A Non-Linear Fracture Mechanics Numerical Solution for Reinforced Concrete Deep Beams

A Theoretical Manual for the Fracture Mechanics Analysis of Reinforced Concrete Beams (FMARCB) Program

Guillermo A. Riveros and Vellore S. Gopalaratnam

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Final report
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Abstract: A significant number of failures in reinforced concrete structures initiate in tension regions promoted by stress risers such as areas of high-stress concentrations or pre-existing cracks. Stable growth of these tensile cracks until peak loads are reached is associated with the development of large zones of fracture [fracture process zone (FPZ)]. The growth of the FPZ introduces the effect of structure size on the failure loads. An energy approach based on fracture mechanics concepts can be used to rationally analyze and design for size effects in brittle failures. Current design equations were developed based on strength analysis (as in the current American Concrete Institute code) where the margin of safety is higher for smaller structures than for larger ones. It is also conceivable that this approach would lead to unconservative designs for some very large structures (e.g., deep slabs for underground storage tanks). Since the empirical formulations of the code are based on data for concrete of normal strength, it places a limit on the maximum strength that can be effectively used in the design equations. As a result, promising high-performance concrete cannot be used to its fullest potential. Revisions to the shear design formulations are needed to ensure a uniform margin of safety for members of all sizes, strength, and geometries. This report describes a finite element analysis of reinforced concrete deep beams using nonlinear fracture mechanics. The development of a numerical model that incorporates compression and tension softening of concrete, bond slip between concrete and reinforcement, and the yielding of the longitudinal steel reinforcement is presented and discussed. The development also incorporates the Delaunay refinement algorithm to create a triangular topology that is then transformed into a quadrilateral mesh by the quad-morphing algorithm. These two techniques allow automatic remeshing using the discrete crack approach. Nonlinear fracture mechanics is incorporated using the fictitious crack model and the principal tensile strength for crack initiation and propagation. The model has been successful in reproducing the load deflections, cracking patterns, and size effects observed in experiments of normal- and high-strength concrete deep beams with and without shear reinforcement.
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Preface

This report presents the non-linear fracture mechanics numerical solution for reinforced concrete deep beams. It will also serve as the theoretical manual of the FMARCB computer program. Funding for this investigation and preparation of this report was provided by the Computer Aided Structural Engineering program (CASE) and the Information Technology Laboratory.

The investigation and report were accomplished under the general supervision of Chris Merrill, Chief, Computational Analysis Branch; Dr. Robert Wallace, Chief, Computational Science and Engineering Division; and Dr. Reed Mosher, Director, Information Technology Laboratory (ITL). This report was prepared by Dr. Guillermo A. Riveros, Engineering and Informatics System Division, ITL; and Dr. Vellore S. Gopalaratnam, Professor, University of Missouri-Columbia.

COL Gary E. Johnston was Commander and Executive Director of ERDC. Dr. James R. Houston was Director.
### Unit Conversion Factors

<table>
<thead>
<tr>
<th>Multiply</th>
<th>By</th>
<th>To Obtain</th>
</tr>
</thead>
<tbody>
<tr>
<td>inches</td>
<td>0.0254</td>
<td>meters</td>
</tr>
<tr>
<td>kips</td>
<td>4448.222</td>
<td>newtons</td>
</tr>
<tr>
<td>pounds (force) per inch</td>
<td>175.1268</td>
<td>newtons per meter</td>
</tr>
<tr>
<td>pounds (force) per square inch</td>
<td>6.894757</td>
<td>kilopascals</td>
</tr>
</tbody>
</table>

The table above lists the conversion factors for units of length, force, and pressure. To convert from one unit to the other, multiply the quantity in the first column by the corresponding value in the By column to obtain the value in the To Obtain column.
1 Introduction

1.1 Introduction

This report presents the non-linear fracture mechanics numerical solution for reinforced concrete deep beams. The model was validated with experimental results of normal and high strength concrete beams with and without shear reinforcement and shear-span-to-depth ratios (a/d) of 2.5 and 1.5. Discussion of results on load displacement, cracking patterns, size effects, and concrete strength is also presented. The report will also serve as the theoretical manual of the FMARCB computer program (Riveros and Gopalaratman 2008).

The primary motivation for this investigation is to study brittle shear failures in reinforced concrete beams from a nonlinear fracture mechanics and finite element point of view and to study the implications with regard to shear design practices. A secondary motivation is to study brittleness in a more general context that includes structural (size and loading geometry) and material contributions to brittleness.

Reinforced concrete deep beams have useful applications in tall buildings, offshore structures, foundations, and military structures. A significant number of failures in reinforced concrete structures initiate in tension regions promoted by stress risers such as areas of high stress concentrations or pre-existing cracks. Stable growth of these tensile cracks until peak loads are reached is associated with the development of large zones of fracture [fracture process zone (FPZ)]. The growth of the FPZ introduces the effect of structure size on failure loads. Hence, if one were to design structures based on equations that were developed based on strength analysis, as in the current American Concrete Institute (ACI) code (ACI-ASCE Committee 326 1962, 1977; ACI-ASCE Committee 445 1998; ACI Committee 318 2005), the margin of safety provided would depend upon the size of the structure. The margin of safety is higher for smaller structures than for larger ones. It is also conceivable that this approach would lead to unconservative designs for some very large structures (e.g., deep slabs for underground storage tanks, dams, and gravity lock walls).

In addition to concerns about the relationship between size and strength and failure loads, there is limited evidence that the deformation capacity
of reinforced concrete elements is also influenced by structure size. This effect is likely to be of significance for high-strength reinforced concrete elements, particularly in applications requiring prescribed levels of ductility.

Early attempts to analyze failure in concrete structures caused by crack growth were not successful, even though it was obvious that a fracture mechanics approach would be realistic to model brittle crack propagation failures (ACI Committee 446 1992). The early attempts to analyze crack propagation failures failed because linear elastic fracture mechanics (LEFM) was used. LEFM assumes that the fracture process zone occurs at the crack tip (point) while the rest of the member volume remains elastic. However, research in the last three decades has resulted in modifications to LEFM to account for the distributed nature of pre-peak micro-cracking and the presence of a large FPZ in concrete. These modifications have allowed better results when fracture mechanics concepts are applied to brittle failure in reinforced concrete. Theories that allow tensile softening and relatively large FPZs are classified as nonlinear fracture mechanics models.

A considerable effort has been committed to developing numerical models that simulate the fracture behavior of materials exhibiting softening, such as mortar, concrete, rock, or bricks used in civil engineering structures. Two numerical methods to simulate fracture exist: the smeared crack approach and the discrete crack approach. In the smeared crack approach, introduced by Rashid (1968), the crack is replaced by a continuous medium with altered mechanical properties. Because it is established through stress computations at integration points, a significant number of cracks with small openings are imagined to be continuously distributed over the finite element. The constitutive laws, defined by stress-strain relations, are nonlinear and may exhibit strain softening. Strain localization instabilities and spurious mesh sensitivity of finite element calculations are likely with modeling strains softening numerically. These difficulties can be overcome by adopting appropriate mathematical techniques (ACI Committee 446 1991, 1992, 1997).

In the discrete crack approach, the crack is formulated as a geometrical change, which requires re-meshing each time a crack is initiated or propagated. The cohesive crack model, developed by Hillerborg et al. (1976), has
been shown to be effective for modeling the fracture process of quasi-brittle materials.

1.2 Objectives

The primary objectives of this research are to:

- Develop a numerical model of reinforced concrete deep beams using finite elements and non-linear fracture mechanics;
- Validate the numerical model with experimental data;
- Develop an automatic multi-crack initiation and propagation pre- and post-processor; and
- Perform parametric studies on:
  - Material properties
  - Loading geometry—Beams with and without shear reinforcement
  - Brittleness and size effects.

1.3 Scope of work

The investigation includes two main components: the numerical model and the parametric studies. The numerical model comprises the development of a finite element pre- and post-processor and the validation of the numerical model with field data. Among the parameters required to develop the numerical model are semi-automatic mesh generation, the implementation of the fictitious crack model (FCM) for multi-crack initiation and propagation (tension softening), compression softening, the debonding characteristics of the longitudinal reinforcement, and yielding of the longitudinal reinforcement. The model will be validated with available field data of four geometric, equal-sized reinforced concrete beams (Ghazavy-Khorasgany and Gopalaratnam 1993). Among the studies to be conducted in the parametric analysis are specimen size, concrete compressive strengths, and load geometry.

1.4 Research significance

The main relevance of this research lies in the practical application of nonlinear fracture mechanics and finite element concepts for developing rational code provisions with regard to shear design of reinforced concrete beams. The results from the study will be useful in initiating a departure in the design philosophy used in the current building and Corps of Engineers codes from a strength-based approach to one based on energy for certain types of brittle failures. In general, the advantages of adopting code
provisions based on fracture mechanics are numerous. Such code provisions are expected to provide uniform margins of safety for structures of all sizes, strengths, and loading configurations. The new procedures can be used to verify that the designs for very large structures are safe. Serviceability-based designs can be undertaken for specific applications for which strength may not be of primary importance. With increased use of high-performance concrete and more demands placed on conventional concrete in newer applications, this departure in design philosophy is not only desirable but also essential.

1.5 Overview

Chapter 2 includes a detailed presentation of shear failure in reinforced concrete, experimental observations, code provisions, and fracture mechanics models to analyze shear failure. Chapter 3 includes a brief review of the basic concepts of linear elastic fracture mechanics and the modifications needed to apply fracture concepts to concrete materials. It also presents a detailed explanation of the nonlinear fracture mechanics models that can be used to perform numerical analysis. The results from this study, along with analysis of these results, are presented in Chapter 4. Conclusions and recommendations for further study are presented in Chapter 5.
2 Shear Behavior of Reinforced Concrete Beams

2.1 Introduction

Beams are subjected to different types of loads, including axial loads, shear loads, wind loads, earthquake loads, snow loads, and others. These types of loads are responsible for the different types of failure mechanisms on reinforced concrete beams. Therefore, it is important to understand how the beams respond to any kind of loading and how changes in material properties and geometry affect the failure mechanism.

Shear strength is one of the most important properties in reinforced concrete beams. In Figure 1, shear strength is plotted in relation to compressive strength (Gaston et al. 1952; Gerstle and Abdalla 1988). Figure 2 shows the effective depth of the beam relative to the shear strength. It is believed that the rate of increase in shear strength with compressive strength decreases at higher strengths (Taylor 1972; Jenq and Shah 1986). Similarly, the reduction in shear strength becomes smaller in very large beam sizes (Bazant 1985; Kani 1966).

![Figure 1. Shear strength of reinforced concrete beams in relation to compressive strength (from Ghazavy-Khorasgany 1994).](image-url)
To properly analyze and design reinforced concrete beams, it is necessary to quantify shear and flexural stresses. The knowledge of how a beam behaves has increased significantly since the ACI-ASCE Committee 326 report 28 issued in 1962. However, the numerous and complex factors influencing the behavior and strength of structural members failing in shear have made it difficult to fully understand the problem of shear strength, even though a significant amount of research has been carried out on this subject.

2.2 Shear Response and Behavior: Experimental Data and Numerical Observations

Compressive strength is the first factor discussed here that influences the shear behavior. The significant amount of experimental observations from early studies and the research efforts on the design of reinforced concrete beams failing due to diagonal tension indicated that the shear resistances of beams were proportional to the ratio of compressive strength to the shear-span-to-depth ratio \( \alpha/d \), where \( \alpha \) is the shear span and \( d \) is the effective depth of the beam. The compressive strength of concrete in these
experiments varied from 2,000 to 6,000 psi (14–41 MPa). Taylor (1972) showed that the influence of concrete strength decreases for higher-strength concrete. For this reason and the limited amount of experimental data available for concrete with a compressive strength higher than 10,000 psi, the ACI Code limits the value of the compressive concrete strength, \( f'_{c} \), that can be used in the design equations to 10,000 psi (ACI Committee 318 2005). Since cracking is governed by tensile strength, and since the tensile strength is often related to \( (f'_{c})^{1/2} \) [for compressive strengths in the range 3,000–6,000 psi (21–41 MPa)], the shear strength in psi of concrete has been assumed to depend on \( (f'_{c})^{1/2} \). This type of dependence is used in the ACI Code as well as in some fracture mechanics models for diagonal tension failure (Bazant and Kim 1985; Bazant and Sun 1987; Bazant and Kazemi 1990). Recent experimental results that include high-strength concrete data suggest a \( (f'_{c})^{1/3} \) and \( (f'_{c})^{1/4} \) dependence (Remmel 1992; Konig et al. 1993).

