edge and a central difference along the edge; and values for nodes in the middle of the mesh were calculated using central differences in both directions.

In general, the forward and central differences for a partial derivative are given by (Abramowitz and Stegun 1972):

$$\text{Forward:} \quad \frac{\partial f_{0,0}}{\partial x} = \left[\frac{f_{1,0} - f_{0,0}}{h}\right] + 0\left(h^2\right)$$

$$\text{Central:} \quad \frac{\partial f_{0,0}}{\partial x} = \left[\frac{f_{1,1} - f_{-1,1} + f_{1,-1} - f_{-1,-1}}{4h}\right] + 0\left(h^2\right)$$

where h is the distance between the sampling points, and $f_{i,j}$ is the value of the function at the (ith, jth) sampling point. These finite difference formulas use equally spaced sampling points, as shown in Figure B1. For a grid with non-uniform spacing, the difference in coordinates must be used instead of the value h.



a. Forward time difference sampling (two points).

b. Central finite difference sampling (four points).

Figure B1. Sampling points for finite difference formulae.

Nine different sets of divergence and curl formulae were used for the two-dimensional, rectangular mesh. Each of the four corner nodes required a set of formulae, as did the nodes on each of the four sides of the mesh. The final set was written for the nodes in the center of the mesh.

For the following equations, variable definitions are given as:

nn:	node number
nei:	number of elements in the *x*-direction of the mesh
diver(*nn*):	divergence at node *nn*
curl(*nn*):	curl at node *nn*
disp(*i*,*nn*):	displacement in the *i*th direction of node *nn*
ynode(*i*,*nn*):	coordinate in the *i*th direction of node *nn*

Node locations for the following formulae are indicated on Figure B2.

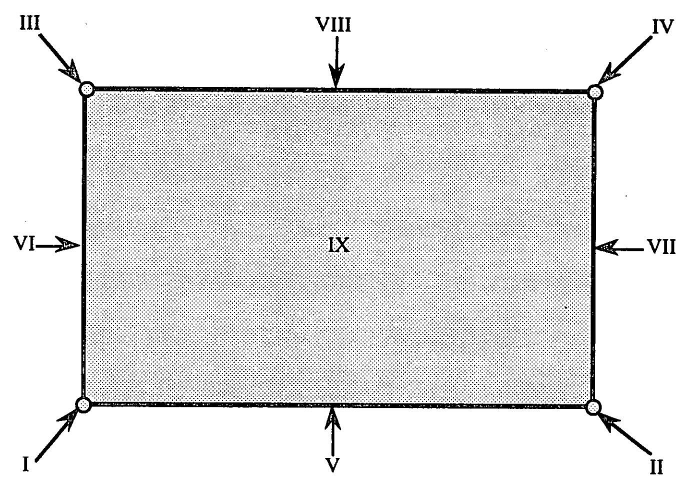


Figure B2. Node locations for finite difference formulae (I—lower left corner; II—lower right corner; III—upper left corner; IV—upper right corner; V—lower edge of mesh; VI—left edge of mesh; VII—right edge of mesh; VIII—upper edge of mesh; IX—middle of mesh).

I. Bottom left corner [*nn* = 1]:

$$\text{diver}\,(nn) = \frac{[\text{disp}(1,\,nn{+}1) - \text{disp}(1,\,nn)]}{[\text{ynode}(1,\,nn{+}1) - \text{ynode}(1,\,nn)]} + \frac{[\text{disp}(2,\,nn{+}nei{+}1) - \text{disp}(2,\,nn)]}{[\text{ynode}(2,\,nn{+}nei{+}1) - \text{ynode}(2,\,nn)]}$$

$$\text{curl}\,(nn) = \frac{[\text{disp}(2,nn{+}nei{+}1) - \text{disp}(2,\,nn)]}{[\text{ynode}(1,\,nn{+}1) - \text{ynode}(1,nn)]} - \frac{[\text{disp}(1,\,nn{+}1) - \text{disp}(1,\,nn)]}{[\text{ynode}(2,\,nn{+}nei{+}1) - \text{ynode}(2,nn)]}$$

II. Bottom right corner [*nn* = *nei*+1]:

$$\text{diver}\,(nn) = \frac{[\text{disp}(1,\,nn) - \text{disp}(1,\,nn{-}1)]}{[\text{ynode}(1,\,nn) - \text{ynode}(1,\,nn{-}1)]} + \frac{[\text{disp}(2,\,nn{+}nei{+}1) - \text{disp}(2,\,nn)]}{[\text{ynode}(2,\,nn{+}nei{+}1) - \text{ynode}(2,\,nn)]}$$

$$\text{curl}\,(nn) = \frac{[\text{disp}(2, nn{+}nei{+}1) - \text{disp}(2, nn)]}{[\text{ynode}(1, nn) - \text{ynode}(1, nn{-}1)]} - \frac{[\text{disp}(1, nn) - \text{disp}(1, nn{-}1)]}{[\text{ynode}(2, nn{+}nei{+}1) - \text{ynode}(2, nn)]}$$