The CEB-FIP (1993) Model Code and the JSCE (1986) Standard Specification relate the shear capacity of concrete as \( (f'_{c})^{1/3} \). It is believed that for the higher-strength concrete, since the fracture surface is relatively flat, shear resistance attributed to aggregate interlock at the diagonal tension crack is reduced. The present code equations for the shear capacity of concrete have been calibrated based on results from experiments on normal-strength concrete. With increased commercial interest in the use of high-strength concrete, it is essential that code provisions be revised to allow routine use of such materials. These revisions should address concerns with regard to the relative brittleness of failures in high-strength concrete. Figure 3 shows the variation of nominal ultimate shear stress with the compressive strength for about 400 tests. For the experiments reported by Bazant and Sun (1987), the upper and lower limits of the linear relationship between \( f'_{c} \) and the ultimate shear strength (\( \nu_{u} \)) are included in the figure. Also shown is the square root relationship presently used by ACI Committee 318 (2005).

The next parameter influencing the shear behavior is the longitudinal reinforcement, which contributes to the overall shear strength of the reinforced concrete elements through dowel action and tensile force transfer (Krefeld and Thurston 1966a, 1966b). When tensile steel is terminated at the shear span, the load capacity of the beam is reduced. However, no such reduction is observed in the case of bent-up bars (where the bars are not cut off but are instead bent up so that they act as compression
reinforcement near the beam supports) (ACI-ASCE Committee 426 1973). This suggests that the compression reinforcement may provide additional resistance.

Clark (1951) reported that the resistance to shear stress is linearly proportional to tensile reinforcement ratio. Taylor (1960) observed that with the increase in the amount of longitudinal reinforcement, the diagonal cracking load for a beam increases, showing an approximately linear relationship between the nominal shear stress and the percentage of reinforcement. Bresler and Scordelis (1963) observed that the shear resistance of reinforced concrete beams increased by using a multi-layered arrangement of tensile steel reinforcement. It has been reported that the ultimate shear capacity of reinforced concrete beams without lateral reinforcement is adversely affected by using longitudinal reinforcement of higher yielding stress, though the cracking capacity is hardly affected (Mphonde and Fractz 1984). It has been suggested that a low-level tension force in the reinforcing bars improves the dowel action (Mathey and Watstein 1963). Larger levels of tension in beams, however, have an adverse effect on the dowel action.
Krefeld and Thurston (1966b) observed that the longitudinal reinforcement resists the external shear force by dowel action after diagonal cracks occur in the reinforcement beam. This resistance depends on the size and spacing of the reinforcing bars, the location of the crack, the concrete cover used, and the concrete strength $f'_c$. Rajagopalan and Ferguson (1968) have proposed the following linear relationship between concrete shear strength and reinforcement content, $\rho$, which is somewhat similar to the linear relationship proposed by Kani (1966):

$$v_u = (0.8 + 100 \rho)(f'_c)^{1/2}$$

where:

- $v_u$ = concrete shear strength (psi)
- $\rho$ = steel ratio
- $f'_c$ = compressive strength (psi).

Bazant and Kazemi (1991) conducted diagonal tension failure experiments on two series of micro-concrete beams and reported interesting results. In both series, geometric similarity was maintained. The only difference was that for one series of experiments, diagonal shear failure was accompanied by bar pull-out, and in the second series, bar pull-out was prevented by hooks. The range of sizes tested was 1:16. The results from the second series showed a strong size effect close to those predicted by LEFM. This was attributed to the hooks used, which reduced slip, and the reduced ductility compared to the first series.

Another parameter influencing the shear behavior is the steel–concrete bond. Forces are transferred from the reinforcing bars to the concrete primarily by inclined compressive force radiating out from the bar. Abrishami and Mitchell (1992) established that the radial components of this inclined compressive force is balanced by circumferential tensile stresses in the concrete surrounding the bars and that this transfer mechanism was recognized by Abrams (1913). They also stated that the ability of a deformed bar to transfer its load to the surrounding concrete is typically limited by the failure of this ring of tension when the thinnest part of the ring splits (splitting failure). The bond stress, $u$, can be expressed as the change in stress in the reinforcement over the length, $dx$, as follows (Abrishami and Mitchell 1992):
\[ u(\pi d_b \partial x) = A_s \times (f_s + \partial f_s) - A_s \times f_s \text{ and hence, } u = \frac{(d_b \times \partial f_s)}{4 \times \partial x} \] (2)

where:

\( d_b \) and \( A_s \) = diameter and area of the reinforced bar

\( f_s = \) stress in the reinforcement.

Equation 2 demonstrates that the bond stress is proportional to the rate of change of the stress in the reinforcement, \( \partial f_s / \partial x \). Hence, if the stress in the reinforcement varies linearly, then the bond stress must be uniform (Abrishami and Mitchell 1992).

Taylor (1960) reported that the diagonal cracking load increases slightly with an increase in the amount of longitudinal reinforcement but that it is apparently not significantly affected by the bond characteristics of the bars. Bresler and Scordelis (1963) reported that multi-layered arrangements of tensile steel reinforcement appear to increase the shear resistance of reinforced concrete beams. This observation is perhaps related to the superior bond characteristics of smaller-diameter bars coupled with improved dowel action.

Kani (1966, 1967) performed experiments on two beams identical in every respect except bond resistance. He observed that the beam with a poor bond, and therefore larger crack spacing, had a higher load-carrying capacity than the beam with a good bond. This research reveals that the better the bond, the lower the diagonal load-carrying capacity (Kani 1964). Only limited data are available to analyze shear failure as influenced by steel–concrete bond characteristics. More specific details of the bond slip behavior and applications to the numeric model will be presented in Chapter 3.

Fenwick and Paulay (1986) reported that experimental work leading to load deformation characteristics of aggregate interlock action indicates that this is an important mode of resistance. It may be expected that about 60 percent of the bond force of beams is resisted by aggregate interlock action. They reported that the flexural resistance of the concrete across the critical section, aggregate interlock action, and dowel forces are the modes by which the bond force acting at the free end of a concrete cantilever is resisted.
Influences and applications of shear reinforcement have been investigated in some detail by Hillerborg (1983), Kiger et al. (1989), and Kim and White (1991). Some of these researchers investigated in more detail the influence of lateral reinforcement ratio, steel distribution, and steel yield strength on shear resistance. Stirrups are used to protect the beams from diagonal tension failure. It has been observed that the behavior of beams with and without stirrups is almost identical before the initiation of diagonal cracking. Stirrups become effective only after the formation of diagonal cracks. However, a small amount of stirrup reinforcement, with $\rho f_y$ ($\rho$ is the steel ratio and $f_y$ is the yield strength in psi) values as low as 50 psi, effectively increases the shearing strength of reinforced concrete beams and prevents brittle (explosive) failures caused by shear (provided the stirrups are spaced approximately $d/2$ apart or closer, where $d$ is the depth of the beam in inches) (Bresler and Scordelis 1963). They acknowledged that web reinforcement effectively prevents sudden failures caused by shear and permits the development of substantial deflections and increases in ductility, providing almost full flexural capacity prior to ultimate collapse.

Kim and White (1991) reported that in ordinary rectangular reinforced concrete beams (without web reinforcement) failing due to shear, inclined cracking started after the development of nearby flexural cracks, near the middle of the shear span, and just above the longitudinal reinforcement. They also stated that the primary reasons for the cracking at a relative low conventionally calculated shear stress were (a) the magnification of the actual shear stress produced by the concentration of bond stress along the longitudinal reinforcement near the middle of the shear span, and (b) the amplification of steel tension force near the middle of the span by the reduction of the internal moment arm length due to development of arch action as flexural cracks extend.

Krefeld and Thurston (1966b) confirmed that the stirrups increase the dowel resistance of longitudinal reinforcement, delaying the failure and retarding the formation of horizontal cracks along the reinforcing bars. Results from Clark's (1951) experiments with varying ratios of web reinforcement show that the shear resistance is proportional to the square root of the web reinforcement ratio.

Moretto (1945) reported that if stirrups are welded to the longitudinal reinforcement, the beam will have about 20 percent more resistance to diagonal shear force than those for which the stirrups were tied. This may
be attributed to improved dowel action with lateral and longitudinal reinforcements. Stirrup reinforcement limits the width of the diagonal tension crack (which increases the aggregate interlock and friction) in addition to carrying part of the shear force, confines the concrete core, and minimizes splitting cracks in the vicinity of the tension reinforcement.

Stirrups also improve the arch action, consequently improving shear resistance, and increase the compressive strength of concrete resulting from the confinement. This confinement also causes additional bond resistance. Consequently, shear reinforcement strengthens the beam and dowel actions of the flexural reinforcement. Shear reinforcement also delays the formation of splitting cracks that often cause a reduction in bond force, and it increases the frictional capacity by creating a rough crack surface and improves the interlock of the aggregates by preventing the increase in the width of the crack.

The presence of shear reinforcement makes the size effect milder. However, it is expected that even with the lateral reinforcement, as long as the overall load-deflection response of such a reinforced concrete beam exhibits softening, there will be some effect of specimen size on the failure load. Systematic information on size effect in the presence of lateral reinforcement is limited. Results from Walraven’s experiments (Walraven 1985, 1990, Walraven and Lehwalter 1994) show that the lateral reinforcement ratio, for the range tested (0.0–0.3 percent), has no significant influence on the size effect.

One of the most important factors affecting the shear behavior is the shear-span-to-depth ratio \((a/d)\). It has been observed that the shear-stress-to-tensile-stress ratio \((v/\sigma_t)\) is proportional to the shear span-to-effective-beam-depth ratio \(a/d\). Shear span, \(a\), is defined in Figure 6 and for a four-point loading configuration is the ratio of flexural moment, \(M\), to shear force, \(V (M/V)\). For simple loading configurations, it is treated as the equivalent length for which the shear force in the beam is constant.

Early experimental work showed the influence of \(a/d\) on a beam’s capacity. It provided valuable evidence that the loading geometry configuration can significantly alter the failure mode and, as a result, the ultimate load. A significantly negative correlation between cracking loads and \(a/d\) was observed and established by Clark (1951). He observed that the shear resistance of reinforced concrete beams is proportional to the inverse of
It seems that he was the first to propose an expression for calculating the shear strength in terms of $a/d$. Later Moody et al. (1954) modified this expression. All the research and experimental data show that the magnitude of loading causes the diagonal tension cracks in reinforced concrete beams without stirrups depending on two parameters: the compressive strength and the cross section of the beam. They also noted that when $a/d$ is large, the cracking strength of concrete governs the strength of the beam, while for smaller values of $a/d$, the load corresponding to crushing of the compression zone governs the strength of the beam. Taylor (1972) reported that the diagonal cracking load reduces slightly with increased shear spans.

Several investigators confirmed the importance of $a/d$ (Kani 1966; Taylor 1972). They observed that for the same beam nominal shear stress, failure changes over a range of $a/d$ of approximately 1.5 to 5. In this range, it has been observed that the full flexural capacity never develops (Figure 4). For an $a/d$ value of approximately 2.5, the flexural resistance is reduced to approximately 50 percent of the full flexural capacity of the beam. The maximum strength reduction for a wide range of reinforcement contents occurs in the vicinity of $a/d = 2.5$ (Figure 4). They also reported that members with $a/d < 2.5$ predominantly fail in shear.

![Figure 4. Ratio of beam strength to its theoretical flexural capacity versus the shear span to depth ratio ($a/d$) and reinforcement ratio ($\rho$) (from Kani 1966).](image_url)
Figure 4 indicates that the nominal flexural capacity for the beam is inversely proportional to the reinforcement ratio $\rho$. It also represents shear failure mode and indicates that the flexural capacity of beams depends not only on the reinforcement ratio $\rho$, but also on $a/d$ as established by Kani (1966).

Kani first observed that diagonal failure was imminent and that full flexural strength was attained outside, in a clearly defined region bounded by limiting values of $\rho$ and $a/d$. The valley of shear failure (Figure 4), known as Kani's shear valley, is now a well-established phenomenon.

Based on the $a/d$ ratio used, one of the following mechanisms will govern failure. It should be noted that the failure mechanism dictates the brittleness of the failure. This type ofbrittleness can be attributed to structural characteristics:

- Slender beams ($a/d > 6$). The moment capacity governs beam failure, and failure is in general ductile.
- Normal beams ($2.5 < a/d < 6$). The general mode of failure is flexural or diagonal tension, depending on the parameters involved.
- Deep or squat beams ($1 < a/d < 2.5$). The mode of failure is either shear compression or shear tension, and failure is generally brittle.
- Very deep beams ($a/d < 1$). Shear capacity governs failure. As soon as diagonal cracks form, tied arch action stabilizes the beam. Failure is relatively ductile, provided the tension steel is adequately anchored.

Kani (1967) affirmed that increasing the beam depth from 6 to 48 in. causes a decrease in the relative beam strength in the order of 40 percent. This is more than the maximum influence over the entire range of all practically important beams. He also reported that without the application of any appropriate correction for the absolute depth of beams, the safety factors derived from experiments on small beams produced dangerously low values. A size factor would not only depend on the depth, but also on the region to which the beam belongs. Kani concluded that increasing the beam depth must result in a considerable reduction in the relative beam strength.

Effective depth is the last factor influencing shear behavior. Because the brittle shear failure in reinforced concrete automatically implies the
existence of a size effect, the shear capacity on reinforced concrete beams is inversely related to the effective depth of the member.

Jenq and Shah (1989) and Shah (1989) observed that for specimens with similar geometry and properties, smaller members with higher compressive strengths failed in a flexural mode, while larger reinforced concrete beams with lower compressive strengths failed in a diagonal tension mode.

Leonhardt and Walther (1962a, 1962b) and Kani (1966) also noted the considerable influence of depth in the strength reduction of reinforced concrete beams. Additionally, Kani tested four series of beams with depths of 6, 12, 24, and 48 in., investigated the influence of \( a/d \) on the shear capacity of concrete, and studied the influence of beam depth. He observed that for similar beams, the shear strength of concrete decreased as the beam size was increased. Using all the data from experiments, he expressed the relative strength in terms of full flexural capacity and a reduction factor that was a function of \( a/d \), beam depth, \( d \), and reinforcement ratio, \( \rho \).

Kani (1966) expressed the relative capacity, \( M_u \), using the concrete strength and the geometric parameters \( (a/d, d, \text{ and } \rho) \) in terms of full flexural capacity \( (M_n) \) and a reduction factor:

\[
M_u = \frac{a}{d} \sqrt{\frac{0.215}{100\rho\sqrt{d}}} M_n. \tag{3}
\]

where \( a/d \) are in inches and \( M_n \) in lb per inch.

Taylor (1960, 1972) and Taylor and Brewer (1963) observed a smaller influence of size in experiments with maximum aggregate sizes proportionally scaled to beam depth. Therefore, the size effect was not entirely eliminated. However, the effect was less than Kani’s (1967). Iguro et al. (1985) obtained results similar to Taylor’s when he tested large beams (up to \( d = 3 \text{ m} \)) with proportionally scaled maximum aggregate sizes. It was speculated at that time that the strength loss resulted from wider cracks in the deeper beams, and, as a result, the contribution due to aggregate interlock was smaller for the deeper beams.
This effect made Chana (1981) and other investigators research in more detail the real effect of the size of reinforced concrete beams. He carried out experiments on prototype beams and beams that were geometrically scaled down in size. The maximum aggregate size was also proportionally scaled down. He observed that model beams (smaller sizes) were relatively stronger in shear than prototype beams. Consequently this scale effect cannot be attributed to the lack of similitude of aggregate interlock action as suspected in the earlier investigations.

As expected, the size effect is more pronounced at ultimate load than at cracking load. A 50 percent reduction in cracking strength was observed when the beam depth was varied from 0.2 to 1.0 m (Walraven and Lehwalter 1994). On the other hand, an approximately 56 percent reduction was observed in the ultimate strength for an identical size reduction. The beams were tested under midpoint loading and had span-to-depth ratios in the range of 2.3–3.4.

The presence of web reinforcement makes the size effect milder (Bazant and Sun 1987). The failure of beams without web reinforcement is more brittle than the failure of similar beams with web reinforcement. In a beam with web reinforcement, cracks are forced to form at closer spacing and over a wider area in the presence of shear reinforcement.

In general, the size effect is caused by the release of strain energy for the beam into the cracking zone; the larger the structure, the greater the energy release (Bazant and Sun 1987). Using experimental observations, Bazant and Kim (1985) derived some of the size effect relations and formulations for shear capacity with and without stirrups in concrete. They assumed that energy loss due to cracking is a function of both the fracture length and the area of the cracking zone, which is assumed to have a constant width at its front, proportional to the maximum aggregate size (Bazant and Sun 1987). Bazant and Kim (1985) stated that size still has an effect on the concrete shear strength in the presence of stirrups, but it is milder than without stirrups.

2.3 Shear stress in reinforced concrete beams

2.3.1 Shear stresses in beams before cracking

The transverse shear stress distribution across a cross section can be obtained by determining the equilibrium of an infinitesimal length of the
beam using the classical elastic theory of stress analysis. Figure 5 shows the free body diagram of the infinitesimal element. The shear stress will be determined at a distance $y$ from the neutral axis. By definition the flexural stress is

\[
\frac{M}{I} \cdot \frac{dy}{dx}
\]
\[ \sigma = \frac{My}{I} \] (4)

where:

- \( M \) = flexural moment
- \( y \) = distance from the neutral axis to the extreme fiber
- \( I \) = moment of inertia.

The resultant force on the left and right sides of the shaded area (Figure 5) induced by the linear stress distribution are

\[ F_{\text{Left}} = \frac{M}{I} \int_{y}^{c} y \, dA \] (5)

\[ F_{\text{Right}} = \frac{(M + \Delta M)}{I} \int_{y}^{c} y \, dA \] (6)

where:

- \( y \) = vertical distance from the neutral axis to the point where the shear stress will be calculated
- \( c \) = vertical distance from the neutral axis to the extreme top fiber of the member
- \( dA \) = infinitesimal area of the cross section \((dx \times h)\).

Then, from equilibrium, \( \sum F_H = 0 \), the shear force \((V_H)\) becomes

\[ V_H = F_{\text{Right}} - F_{\text{Left}} = \frac{\Delta M}{I} \int_{y}^{c} y \, dA. \] (7)

Substituting \( V_H = vb \Delta x \) into Equation 7 and solving for the shear stress \((v)\), we obtain

\[ v = \left( \frac{\Delta M}{\Delta X} \right) \frac{1}{bI} \int y \, dA = \frac{V}{bI} \int_{y}^{c} y \, dA = \frac{VQ}{bI} \] (8)
where:

- \( b \) = width of the beam
- \( I \) = moment of inertia
- \( V \) = shear force
- \( Q \) = first moment of the shaded area with respect to the neutral axis.

The principal tensile stress trajectory in an uncracked reinforced concrete beam can be obtained from the normal and shear strengths at each point on the beam. At the neutral axis where the normal stress is zero, a state of pure shear stress exists, giving rise to a principal tensile stress oriented at 45 degrees with respect to the longitudinal axis of the beam. This principal stress causes diagonal tension cracking.

The elastic shear stress equation can be derived by linear stress distribution (Wang and Salmon 1979) and has the following form:

\[
\nu = \frac{V}{bjd} \left[ 1 - \frac{y^2}{(Kd)^2} \right]
\]

where:

- \( jd \) = distance between the tension and compression forces
- \( Kd \) = neutral axis distance.

This equation shows that the maximum elastic shear stress occurs at the neutral axis (\( y = 0 \)):

\[
\nu = \frac{V}{bjd} = \frac{VQ_{\text{max}}}{Ib}.
\]

2.3.2 Diagonal tension cracks induced by shear

If \( \nu \) is the shear stress at a point below the neutral axis, and \( \sigma_t \) is the tensile stress in concrete, the amount (\( \sigma_{t,II} \)) and inclination (\( \varphi \)) of principal tension and the compression trajectories are given by Equations 11 and 12:
Equations 11 and 12 clearly show the reason why the cracks are vertical in regions of high bending stresses and the cracks are inclined in regions of high shear stresses (Park and Paulay 1975; Wang and Salmon 1979).

According to the conventional strength criterion, if the principal tensile stress, \( \sigma_t \), exceeds the concrete tensile strength, \( f'_{t} \), it will lead to diagonal cracking. This type of principal-stress-based criterion for cracking is used in some fracture-mechanics-based finite element codes for concrete structures, although an energy-based criterion for cracking used in other fracture models may be more appropriate.

Considering the equilibrium of an infinitesimal element, \( dx \), in Figure 5, the horizontal shear stress, \( v \), can be related to the differential tensile force, \( dT \), in the bar across the element, \( dx \), and is given by

\[
v \times b \times dx = dT \quad \text{or} \quad v \times b = \frac{dT}{dx}.
\]  

(13)

If the tensile force in the reinforcing bar is not varying, then \( \frac{dT}{dx} = 0 \).

This means that there is no force transfer between the concrete and the reinforcement.

**2.3.3 Mechanics of shear resistance**

The internal forces that resist the applied shear force across an inclined crack are shown in Figure 6. The nominal shear capacity (\( V_n \)) in a reinforced concrete beam is represented as the summation of forces shown in the free body diagram. The internal shear resisting components are the shear stresses transferred by uncracked concrete in the compression zone, \( V_{cz} \), the vertical component, \( V_{av} \), of the frictional forces along the diagonal crack attributed largely to aggregate interlock, \( V_a \), the shear resistance due to dowel action of the longitudinal steel reinforcement, \( V_d \) and the shear force carried by the web reinforcement (if any), \( V_s \) (ACI-ASCE Committee 326 1977):

\[
\sigma_{I,II} = \frac{\sigma_t}{2} \pm \sqrt{\left( \frac{\sigma_t}{2} \right)^2 + v^2}
\]

(11)

\[
\tan(2\phi_{\max}) = \frac{2v}{\sigma_t}.
\]

(12)
The first three terms on the right-hand side of Equation 14 are often lumped into one term and denoted as $V_c$ for design purposes (and attributed to the shear capacity of concrete, in addition to the above four mechanisms included in Equation 14). For deep beams, shear transfer due to inclined compression results in arch action, $V_c$, in Figure 6. The development of arch action is to a large extent dependent on the anchorage capacity of longitudinal reinforcement.

$$V_n = V_{cz} + V_{ay} + V_d + V_s. \quad (14)$$

**Figure 6.** Beam loaded in a four-point bending configuration showing the definition of shear span. The enlarged detail of the shear span highlights the components of shear resistance (from ACI-ASCE Committee 326 1977).