III. Top left corner $[nn = nei * (nei + 1) + 1]$:

$$\text{diver}\,(nn) = \frac{[\text{disp}(1, nn{+}1) - \text{disp}(1, nn)]}{[\text{ynode}(1, nn{+}1) - \text{ynode}(1, nn)]} + \frac{[\text{disp}(2, nn) - \text{disp}(2, nn{-}nei{-}1)]}{[\text{ynode}(2, nn) - \text{ynode}(2, nn{-}nei{-}1)]}$$

$$\text{curl}\,(nn) = \frac{[\text{disp}(2, nn) - \text{disp}(2, nn{-}nei{-}1)]}{[\text{ynode}(1, nn{+}1) - \text{ynode}(1, nn)]} - \frac{[\text{disp}(1, nn{+}1) - \text{disp}(1, nn)]}{[\text{ynode}(2, nn) - \text{ynode}(2, nn{-}nei{-}1)]}$$

IV. Top right corner $[nn = (nei + 1) * nei + 1)]$:

$$\text{diver}\,(nn) = \frac{[\text{disp}(1, nn) - \text{disp}(1, nn{-}1)]}{[\text{ynode}(1, nn) - \text{ynode}(1, nn{-}1)]} + \frac{[\text{disp}(2, nn) - \text{disp}(2, nn{-}nei{-}1)]}{[\text{ynode}(2, nn) - \text{ynode}(2, nn{-}nei{-}1)]}$$

$$\text{curl}\,(nn) = \frac{[\text{disp}(2, nn) - \text{disp}(2, nn{-}nei{-}1)]}{[\text{ynode}(1, nn) - \text{ynode}(1, nn{-}1)]} - \frac{[\text{disp}(1, nn) - \text{disp}(1, nn{-}1)]}{[\text{ynode}(2, nn) - \text{ynode}(2, nn{-}nei{-}1)]}$$

V. Nodes located on bottom edge of mesh $[1 < nn < (nei + 1)]$:

$$\text{diver}\,(nn) = \frac{[\text{disp}(1, nn{+}1) - \text{disp}(1, nn)]}{2 * [\text{ynode}(1, nn{+}1) - \text{ynode}(1, nn)]} + \frac{[\text{disp}(1, nn) - \text{disp}(1, nn{-}1)]}{2 * [\text{ynode}(1, nn) - \text{ynode}(1, nn{-}1)]}$$

$$+ \frac{[\text{disp}(2, nn{+}nei{+}1) - \text{disp}(2, nn)]}{[\text{ynode}(2, nn{+}nei{+}1) - \text{ynode}(2, nn)]}$$

$$\text{curl}\,(nn) = \frac{2 * [\text{disp}(2, nn{+}nei{+}1) - \text{disp}(2, nn)]}{[\text{ynode}(1, nn{+}1) - \text{ynode}(1, nn{-}1)]} - \frac{[\text{disp}(1, nn{+}1) - \text{disp}(1, nn{-}1)]}{2 * [\text{ynode}(2, nn{+}nei{+}1) - \text{ynode}(2, nn)]}$$

VI. Nodes located on the left edge of the mesh $[\text{mod}(nn, nei + 1) = 1]$:

$$\text{diver}\,(nn) = \frac{[\text{disp}(1, nn{+}1) - \text{disp}(1, nn)]}{[\text{ynode}(1, nn{+}1) - \text{ynode}(1, nn)]} + \frac{[\text{disp}(2, nn{+}nei{+}1) - \text{disp}(2, nn)]}{2 * [\text{ynode}(2, nn{+}nei{+}1) - \text{ynode}(2, nn)]}$$

$$+ \frac{[\text{disp}(2, nn) - \text{disp}(2, nn{-}nei{-}1)]}{2 * [\text{ynode}(2, nn) - \text{ynode}(2, nn{-}nei{-}1)]}$$

$$\text{curl}\,(nn) = \frac{[\text{disp}(2, nn{+}nei{+}1) - \text{disp}(2, nn{-}nei{-}1)]}{2 * [\text{ynode}(1, nn{+}1) - \text{ynode}(1, nn)]} - \frac{2 * [\text{disp}(1, nn{+}1) - \text{disp}(1, nn)]}{[\text{ynode}(2, nn{+}nei{+}1) - \text{ynode}(2, nn{-}nei{-}1)]}$$

VII. Nodes located on the right edge of the mesh $[\text{mod}(nn, nei + 1) = 0]$:

$$\text{diver}\,(nn) = \frac{[\text{disp}(1, nn) - \text{disp}(1, nn{-}1)]}{[\text{ynode}(1, nn) - \text{ynode}(1, nn{-}1)]} + \frac{[\text{disp}(2, nn{+}nei{+}1) - \text{disp}(2, nn)]}{2 * [\text{ynode}(2, nn\,{+}nei{+}1) - \text{ynode}(2, nn)]}$$

$$+ \frac{[\text{disp}(2, nn) - \text{disp}(2, nn{-}nei{-}1)]}{2 * [\text{ynode}(2, nn) - \text{ynode}(2, nn{-}nei{-}1)]}$$

$$\text{curl}\,(nn) = \frac{[\text{disp}(2, nn{+}nei{+}1) - \text{disp}(2, nn{-}nei{-}1)]}{2*\,[\text{ynode}(1, nn) - \text{ynode}(1, nn{-}1)]} - \frac{2*\,[\text{disp}(1, nn) - \text{disp}(1, nn{-}1)]}{[\text{ynode}(2, nn\,{+}nei{+}1) - \text{ynode}(2, nn{-}nei{-}1)]}$$