For discussions on how fracture mechanics can be rationally applied to brittle shear failures in reinforced concrete, it is necessary to understand not only how the major parameters influence these shear-resisting mechanisms, but also what fraction of shear resistance results from each of these mechanisms at various loading stages (particularly crack initiation and ultimate load). The implicit assumption here is that these mechanisms are
independent of one another. Of the four mechanisms included in Equation 14, the following observations are in order:

- **Shear resistance of un-cracked concrete:** This mechanism of shear resistance is effective in the compression zone after diagonal cracking and is primarily related to concrete strength. Confinement provided by shear reinforcement is expected to contribute to increased shear resistance resulting from the increased concrete strength.

- **Shear resistance due to aggregate interlock and friction:** By the roughness of the two crack surfaces in cracked concrete, some shear resistance due to aggregate interlock and friction will exist. Interface shear transfer becomes effective after diagonal tension cracking occurs, and this transfer is pronounced when slip occurs. This mechanism is related to the concrete microstructure (and thus indirectly to the strength) and fracture energy of concrete. This mechanism is one of the bases for the non-linear fracture mechanics analysis of reinforced concrete structures. By limiting the width of diagonal tension cracks, shear reinforcement serves to enhance the energy dissipation due to aggregate interlock.

- **Dowel action:** When longitudinal steel reinforcement crosses a crack, a dowel force in the bar resists part of the shearing displacements along the crack. The dowel force induces tensile stresses in the surrounding concrete. The splitting cracks in turn decrease the stiffness of the concrete around the bar, so the dowel force also decreases. Following splitting, the doweling force is a function of the stiffness of the concrete directly under the bars and the distance from the point where the doweling shear is applied to the first stirrups supporting the dowel (ASTM 1988).

- **Shear reinforcement:** When an inclined crack opens, either a combination of horizontal flexural reinforcement and inclined reinforcement or a combination of horizontal and vertical reinforcement is required to restrain it from opening. The inclined or vertical reinforcement is referred to as shear reinforcement or web reinforcement and may be provided by inclined or vertical stirrups (Figure 7). The direct contribution provided by the shear reinforcement is perhaps the least complicated to determine, compared with the other mechanisms. Its indirect contribution, however, which affects the other three shear resisting mechanisms as described above, is more complicated and, as a result, ignored in most design procedures and analytical models.
Arch action: Shear transfer due to an inclined compressive load path resulting in arch action is another major mechanism of shear load transfer. The vertical component of this compression is the primary contributor to shear resistance in many instances. For arch action to develop, the horizontal reaction at the base of the arch should be somehow supported. The longitudinal tensile reinforcement provides this support if it is adequately anchored to the beam. As a result, the development of arch action is to a large extent dependent on the anchorage and capacity of the longitudinal reinforcement.

The magnitude of the shear force \( V \) attributed to concrete has earlier been expressed as the rate of change of moment capacity \( dM/dx \). Using the internal force, \( T \), (Figure 6) equilibrium, it is possible to obtain \( V \) as follows:

\[
M = T(jd)
\]

\[
V = \frac{d}{dx} \left[ T(dj) \right]
\]

\[
V = \frac{d(T)}{dx} (jd) + \frac{d(jd)}{dx} T.
\]

Two situations are of interest with regard to Equation 17.

First, if \( jd \) is assumed to remain constant, then

\[
V = \frac{d(T)}{dx} (jd).
\]
The development of this type of shear force is caused by flexural action of the beam. The rate of change of the internal tensile force in the steel depends on the ability of the steel-concrete interface to transfer shear stresses (bond stresses). It is well known that the transfer of tension from concrete to reinforcing steel occurs through bond forces, and the relation between the steel tensile force and bond stress is given by

\[
\frac{dT}{dx} = u\Sigma D = q
\]  

(19)

where:

- \( u \) = average bond stress along \( dx \)
- \( \Sigma D \) = sum of diameters of all the tensile steel reinforcing bars
- \( q \) = bond force at the bar-concrete interface per unit length of the beam.

In this case the beam action is responsible for shear force resistance. It is clear from Figure 6 that the moment due to the tensile force in the reinforcing bar is resisted by three major mechanisms of beam action: aggregate interlock, friction between the two inclined crack surfaces, and dowel action of the reinforcing bars.

Second, if \( T \) is constant, this also implies that bond stresses do not exist at the interface, so

\[
V = \frac{d(jd)}{dx}T = \frac{d(jd)}{dx}C
\]  

(20)

where \( C \) is the total compression force in the cross section (which also equals the tensile force in the reinforcing steel if one neglects the tensile capacity of concrete). The development of this type of shear force is attributed to the arch action (the locus of the \( C \) force along the span is in the form of an arch).

The major shear resistance mechanism in deep beams is arch action. Arch failure usually happens after beam action fails in resisting the shear force. Large interface slips and a loss of bond also accompany this failure. Arch failure is manifested as shear compression failure or diagonal compression failure. Typically this failure is more brittle than the beam failure.
mechanism. In general, a combination of the two resistance mechanisms may be active in providing shear resistance to the cross section. The transition from beam action to arch action is gradual. Alternative explanations for these mechanisms have also been used to explain shear failure in reinforced concrete beams. These explanations, although termed differently (e.g., truss mechanism), result from the same basic contributions of force transfer.

2.3.4 Shear failure modes

Shear failure modes are more complicated than typical flexural failures (Reinhardt 1989). This section will describe the most generally accepted shear failure modes recognized by the Joint ACI-ASCE Committee 426 on shear and torsion failure (1973, ACI-ASCE Committee 326 1962, 1977). Diagonal tension, shear tension, and shear compression are the three most important shear failure modes. These failure modes are shown schematically in Figure 8 and described below.

Diagonal tension

When the main diagonal crack extends along both its tips to cover the full depth of the beam, a diagonal tension failure usually occurs (Figure 8a). For a particular combination of material, reinforcement, and geometric properties, diagonal tension failure occurs at the initiation of diagonal cracks. In those cases the load at crack initiation is not significantly different from the load at ultimate failure. It has been observed for other cases that many diagonal cracks develop in a stable manner until one of them becomes dominant. In these cases, the load at ultimate failure may be significantly larger than that at the initiation of the first diagonal tension crack. This type of failure is normally brittle and occurs when the shear-span-to-depth ratio is more than 2.5.

Shear tension

Shear tension failure occurs when a diagonal crack in the tension zone extends along the tensile reinforcement (Figure 8b). This failure is accompanied with debonding and splitting of the cover at the level of steel reinforcement. The diagonal tension crack is the initiation of shear-tension failure and bond stress failure, and the resultant drop in dowel resistance is suspected to be responsible for final failure. This type of failure in practice is generally brittle and explosive.
Figure 8. Modes of shear failure (a) diagonal tension failure, (b) shear tension failure, and (c) shear compression failure (ACI-ASCE Committee 326 1977).

Shear compression

When the stresses in the compression zone of the beam cross section (at the tip of the diagonal tension crack) exceed the compression capacity of concrete, and when good bond transfer exists between the concrete and the reinforcement, shear compression failure occurs (Figure 8c). In this case the diagonal crack first grows in a stable manner while the arch action starts governing the beam response. On further loading, the diagonal crack penetrates into the compression zone, reducing the depth of this zone. This type of failure usually occurs at a shear-span-to-depth ratio of less than 2.5. The crushing of the concrete at the tip of the diagonal crack leads to a brittle failure at ultimate loads.
Cracks induced by shear stress

Vertical flexural cracks appear (perpendicular to the beam axis) when large quantities of bending moment exist and the effects of shear stresses are either negligible or do not exist (i.e., extreme tensile fiber of the beam). On the other hand, when significant shear forces act in combination with bending moments, diagonal cracks inclined to the beam axis appear near the neutral axis of the beam (Figure 9a). The magnitude of the principal tensile stress and the inclination of the crack depend on the ratio of the shear and normal stresses at the particular location as described by Equations 11 and 12.

![Diagram of flexural-shear cracking](image)

**Figure 9.** Two important types of diagonal inclined shear cracks.

If web shear cracks and flexural shear cracks appear, failure caused by shear forces in reinforced concrete beams occur. Those cracks are known as the two primary types of inclined cracks (Figure 9). Web shear cracks are formed initially by diagonal tension caused by shear stresses. They mostly occur in pre-stressed concrete members, particularly in I-shaped beams with relatively thin webs. This kind of crack may also occur at the inflexion points (points of zero moment), where some of the longitudinal reinforcing bars are generally cut off. Flexural-shear cracks are the shear cracks developed from previously formed vertical flexural cracks. The
magnitude of the shear stress in relation to the magnitude of the flexural stresses defines the shape of the flexural-shear cracks.

2.4 Fracture mechanics approach for predicting the shear capacity of reinforced concrete beams

There are a limited number of semi-empirical nonlinear fracture models available for predicting the shear capacity in reinforced concrete beams. The present state-of-the-art measures for modeling the influence of reinforcement on the fracture characteristics of concrete are inadequate. Despite this drawback, these models are useful in understanding the mechanics of size-dependent brittle failures in reinforced concrete.

2.4.1 Law-based model of Bazant

To incorporate the effects of steel ratio and shear span in their governing equation, Bazant and Kim (1985) incorporated beam and arch action observed in shear failure. Shear strength, according to their proposal, is given by

\[ v = K_1 \times p^p \left( f'_c \sqrt{\frac{\rho}{(a/d)^r}} \left( 1 + \frac{d}{\lambda_0 \times d_a} \right) \right)^{-1/2} \]  

(21)

in which \( K_1, K_2, \lambda_0, r, q, \) and \( p \) are empirical parameters, \( f'_c \) is the concrete compressive strength, \( \rho \) is the steel ratio, \( a/d \) is the shear-span-to-depth ratio, and \( d_a \) is the aggregate size.

Bazant and Kim (1985) reported that their equation matches all the available experimental data better than the ACI (ACI Committee 318 1989) and CEB-FIP (1993) code equations for shear capacity. However, the scatter in the data is still significant. This scatter may be caused by a combination of factors, including normal scatter in concrete strength, results from experiments with beam sizes that are not geometrically proportional, data from many different laboratories where levels of resolution of measurements of the various test parameters are different and significant differences in aggregate types and sizes as well as reinforcing bar types and sizes.

Bazant and Sun (1987) incorporated the effects of shear reinforcement and aggregate size in the Bazant-Kim model. They also calibrated their revised equation with a larger set of experimental data. Although for beams with
shear reinforcement, the size effect may be milder than that for beams without shear reinforcement, they noted that its presence enhances the ultimate shear strength of concrete elements. They reported that Equation 22 provides a better size effect law than Equation 21:

$$v_c = v_0^c \left(1 + \beta^r\right)^{-1/2r} \quad \text{where} \quad \beta = \frac{d}{\lambda_0 \times d_a}$$  \hspace{1cm} (22)

where $r$ is an empirical coefficient with an optimum value between 0.75 and 1.0. To incorporate the effect of maximum aggregate size, they used the equation proposed by Bazant (1985):

$$v_c = v_0^c \left(1 + \sqrt{\frac{c_0}{d_a}} \times (1 + \beta)^{-1/2}\right).$$  \hspace{1cm} (23)

Linearization has also been used to obtain the empirical coefficients in Equation 23. The effect of shear stirrups has been incorporated through $\lambda_0$ (defined as a function of the stirrup reinforcement ratio, $\rho_v$):

$$\lambda_0 = 25 \left(1 + \frac{\rho_v}{\rho_0}\right)$$  \hspace{1cm} (24)

where $\rho_0$ is a function of the shear-span-to-depth ratio and has the following form:

$$\frac{1}{\rho_0} = a_0 \left[1 + \tanh \left(\frac{2a}{d} - 5.6\right)\right]$$  \hspace{1cm} (25)

where $a_0$ is an empirical constant. Comparing the different available procedures, Bazant and Sun (1987) showed that the current ACI Code provisions result in an over-design of most reinforced concrete beams.