VIII. Nodes located on the top edge of the mesh $[nei * (nei + 1) + 1 < nn < (nei + 1)^2]$

$$\text{diver}\,(nn) = \frac{[\text{disp}(1, nn{+}1) - \text{disp}(1, nn)]}{2*\,[\text{ynode}(1, nn{+}1) - \text{ynode}(1, nn)]} + \frac{[\text{disp}(1, nn) - \text{disp}(1, nn{-}1)]}{2*\,[\text{ynode}(1, nn) - \text{ynode}(1, nn{-}1)]}$$

$$+ \frac{[\text{disp}(2, nn) - \text{disp}(2, nn{-}nei{-}1)]}{[\text{ynode}(2, nn) - \text{ynode}(2, nn{-}nei{-}1)]}$$

$$\text{curl}\,(nn) = \frac{2*\,[\text{disp}(2, nn) - \text{disp}(2, nn{-}nei{-}1)]}{[\text{ynode}(1, nn{+}1) - \text{ynode}(1, nn{-}1)]} - \frac{[\text{disp}(1, nn{+}1) - \text{disp}(1, nn{-}1)]}{2*\,[\text{ynode}(2, nn) - \text{ynode}(2, nn{-}nei{-}1)]}$$

IX. Nodes in the center of the mesh:

$$\text{diver}(nn) = \frac{[\text{disp}(1, nn{+}nei{+}2) - \text{disp}(1, nn{+}nei)]}{2*\,[\text{ynode}(1, nn{+}nei{+}2) - \text{ynode}(1, nn{+}nei)]} + \frac{[\text{disp}(1, nn{-}nei) - \text{disp}(1, nn{-}nei{-}2)]}{2*\,[\text{ynode}(1, nn{-}nei) - \text{ynode}(1, nn{-}nei{-}2)]}$$

$$+ \frac{[\text{disp}(2, nn{+}nei{+}2) - \text{disp}(2, nn{-}nei)]}{2*\,[\text{ynode}(2, nn{+}nei{+}2) - \text{ynode}(2, nn{-}nei)]} + \frac{[\text{disp}(2, nn{+}nei) - \text{disp}(2, nn{-}nei{-}2)]}{2*\,[\text{ynode}(2, nn{+}nei) - \text{ynode}(2, nn{-}nei{-}2)]}$$

$$\text{curl}\,(nn) = \frac{[\text{disp}(2, nn{+}nei{+}2) - \text{disp}(2, nn{-}nei)]}{2*\,[\text{ynode}(1, nn{+}nei{+}2) - \text{ynode}(1, nn{+}nei)]} + \frac{[\text{disp}(2, nn{+}nei) - \text{disp}(2, nn{-}nei{-}2)]}{2*\,[\text{ynode}(1, nn{-}nei) - \text{ynode}(1, nn{-}nei{-}2)]}$$

$$- \frac{[\text{disp}(1, nn{+}nei{+}2) - \text{disp}(1, nn{+}nei)]}{2*\,[\text{ynode}(2, nn{+}nei{+}2) - \text{ynode}(2, nn{-}nei)]} - \frac{[\text{disp}(1, nn{-}nei) - \text{disp}(1, nn{-}nei{-}2)]}{2*\,[\text{ynode}(2, nn{+}nei) - \text{ynode}(2, nn{-}nei{-}2)]}$$

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (Leave blank)

2. REPORT DATE
November 1991

3. REPORT TYPE AND DATES COVERED

4. TITLE AND SUBTITLE
An Analysis of the Stress Wave in Solids (SWIS) Finite Element Code

5. FUNDING NUMBERS
PE: 6.11.02A
PR: 4A161102AT24
TA: FS
WU: 017

6. AUTHORS
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Hanover, New Hampshire 03755-1290

8. PERFORMING ORGANIZATION REPORT NUMBER
Special Report 91-21

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Office of the Chief of Engineers
Washington, D.C. 20314-1000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

Available from NTIS, Springfield, Virginia 22161

12b. DISTRIBUTION CODE

13. ABSTRACT (*Maximum 200 words*)

The Stress Wave in Solids (SWIS) finite element code is a versatile program in that it can solve problems in one, two or three spatial dimensions. Although the code assumes linear elasticity and isotropic materials, it can solve problems in regions containing up to nine different material types. To demonstrate its utility, SWIS has been used to solve three classical wave propagation problems: one-dimensional longitudinal displacement, impulse along the length of a cantilevered beam and Lamb's problem. This report describes how to use SWIS by summarizing the contents of the input and output files. Discussions of damping factors, computation times and comparisons to other solutions are also included.

14. SUBJECT TERMS
Finite element Numerical modeling Stress waves Wave propagation

15. NUMBER OF PAGES
37

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. Z39-18
298-102