### 2.4.2 Empirical approaches using fracture mechanics

Gustafsson and Hillerborg (1988) reported their results on the sensitivity in shear strength of longitudinally reinforced concrete beams to the fracture energy of concrete. This result was based on the non-linear fracture mechanism model often called the fictitious crack model. This model is basically a description of stress versus deformation properties of materials
in tension. A semi-analytical study on the shear strength of longitudinally reinforced concrete beams without shear reinforcement reported by Gustafsson and Hillerborg (Gustafsson 1985; Gustafsson and Hillerborg 1988; Hillerborg 1989) showed that the nominal shear strength at failure decreases as the depth of the beam or the span-to-depth ratio increases. These predictions are consistent with results from experiments on shear failure of reinforced concrete beams.

Gustafsson and Hillerborg have suggested that the shear strength of concrete is proportional to $f_i(d/l_{ch})^{0.25}$. Consideration of different characteristic length $l_{ch} = \left( \frac{E \times G_f}{f_t^2} \right)$ values for different types of concrete might reduce the large scatter in shear strength test data and contribute to a safer and more accurate shear strength design of concrete beams and slabs. They suggested that a consideration of $l_{ch}$ in the shear strength formulas existing at that time may be achieved by replacing the measure of the beam size, $d$, in the actual size-effect reduction factor by ratio $(d/l_{ch})$ and that Weibull theory does not provide a probable explanation of the size effect in shear strength.

From this empirical observation, Gustafsson and Hillerborg derived an expression for shear strength ($v_u$) based on the assumption that tensile strength ($f_t$) is proportional to the square root of compressive strength ($f_c$):

$$v_u \approx f_t \left( \frac{E_c \times G_f}{f_t^2} \right)^{1/4} \approx \left( \frac{E_c \times G_f \times f_t^2}{d} \right)^{1/4} \approx \left( \frac{E_c \times G_f \times f_c}{d} \right)^{1/4}$$  \hspace{1cm} (26)$$

in which $E_c$ is the modulus of elasticity and $G_f$ is the fracture energy. The significance of fracture energy and beam depth is readily apparent from Equation 26. The size dependence predicted by their formula is similar to that experimentally observed by Iguro et al. (1985) and less than the inverse square root dependence on $d$ as predicted by linear elastic fracture mechanics (LEFM) theories. More recently Remmel (1992) has, like Gustafsson and Hillerborg (1988) and Hillerborg (1989), suggested a fracture-parameter-based empirical equation for the shear capacity of concrete:

$$v_u = C^2 \left( \frac{G_f \times E}{425 \times f_{ct}^2} \right)^{0.2}$$  \hspace{1cm} (27)$$
where $d$ is in meters, and $f_{ct}$ is the tensile strength of concrete. Since these proposals (Gustafsson and Hillerborg 1988; Hillerborg 1989; Remmel 1992) are quite recent, they have not been independently validated. Also, their range of applicability is not known. Theoretical calculations indicate that fracture energy, $G_f$, is an important material property parameter affecting the shear strength in reinforced concrete beams (Gustafsson 1988).

2.5 Conventional approaches for predicting shear failure

The factors influencing the behavior and strength of concrete beams failing in shear are numerous and complex. These factors include portions and shape of the beam; structural restraints and the interaction of the beam with other components in the system; the amount and arrangement of tensile, compressive, and transverse reinforcement; the degree of pre-stress; the load distribution history; the properties of the concrete and the steel; the concrete placement and curing; and the environmental history (Bresler and MacGregor 1966).

Other investigators on this area, Reinhardt and Walraven (1982), analyzed and studied the mechanism of shear transfer across cracks. Using their investigation and earlier experimental observations, they reported that aggregate interlock in cracks is not merely a relation between shear stresses and shear displacement, but it is also an interaction between normal and shear displacement on one hand and normal and shear stresses on the other. They stated that when the reinforcement ratio remains constant, it is not a significant influence on the behavior, and they established that a fundamental difference exists between the behavior of reinforced cracks and un-reinforced cracks that are restrained by external bars. They reported that the crack opening path for reinforced cracks is approximately constant, independent of the reinforced ratio, but dependent on the external restraint stiffness for cracks in plain concrete.

Several semi-analytical models have been proposed for predicting the shear capacity of reinforced concrete. None of these models, however, has gained universal acceptance (Brock 1960; Bresler and MacGregor 1966; Reinhardt and Walraven 1982; Reinhardt 1989; Reineck 1991). Though significantly different assumptions are used in many of these models, they can be broadly classified into two categories. The first category consists of models based on the hypothesis that the major shear resistance mechanisms are in the tension zone of the beams (attributed to aggregate
interlock, friction along the crack surface, and dowel action of the reinforcement). Some truss analogy theories fall in this group of models. The second category consists of models based on the hypothesis that the major mechanism of shear transfer can be attributed to the compression zone of the cross section. The compressive force path theory (Kotsovos 1983, 1984, 1986, 1988; Kotsovos and Bodrowski 1993) and the shear compression theory (Moody et al. 1954) are examples of this type of model. Additionally there are more compressive models that incorporate the contribution to shear resistance from both the compression and tension zones in a more rational manner. The recently proposed unified theory (Hsu 1993) falls into this category.

The behavior and the shear resisting mechanism are very different in regions displaying primarily beam action, referred to as B-regions, and those displaying arch action, referred as D-regions, where D implies discontinuity or disturbance, which results in load transfer by in-plane forces such as arch action (Macgregor 1992). Discontinuity regions result from any sudden changes in loading, shape, or cross-sectional dimensions of the beam. These regions are sometimes called disturbed regions because the flow of stress in and out of the vicinity of these regions is irregular and disturbed. In other words, the region of discontinuity formed by a sudden change in internal resultant stresses or the regions where concentrated external loads are applied or where reactions are imposed are called D-regions or local regions. In these regions the internal stress and strain fields are not continuous and Saint-Venant’s principle does not hold (Figure 10).

![Figure 10. B- and D-regions (from MacGregor 1992).](Image)
D-regions extend about one member depth each way for concentrated loads, reactions, or abrupt changes in section or direction. The area between D-regions can be treated as B-regions. Examples of D-regions are locations of concentrated loads, reaction points, cross sections where openings exist, corbels, brackets, and joints. Collins (1978) attempted to analyze these regions using the strut-and-tie model. The Bernoulli’s compatibility condition that assumes that a plane’s cross sections remain plane is not valid for these regions. Figure 10 shows D- and B-regions in beams subjected to different types of loads.

D-regions’ stresses and strains are extremely irregular and cannot be expressed as a mathematical function. Compatibility conditions are usually ignored in the D-regions. Only equilibrium equations are applied in the design of these regions. More recently the finite element method has been employed with limited success to evaluate the stress and strain states in these regions (Hsu 1993). The other portions of a structure where internal stress fields are relatively continuous (where Saint-Venant’s principle is applicable) are called B- or “main” regions. B-regions are typically very small in deep beams because of low shear-span-to-depth ratio, while for a slender beam, the lengths of the D-regions are considerably small.

2.5.1 Tooth model

The concrete blocks between the adjacent diagonal cracks are termed “teeth.” This model was attributed to Kani (1966), Fenwick and Paulay (1986), Taylor (1972), and Chana (1981). The model is ideally suited for reinforced concrete members without transverse reinforcement. It is assumed that the shear force is in principle transferred through the cracked tension zone of the beam. Aggregate interlock, friction along the crack surface, and dowel action of the reinforcement are responsible for this stress transfer. If one tooth (a concrete cantilever) (Figure 6) is considered, the bond forces at the steel-concrete interfaces cause change in the tension force in the reinforcement. This variation in tension force imposes a displacement on the cantilever, which is assumed to cause the sliding of the crack interface, thus affecting aggregate interlock and dowel action. It is also assumed that the flexural resistance at the base of the concrete cantilever is responsible for failure.

The tooth model may be applied to reinforced concrete beams where there is no effective anchorage for reinforcement and where the shear-span-to-depth ratio is more than 2.5. This model is not suitable for the analysis of
relatively deep beams where no tooth forms or for beams where the anchorage of the longitudinal reinforcement results in a mechanism of load transfer similar to a tied arch.

2.5.2 Arch theory

The development of this theory was accredited to Kani (1966). It is mainly intended for reinforced concrete beams that do not have stirrups but where tied arch action develops due to anchorage of the reinforcement. The main assumption of this theory is that cracks develop and grow in the direction perpendicular to the direction of the principal tensile stresses. As a result, one or a number of tied arches form that are mainly parallel to the trajectory of the principal compressive stresses. The ends of the arch are supported vertically at the beam supports. According to this theory, failure is assumed to occur when the compressive capacity of the arch is abruptly reduced because of crack penetration into the compression zone. While conceptually sound, this theory has found very little practical acceptance, perhaps because of the complexity in defining the arch zone.

Fenwick and Paulay (1986) established the Arching Index for arch action on beams:

\[
Arching\ Index = \frac{V - v_t}{v} \tag{28}
\]

where:

\[
v = \frac{V}{(b \times jd)} = \text{nominal shearing stress}
\]

\[
b = \text{width of the beam}
\]

\[
jd = \text{distance between the tension and compression forces}
\]

\[
v_t = \text{average horizontal shear stress in the tension zone of the beam.}
\]

An arching index of unity would indicate that the whole of the external shear is resisted by arch action.

2.5.3 Shear compression theory

This theory, based on the assumption that shear failure is caused by stresses in the compression zone, is attributed to Moody et al. (1954) and Zwoyer and Siess (1954). The theory assumes that the diagonal tension
crack is stress-free, that concrete cannot resist tensile stress, that tensile reinforcement does not transfer shear, and that failure occurs by crushing the concrete in the compression zone at the critical section. The model focuses only on shear compression failure.

2.5.4 Compressive force path method

It has been demonstrated that the objective for safe and efficient design solutions has not yet been achieved. The reason for this has been shown to be associated with the lack of valid concepts underlying the methods currently used for shear design (Kotsovos and Bobrowski 1993). Kotsovos and Bobrowski (1993) proposed a design method suitable for any skeletal structural concrete configuration. This method is based on (a) modeling a simple reinforced concrete beams in compliance with the concept of the compressive force path, (b) using a failure criterion that describes the well-known experimental relationship between strength and shear-span-to-depth ratio, and (c) preventing brittle fracture by designing transverse reinforcement in localized regions identified by the concept of the compressive force path as critical while keeping the amount of such reinforcement in the remainder of the beam to nominal values.

This model provides a realistic description of the elements of any skeletal structural concrete configuration between consecutives points of inflection, with the connection between such elements being modeled as an internal support affected by the provision of suitable web reinforcement.

Proponents of this theory (Kotsovos 1984, 1986; Kotsovos and Bobrowski 1993) do not accept the hypothesis that shear failure results from the shear force exceeding the shear capacity of critical cross section. Consequently, they argue that the current codes of practice based on truss analogy and aggregate interlock are in conflict with reality. The cause of failure based on an assumed mechanism is attributed in this model to the tensile stresses developed in the compressive force path (above the neutral axis). The compressive force path is empirically assumed to be a bilinear path along which the compressive force is transmitted from loading points (or the middle cross section) to the supports. In other words, the compressive path is approximately defined by the direction of the resultant compressive force along the axis of the beam. The different modes of diagonal failures observed are assumed to be related to different multi-axial states of stress along the compressive path.
Kotsovos (1988) has shown that shear designs based on the current codes of practice might lead to a sudden and brittle failure. According to this theory, the shear force is only partly responsible for diagonal tension failure. It is assumed that the mechanism of transfer of the compressive force due to bending to the supports is more crucial to diagonal failure than the mechanism of transfer of the shear force. The dependence of shear resistance and mode of diagonal failure on the shear-span-to-depth ratio is considered to be the reflection of the dependence of the compressive force path on the geometry of the beam element. The model is further justified by arguing that improving the bond properties of concrete (Kotsovos 1988) has a significant adverse effect on the diagonal failure capacity (instead of increasing the diagonal failure capacity).

Kotsovos stated that shear resistance appears to be associated with the region of the path along which the compressive force is transmitted to the supports and not, as widely considered, the region of the beam below the neutral axis. Because of triaxial stress conditions in the region of the compressive force path, concrete is significantly stronger than it is widely considered to be (Kotsovos 1988).

2.5.5 Diagonal compression field theory and modified diagonal compression field theory

The diagonal compression field theory was developed from the use of a truss model with a variable angle of inclination of diagonal cracks. In this model, both the reinforcement and the cracking are considered in a smeared sense. It is assumed that reinforcement and cracking are distributed uniformly within the structure. Cracked concrete is treated as a new material with mechanical properties different from those of the uncrushed concrete. Diagonal compression struts form along the compression field. Yielding of stirrups, yielding of longitudinal (tensile) reinforcement, or crushing in the compression field might be the cause of failure. This theory does not explain the effect of shear force on the reduction of flexural capacity of beams.

Later modifications of the theory make it more practically relevant. The theory has been developed for reinforced concrete elements subjected to in-plane shear and axial normal stresses (Vecchio and Collins 1986, 1988). Vecchio and Collins (1988) reported that the modified compression field theory relates average stresses to average strains in a cracked reinforced concrete element, satisfying conditions of compatibility and equilibrium.
They stated that the modified compression field theory enables realistic predictions of the response of reinforced concrete membrane elements subjected to in-plane shear and axial forces. It is formulated to satisfy general conditions of compatibility and equilibrium while incorporating a realistic constitutive relation for cracked concrete in tension and in compression, as determined from extensive experimental data (Vecchio and Collins 1988). From the analysis, the experimental data suggest that the method provides an enhanced ability to design and analyze the shear response of beams in a rational manner, rather than having to rely on restrictive, narrow-ranging, and often overly conservative empirical formulations.

Equilibrium, compatibility, and constitutive laws for the composite are formulated using the average stress and the average strain over the matrix concrete and reinforcement. The theory-modified version (Gustafsson and Hillerborg 1985) considers the contribution of the tensile load-carrying capability of the concrete between cracks, while in the original theory this contribution was ignored. The angle of inclination of concrete struts or diagonal cracks is assumed to coincide with the direction of the principal compressive stresses and strains. The average strain must satisfy the Mohr compatibility condition.

The applied stresses are related to the average steel and concrete stresses. Compatibility equations are obtained using Mohr's circle. The constitutive equations before and after cracking are established empirically. An iterative procedure to determine the local stresses has been developed by noting that the two systems of internal forces—one based on the averaging procedure of the theory and one based on the actual state of stress at the crack—are equivalent and by establishing the relationship between the crack width, the compressive force on the crack, and the shear stress along the crack.

Based on this theory and a limited number of related experimental results, it has been observed that a significant amount of tensile stress exists in concrete between cracks, even at very high levels of average tensile strain.

2.5.6 Truss analogy and truss models

A modified truss model approach, with the diagonals at variable angles of inclination and the concrete contribution for beams with web reinforcement, is proposed as a viable and economical design tool. This statement
was supported by the computed values compared with a wide range of experimental results of reinforcement and prestressed concrete beams failing in shear (Ramirez and Breen 1991). Ramirez and Breen (1991) reported that this method is conservative and in good agreement by analyzing their experiments.

The proposed modified truss model (Ramirez and Breen 1991) with a concrete contribution offers a clear behavioral concept and covers the design of beams with no or low amounts of shear reinforcement. In this method of analysis, the concrete compression zone and the tensile steel form two parallel chords of a hypothetical truss, while the stirrups and concrete cantilevers between the diagonal cracks form the vertical tensile elements and inclined compression struts of this truss, respectively (Ramirez and Breen 1991). Assuming a pin-jointed truss formed by the elements mentioned above, the equilibrium consideration provides the relationships among all the forces in the system. The amount of shear reinforcement required and the resultant diagonal compression force in concrete cantilevers are determined based on the above analogy. This analogy has led to the widely popular concept of truss models that also constitutes the basis for some code equations. Ramirez and Breen (1991) reported that the ACI method and proposed modified truss model procedures could be improved to minimize scatter and achieve better agreement with experimental evidence.

Figure 11 shows the typical crack formation in a beam (Ramirez and Breen 1991). The fundamental assumption is that when diagonal cracks occur in the reinforced concrete beam, the beam acts much like an ordinary truss. It is assumed that the stirrups yield at failure. Consideration of the concrete contribution to shear and the assumption providing for a variable angle of inclination of the plane of failure (instead of the assumption of a 45 deg crack) have been used in more recent modified truss models. The truss models, however, do not have the capability of incorporating the effects of the shear-span-to-depth ratio and the size effect. Conventional truss models can only be applied to certain portions of concrete structures. The application of the classical truss model to D-regions of the beam is not easily possible. To overcome this drawback, some investigators (Collins 1978) have attempted to analyze these local regions using the strut-and-tie model.
2.5.7 Unified theory of reinforced concrete

The unified theory of reinforced concrete is based on a conventional strength analysis. This theory (Hsu 1993) has been intended for applications where all four basic actions—bending, axial loading, shear, and torsion—are expected. The basic concept of the classical truss model is utilized in developing this integrated approach to design for combination loading including flexure, axial force, shear force, and torsion.

There are five models used in the unified theory. Each of these is suited for certain specific types of problems. The strut-and-tie model, the Bernoulli compatibility truss model, the equilibrium (plasticity) truss model, the Mohr’s compatibility truss model, and the softened truss model are the models mentioned above. The equilibrium (plasticity) truss model, the Mohr compatibility truss model, and the softened truss model are classified as rotating-angle models. The other two models are based on fixed-angle assumptions. The difference between these two classes of models is the assumption of direction of diagonal cracks with regard to post- and pre-cracking principal stresses. The rotating-angle models are not capable of predicting the concrete contribution to shear resistance, unlike the fixed-angle models.
2.5.8 Strut-and-tie model

Another method developed to improve the analysis of reinforced concrete beams is the strut-and-tie model. This model is based on idealizing the flow of stress and visualizing the reinforced beam as a hypothetical truss in which generally concrete struts are compression members and reinforcing steels bars are tensile members. However, a limited role is also allowed for concrete in tension and steel in compression. Treatment of the D-region is one of the applications of this model.

The compressive strength of the struts is assumed to be a prescribed percentage of the concrete strength in uniaxial compressive; this was observed from recent strut-and-tie model analyses. Collins (1978) incorporated the concept of strain softening in evaluating the compressive strength of struts. Though it is generally understood that this model is applicable only to D-regions, relatively deep beams or beams with shear-span-to-depth ratios of less than two can effectively be analyzed using this approach.

Even though this model has been applied to simply supported beams under combined shear and torsion loading, its major impact is in design of D-regions where dimensioning of struts and ties involves satisfying complicated compatibility conditions and material laws. Specifically, knee joints under closing and opening moments can be reasonably modeled by this theory (Hsu 1993). These regions can be assumed to form hypothetical trusses with concrete struts and reinforcement ties in the direction of stress trajectories and connected at joints. The whole region can be assumed to undergo the boundary forces as an external force is imposed on its free body diagram. A reinforced concrete beam under the combined action of bending moment and shear force can be simulated as a parallel chord truss where the compressive forces are resisted by the top chord and diagonal struts while the tensile reinforcement and stirrups resist the tensile forces. When the strut-and-tie model is applied to B-regions, it usually leads to the classical truss model.

2.5.9 Bernoulli compatibility truss model

The strut-and-tie model only uses the equilibrium equations; therefore, it is inadequate when the distribution of the stresses along the depth of the beam is required. In such a case, the strain compatibility condition and the constitutive laws of the component materials are needed to relate the
stresses and strains. The model is best suited for the analysis and design of bending problems and the combination of bending and axial loads to predict both strength and deformation of the elements (moment curvature diagrams). It also can be applied to evaluate the cracked moment of inertia and the effective moment of inertia, as well as evaluation of the bending rigidity of singly or doubly reinforced, cracked, and uncracked sections (the ratio of moment to curvature). More complicated problems, such as asymmetrical reinforced columns with eccentric loads (combined bending and axial loads), can be solved by this theory by determining the plastic centroid. Balanced, over-reinforced, and under-reinforced cross sections (in other words, compression failure, tension failure, and moment-axial force interaction) all can be treated by using this theory. Applying the same truss model to this theory is based on the similar action of reinforced concrete beams and trusses. The compression chord of a parallel chord truss at the top and the tension chord at the bottom are simulated, respectively, with concrete under compression at the top and reinforcement under tension at the bottom of the beam. Because of the strain discontinuity caused by cracking, Bernoulli's hypothesis of a linear compatibility condition is only an approximation for reinforced concrete beams.

2.5.10 Equilibrium (plasticity) truss model

Yielding of steel is the basic assumption of this model. The model incorporates the equilibrium equation and uses plasticity theories. In the unified theory, the role of compatibility, condition, and constitutive laws of materials is not taken into consideration yet. This model can be applied when the interaction of shear and torsion with bending and axial load is needed. The model does not satisfy compatibility conditions. Constitutive laws of the component materials are not considered. This theory leads to staggered shear force diagrams for uniformly loaded beams, which have been adopted by the CEB code (CEB-FIP 1993) for the shear design of reinforced concrete beams. The ACI Code (ACI Committee 318 1986) has not incorporated this model into its shear design provisions because of its inadequacy in some instances (such as over-reinforced concrete beams).

The other shortcoming of this theory is with regard to assumptions of bond. The equilibrium (plasticity) truss model is characterized by the assumption of the yielding of steel. Because of this assumption, the compatibility condition is automatically satisfied. As the stress in the steel is not unknown, the equilibrium condition in the cross section leads to depth of the compression zone in flexural beams. When it comes to concrete...
reinforced elements in both the longitudinal and transverse directions subjected to shear, the assumption of yielding in both types of reinforcement in conjunction with the equilibrium equations leads to the relation between shear and internal stresses in concrete and steel. The model also highlights the fact that the orientation of diagonal cracks depends on the ratio of transverse to longitudinal reinforcement. This model can be applied effectively only to balanced or under-reinforced elements.

2.5.11 Mohr compatibility truss model

This model, based on elasticity, Mohr's compatibility condition (Mohr's circle for strains), and the equilibrium equation in two dimensions (Mohr's circle for membrane stresses), is used to compute the service load of a beam when both concrete and steel are elastic. Basically it is the compression field theory developed by Collins (1978) applicable to membrane elements. It can also be applied when the interaction of shear and torsion with bending and axial load is required.

2.5.12 Softened truss model

This model is based on Mohr's compatibility condition, the equilibrium equation in two dimensions, and the constitutive law of softening in concrete. In this model the nonlinear behavior of concrete and reinforcement are considered, specifically the biaxial softening of reinforced concrete and the influence of tensile stresses between the cracks. The model can be applied when shear and tension stresses are involved in combination with axial and flexure loads.

2.6 Shear design requirements

The initial known practice for determining shear stress requirements has been to compare the nominal shear stress with a percentage or a specific function of the concrete compressive strength (ACI-ASCE Committee 326 1962). The nominal shear stress is calculated by dividing the shear force at the desired cross section by a nominal area:

\[ \nu = \frac{V}{b(jd)} \]  \hspace{1cm} (29)
where:

- \( v \) = shear stress, psi
- \( V \) = shear force, lb
- \( b \) = width of the beam web, in.
- \( jd \) = internal lever arm (Figure 5), in.

Shear provisions in the design codes are usually based on the concept of shear capacity (at cracking) of the critical cross section. The aggregate interlock and the resultant friction are considered majors contributors to shear resistance in this method. It is assumed that shear failure will occur when the shear force exceeds the shear resistance of the critical cross section. The amount of shear reinforcement needed is evaluated based on the shear force in excess of the shear capacity of the concrete. Shear reinforcement, in addition to enhancement of the shear capacity of the reinforced beam, is also typically responsible for changing the mode of failure.

### 2.6.1 Shear provisions in the ACI building code

ACI Code (ACI Committee 318 2005) procedures allow for the assumption of the nominal shear stress at any cross section as

\[
v = \frac{V}{b_w d}
\]  

where \( b_w \) is the width of the web of the beam, instead of the actual shear stress distribution. In Equation 30, \( v \) is the average nominal value of the shear stress, which simplifies the calculation of shear stress distribution on the cross section.

In the current ACI approach (ACI Committee 318 2005), the nominal shear strength provided by the concrete at ultimate is taken to be equal to the shear stress at which diagonal tension cracking first occurs. The expression for the nominal shear strength is:

\[
V_c = \left[ 1.9\sqrt{f'_c} + 2500\rho_w \frac{V_u d}{M_u} \right] b_w d \leq 3.5\sqrt{f'_c} b_w d
\]  

(31)
or

\[ v_c = \left[ 1.9\sqrt{f'_c} + 2500\rho_w \frac{V_u d}{M_u} \right] \leq 3.5\sqrt{f'_c} \]  \hspace{1cm} (32)

where:

- \( f'_c \) = compressive strength of concrete in psi
- \( \rho_w \) = longitudinal steel reinforcement ratio
- \( V_u \) = factored shear force at ultimate, lb
- \( d \) = effective depth of the longitudinal steel, in.
- \( M_u \) = factored moment at ultimate, lb/in.
- \( b_w \) = width of the beam web, in.
- \( v_c \) = nominal shear strength provided by concrete in lb (or \( v_c \) in psi).

Equation 32 provides for a transition in strength from flexural shear failure, when the shear force is small compared with the flexural moment (\( V_u < M_u \)), to a web shear failure, when the shear force is appreciably larger than the flexural moment (\( V_u > M_u \)).

The commentary to the ACI Code (Section RII.3.2.1) acknowledges that Equation 31 might overestimate the influence of the compressive strength (\( f'_c \)) and underestimate the influences of the steel ratio (\( \rho_w \)) and \( (V_u \times d)/M_u \). Kani’s (1966) report is based on extensive experimental data and also acknowledges that the shear strength decreases with increases in the overall depth of the member. The code does not provide specific recommendations to take into account such dependence. Revisions to the code that partially address some of the concerns discussed here were proposed by the ACI-ASCE Committee 426 (1973) for general members subjected to shear and by MacGregor and Gergely (1977) for beams but have not been incorporated into the ACI Code.

The ACI Code provisions limit \( \sqrt{f'_c} \) values in Equation 31 to 100 psi (0.69 MPa), except for beams with code-specified amounts of stirrup reinforcement. This is because the limited practical data available for shear failure of beams made with concrete compressive strengths in excess of 10,000 psi (69 MPa) suggest that, at these strength levels, the influence of concrete compressive strength on the shear strength diminishes.
The presence of stirrups has generally been observed to enhance the ultimate shear strength of concrete by introducing confining pressures that increase the shear strength of the concrete members. However, the present practice adopted by the ACI Code neglects this contribution. The ACI Code also suggests the use of the simplified formula for determining the concrete capacity in shear in case more detailed analysis is not deemed necessary. $V_c$ is given by:

$$V_c = 2b_w d \sqrt{f'_c}.$$  \hfill (33)

The ACI procedure is generally conservative. In a few instances, however, the margin of safety may be significantly diminished. This includes cases where very small amounts of tensile reinforcement are used (Rajagopalan and Ferguson 1968) or when high-strength concrete beams or very deep concrete beams are used.

The ACI Code equation is supposed to predict the shear strength of beams at diagonal crack formation. Consequently, it is expected that the experimental ultimate shear strength is always greater than the shear strength predicted based on this formula. When the experimental results at ultimate are compared with the ACI Code formula for the shear capacity of concrete, it appears that the formula is not conservative in some cases (Bazant and Sun 1987) and does not represent the trend in the experimental data adequately. Bazant and Sun (1987) suggested that it would make no sense to introduce the size effect into the ACI formula without improving the form of the existing formula itself, which was previously shown possible for beams without stirrups. They also reported that the formula for diagonal shear failure of beams without stirrups incorporating the size effect law was developed and shown to offer significantly reduced error in comparison to the existing formulas of ACI or CEB-FIP, which were developed on the basis of crack initiation rather than failure.

### 2.6.2 Shear capacity in the CEB model code

The CEB code (CEB-FIP 1993), which differs from ACI Code, incorporates the effect of specimen size in the shear capacity provided by concrete. The size effects in these provisions are empirical and are not based on systematic fracture-mechanics-based development. The shear strength of reinforced concrete beams without shear reinforcement at diagonal cracking according to CEB Code (1978) is given by
\[ v = k_1 \tau_{Rd} \kappa (1 + k_2 \rho) \]  

\[ \tau_{Rd} = 0.001 f_{ck} + 0.06 \text{ for } f_{ck} \leq 20 \text{ MPa} \]  

\[ \tau_{Rd} = 0.008 f_{ck} + 0.1 \text{ for } f_{ck} > 20 \text{ MPa} \]  

\[ \kappa = \text{Max} \left| \frac{1.6 - d}{1.0} \right| \]  

\[ \rho = \text{Min} \left| \frac{A_s}{bd} \right| \frac{0.02}{A_s} \]  

in which \( d \) is the beam depth in meters, \( f_{ck} \) is the concrete compressive strength in MPa, \( v \) is the shear capacity in MPa, \( k_1 \) and \( k_2 \) are empirical coefficients equal to 1.31 and 54.7, respectively, \( \rho \) is the reinforcement ratio, \( A_s \) is the reinforcement area, and \( b \) is the beam width.

Recently, the CEB-FIP Model Code 1990 (CEB-FIP 1993) explicitly includes the size effect in its recommendations for the shear strength of reinforced concrete beams. The shear strength of concrete is given by

\[ v = 0.15 \left( \frac{3d}{a_v} \right)^{1/3} \left( 100 \rho_w f'_w \right)^{1/3} \left( 1 + \sqrt{200d} \right) \]  

\[ V_c = \left[ 0.15 \left( \frac{3d}{a_v} \right)^{1/3} \left( 100 \rho_w f'_w \right)^{1/3} \left( 1 + \sqrt{200d} \right) \right] b_w d \]  

where \( f'_w \) and \( v \) are in MPa, \( d \) and \( a \) are in mm, \( a_v \) equals the distance from the major load to the support (analogous to the shear span), and all other terms are as defined earlier. For slabs, the term \( 0.15 \left( 3d/a_v \right)^{1/3} \) is replaced by 0.12. As with the ACI Code, the concrete compressive strength to be used in the shear capacity computation for slabs is restricted to 50 MPa.

In Equation 39, the size effect incorporated is not derived from fracture mechanics principles but is based on empirical data. It is interesting to note that the above equation, though perhaps valid in the calibrated range of specimen depth, \( d \), results in a size effect that is opposite of that used in
the Bazant size-effect law. This equation suggests a strong size dependence for small values of \( d \) and a relatively small size dependence for larger values of \( d \).

### 2.6.3 JSCE standard specification for shear capacity

The shear capacity of concrete of the JSCE (1986) recommendation is based on results from tests of relatively large reinforced concrete beams. The equations to be used for shear capacity of concrete are

\[
v = \frac{a \beta_d \beta_p \beta_n \sqrt{f'_c}}{V_b}
\]

\[
V_c = \left[ \frac{a \beta_d \beta_p \beta_n \sqrt{f'_c}}{V_b} \right] b_w d
\]

where \( \beta_d = \sqrt{1/d} \leq 1.5 \), \( \beta_p = \sqrt[3]{100 \rho_w} \leq 1.5 \), \( \beta_n \) is a coefficient that takes into account the effect of axial force on the shear capacity (the value is limited to 2 for large axial tension and 0 for large axial compression), \( V_b = 1.3 \), \( f'_c \) is in MPa, and \( d \) is in meters. The size dependence factor adopted by the JSCE Standard \( (V_c \approx d^{-1/4}) \) is identical to experimental observations of Iguro et al. (1985) based on tests of large beams up to 3 m in depth and the model proposed by Gustafsson and Hillerborg (1988) and Hillerborg (1989).

### 2.6.4 Other empirical shear design equations

Some researchers (Gustafsson 1985; Gustafsson and Hillerborg 1985; Bazant and Sun 1987; Hillerborg 1989; Remmel 1992) have reported that the ACI Code and the CEB Code provisions are not always conservative. Alternative empirical equations that are reported to be in better agreement with test results have been proposed and include Zsutty’s equation (Zsutty 1968, 1971), which does not take the size effect into consideration, and Bazant’s equation based on the size effect law. The CEB Model Code (CEB-FIP 1993) formula is similar to Zsutty’s equation for concrete shear design.

The empirical equations developed by Zsutty (1968, 1971) gives the shear strength of concrete, \( v \), as:
where \( a \) is the shear span, in.

A better agreement with experimental results is obtained when the size effect is explicitly taken into consideration, as in Bazant’s size effect law empirical equation (Bazant and Kim 1984; Bazant and Sun 1987), where \( d \) is the size (depth) of the beam, \( d_a \) is the maximum aggregate size, \( \rho \) is the tensile reinforcement ratio, \( a/d \) is the shear-span-to-depth ratio, and \( f'_c \) is the compressive strength of concrete. All other terms are empirical coefficients calibrated from experimental results. These constants for the English units are as follows: \( K_1 = 4 - 10 \), \( P = 1/3 \), \( q = 1/2 \), \( K_2 = 3000 \), \( r = 2.5 \), and \( \lambda_0 = 25 \).

Bazant reported an equation that is based on fracture mechanics, and it has been discussed in greater detail in this chapter. With respect to this proposed formula, the scatter of the data available in literature is reported to be much less than that with respect to the ACI and CEB Code equations. In addition, Bazant and Sun (1987) demonstrated that for the very large beam sizes, the ACI Code provision may indeed not be conservative.

### 2.6.5 Shear strength of reinforced concrete beams with stirrups

Conventionally, and according to many proposed analytical methods and codes of practice, the design of shear reinforcement can be based on superposition principle if conventional stirrups are used:

\[
V_n = V_c + V_s \quad \text{(or) } \quad V_n = v_c + v_s
\]

where \( V_s = (A_v \times f_y \times d) / \bar{s}, \quad v_s = V_s / (b \times d) = \rho \times f_y, \quad \rho = A_v / (b \times d) \times (d / \bar{s}). \) It should be noted that none of the codes consider enhancement in the \( V_c \) caused by the presence of lateral (stirrup) reinforcement (Johnson and Ramirez 1989).
3 Fracture Mechanics and Its Application to Finite Elements in Concrete

3.1 Introduction

The concepts of fracture mechanics were used in the World War II era to predict the failure of brittle steel materials in boat hulls. Since then, linear elastic fracture mechanics (LEFM) has been increasingly used to predict brittle fracture of high-strength, low-toughness materials. Modifications to conventional LEFM are needed to deal with fracture in inelastic materials or materials that develop a pseudo-plastic zone (fracture process zone) ahead of the crack tip. Fracture mechanics in the latter type of materials is of particular interest in this study. Also, for concrete composites, there is a need for modifications to relate results from small-scale laboratory tests to full-size structures.

Fracture mechanics considers the existence of defects and flaws for the analysis and design of concrete structures, an advantage over the conventional methods. The presence of pre-existing flaws is inherent to the properties of concrete. Flaws exist as either micro-cracks due to shrinkage or debonded aggregate-paste interfaces (Kaplan 1961; Cho et al. 1984). Another important advantage of using fracture mechanics is that it automatically incorporates size effects typically observed in the brittle failure of concrete structures.

LEFM uses a crack-size-dependent energy criterion to predict catastrophic crack propagation. Fracture mechanics for concrete and reinforced concrete, will, because of its inelastic nature of failure, need the use of additional criteria for the initiation, growth, and stability of cracks.

3.2 Modes of fracture

There are three principal modes of fracture, commonly referred to as the opening mode, the sliding mode, and the tearing mode. The differences between these modes of fracture are in the type of relative displacement between the surfaces of a crack.
3.2.1 Mode I

In the opening mode, the instantaneous displacement of two crack surfaces is in opposite directions and perpendicular to the crack surface, as shown in Figure 12a. Direct tension and flexural action usually cause structural failure in this mode.

3.2.2 Mode II

In the sliding or shearing mode, the instantaneous movements of the crack surfaces are in the opposite direction and perpendicular to the crack front line but in the plane of the crack surface. Plane xy is the plane of symmetry, but cracking with respect to plane xz is asymmetric. This mode of fracture is usually observed when an in-plane shear force is imposed on the structure (Figure 12b).

Figure 12. Fracture failure modes. (a) Mode I – opening mode; (b) Mode II – sliding or shear mode; (c) Mode III – tearing or out-of-plane shearing mode.
3.2.3 Mode III

Figure 12c shows the tearing mode. In this mode, the instantaneous movements of the two crack surfaces are in opposite directions but in the direction of crack front line and in the plane of crack surface. It is asymmetric with respect to both planes \( xy \) and \( xz \). Torsion loading causes this mode of fracture.

3.2.4 Mixed Mode

This mode of fracture depends on the configuration of external loads in relation to the crack location and geometry. Depending on the resultant stress field in the vicinity of the crack tip, one or more of the principal modes may govern fracture. Figure 13 shows a diagonal tension failure in a reinforced concrete beam. This type of failure involves both opening and in-place shearing (sliding) modes of fracture (mixed-modes I and II).

The existence of mixed-mode fracture is uncertain, so an inconclusive debate continues (Jenq and Shah 1986, 1988; Taha and Swartz 1989; Swartz and Taha 1990; Bocca et al. 1991). Many practical failures in concrete structures can be explained at a local level, even if at a macroscopic level some failures appear to be due to mixed mode conditions. Flexural shear cracking, shown in Figure 13, is a typical example of a mixed-mode fracture problem (Modes I and II). In this cracking pattern, the inclined diagonal tension cracks are subjected to normal and shear stresses along the crack face.

Bocca et al. (1991) reported that the mixed-mode fracture energy is found to be of the same order of magnitude as the Mode I fracture energy and
that each elementary crack growth step begins by a Mode I (or opening) mechanism along the curvilinear trajectory. This is particularly true for the larger specimens, where energy dissipation due to friction and interlocking is negligible compared with the energy dissipated by separation.

3.3 Linear elastic fracture mechanics (LEFM)

LEFM is the basic theory of fracture, originated by Griffith (1921) and completed by Irwin (1957) and Rice (1968a, 1968b). LEFM is a highly simplified theory that is applicable to any material as long as certain conditions are met. These conditions are related to the basic ideal situation analyzed in LEFM in which all the material is elastic except in a vanishingly small region (a point) at the crack tip (Bazant and Planas 1998).

Fracture mechanics is a method of characterizing the fracture behavior in structural parameters, stress, and flaw size, which can be used directly as Barsom and Rolfe (1987) stated. They also stated that the science of fracture mechanics can be used to describe quantitatively the trade-offs among stresses, material toughness, and flaw size.

3.3.1 Griffith's fracture criterion

If the amount of energy supplied to the crack tip for an increment of crack growth is equal to or greater than the amount of energy required for crack growth, then this results in catastrophic crack growth (Broek 1987; Anderson 1991). This concept forms the basis of LEFM. The energy supplied to the crack tip, $U$, is usually either the energy released by crack formation or the work done by the applied forces due to additional displacement caused by introducing a crack. $W$ denotes the amount of energy absorbed in the creation of new free surfaces as a result of crack propagation. The Griffith criterion for catastrophic crack growth is then given by

$$\frac{dU}{da} > \frac{dW}{da}$$

(46)

where $da$ is incremental crack growth and $dU/da$ is the energy release rate, $G$. For a center crack plate (crack length $2a$) subjected to a far-field tensile stress, $\sigma$, it can be shown that

$$G = \frac{dU}{da} = \frac{\sigma^2 \pi a}{E}$$

(47)
where $E$ is the modulus of elasticity, and $dW/da$ is crack resistance, $R$, which is given by:

$$R = \frac{dW}{da} = 2\gamma$$

(48)

where $\gamma$ is the specific surface energy, considered a material property. Consequently, if $\sigma^2 a/E > 2\gamma$ or $\sigma > \sqrt{2E\gamma/\pi a}$, the crack will grow catastrophically.

The energy release rate, $G$, for Mode I fracture is denoted as $G_I$. At incipient catastrophic crack growth, $G_I$ equals $G_{IC}$, which is the critical energy release rate ($G_{IC} = 2\gamma$). $G_{IC}$ is considered a material property.

3.3.2 Stress distribution ahead of the crack tip

In 1898, Kirsch determined that the stress concentration at the edge of a circular hole is three times the nominal stress (Figure 14a) (Barsom and Rolfe 1987). Later, in 1913, Ingliss solved the same problem for a more general case of an elliptical hole (Figure 14b) (Barsom and Rolfe 1987; Broek 1987). For that case, the stress concentration factor was obtained as $1 + 2 \times a/b$. A sharp crack in a solid body can be approximated in the limiting case with an elliptical crack where $b \to 0$ (Figure 14c). The related expression for the stress distribution shows that the normal stress at the crack tip will be infinity for a line crack.

Irwin was credited with obtaining mathematical expressions for the stress field near the crack tip (Broek 1987). The stress field ($\sigma_{ij}$) at the crack tip in each of the principal modes of fracture can be obtained in the form

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\phi)$$

(49)

where:

$r$ = distance from the crack tip

$\phi$ = angle of the stress tensor $f_{ij}(\phi)$.
They are shown in Figure 15. Equation 49 is valid for small values of \( r \). The form of \( f_{ij} \) depends on the specimen and the crack geometry. \( K \) in Equation 49 is called the stress intensity factor, and it equals \( K_I, K_{II}, \) or \( K_{III} \), depending on the applicable mode of fracture (I, II, or III, respectively).
Irwin found that the stress and displacement fields in the vicinity of crack tips subjected to the three modes of deformation are given by Equations 50, 51, and 52. Irwin developed these equations using the Westergaard method. The stress distribution for a Mode I fracture (Barsom and Rolfe 1987) can be obtained from

\[ \sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos \left( \frac{\varphi}{2} \right) \left[ 1 - \sin \left( \frac{\varphi}{2} \right) \sin \left( \frac{3\varphi}{2} \right) \right] \]  

(50)

\[ \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos \left( \frac{\varphi}{2} \right) \left[ 1 + \sin \left( \frac{\varphi}{2} \right) \sin \left( \frac{3\varphi}{2} \right) \right] \]  

(51)

\[ \tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos \left( \frac{\varphi}{2} \right) \sin \left( \frac{\varphi}{2} \right) \sin \left( \frac{3\varphi}{2} \right) . \]  

(51)

At \( \varphi = 0 \), one obtains \( \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \). The distribution obtained for \( \sigma_{yy} \) is also shown in Figure 14.

### 3.3.3 Stress intensity factor

The second major achievement in the theoretical foundation of LEFM was the introduction of the stress intensity factor, \( K \), as a parameter for the intensity of stresses close to the crack tip and related to the energy release rate (Bazant and Planas 1998). Ingliss (1913) studied the unexpected failure of naval ships, and Griffith (1921) extended this work using thermodynamical criteria. From this work Irwing (1957) developed the concept of the stress intensity factor.

The stress intensity factor is a measure of the change in stress within the vicinity of the crack tip. Therefore, it is important to know the crack direction and when the crack stops propagating. The stress intensity factor is compared with the critical stress intensity factor to determine whether or not the crack will propagate.

Dimensional analysis can be used to show that the stress intensity factor for Mode I fracture, \( K_I \), has the following form:

\[ K_I = g \sigma \sqrt{\pi a} \]  

(53)
where \( g \) is a non-dimensional function depending on the size and geometry of the crack, the size and geometry of the structural component, and the type of loading. For normal cracks, its value ranges between 1 and 2, but it may be larger for longer cracks. Functions defined for common geometries and loading conditions are available in the published literature (Barsom and Rolfe 1987). The variable \( \sigma \) is the nominal far-field stress and \( 2a \) is the crack length. If \( K_I \) is the same for two cracked bodies, then based on the equations, the same stress field will exist at their crack tips. If the two bodies are made of the same material, an identical response is expected. This fact results in the important conclusion that \( K_I \) can be used as a similarity parameter to compare the response of the same material at the crack tip and also to compare the degree to which materials are influenced by the stress fields.

### 3.3.4 Fracture toughness: critical stress intensity factor

Another important parameter of the linear elastic fracture mechanics is the fracture toughness, \( K_{IC} \) (the capacity). Fracture toughness is a material fracture property (Swamy 1979). If a plate is loaded (Figure 16) to the failure stress, \( \sigma_f \), and if \( \sigma = \sigma_f \) is the value of the nominal far-field stress at failure, then a value of \( K_I \) associated with \( \sigma_f \) could be determined and referred to as \( K_{IC} \). If this toughness property is available for the material, then the failure state stress could be expressed as (Swamy 1979):

\[
\sigma_f = \frac{K_{IC}}{\sqrt{\pi a}}. \tag{54}
\]

![Figure 16. Failure stress related to crack size for an infinitely wide plate subjected to tension (from Shah et al. 1995).](image)
From Figure 16 it can be observed that when the obtained value of $\sigma_f$ exceeds the tensile strength of the material, $f_t$, (or when $a < a_c$, the critical crack length) for a small crack length, the LEFM is not valid. The relation between the energy release rate, $G_I$, and the stress intensity factor, $K_I$, can be shown (Barsom and Rolfe 1987; Bocca et al. 1991) to be of the form

$$G_I = \frac{K_I^2}{E'}$$  \hspace{1cm} (55)

where for plane stress conditions $E' = E$ and for plane strain conditions $E' = E/(1 - v^2)$. Instead of writing the catastrophic crack growth condition in terms of $G_I$ (as $G_I = G_{IC}$), one may alternatively use the condition $K_I = K_{IC}$ ($K_{IC}$ can be obtained using the above relation by substituting $G_I = G_{IC}$). $K_{IC}$ and $G_{IC}$ are not dependent on loading or crack geometry. Like $G_{IC}, K_{IC}$, which is also termed the fracture toughness of the material, is a material property.

3.3.5 Plastic zone and influence of constraint at the crack tip

Realistically, using elastic theories, the materials cannot be expected to sustain the extremely large stresses predicted at the crack tip. The theoretical shape of the local yield (plastic) zone can be determined using the elastic stress distribution obtained in the vicinity of the crack tip and the yield strength of metals (the shaded zone in Figure 15). Elongation in the direction of applied stress is accompanied by negative strains in the other perpendicular directions, because of the influence of Poisson’s ratio. In other words, an elongation occurs in the applied stress direction, and shrinkage occurs in the other direction. Plane stress or plane strain conditions may exist at the crack tip, depending on the type of constraint at the crack tip.

Usually at the surface, the plane stress condition results in a large plastic zone because of the low level of constraint. Also, with the large plastic deformation and more energy absorption, there is more resistance to crack growth. Plane strain conditions prevail in the interior of the specimen, where constraints are more severe. A smaller plastic zone with less energy absorption is typical under the plane strain conditions of crack growth. The thickness of the specimen, as a result, plays a major role in determining the state of stress at the crack tip.
3.4 Nonlinear fracture mechanics for concrete fracture

3.4.1 Characterization of crack growth and associated fracture mechanisms in plain concrete

To study the morphology of the cracks and fracture surfaces in concrete specimens, investigators have used scanning electron microscopy (Du et al. 1987, 1990; Maji and Shah 1989) and other similar high-resolution methods (Shah 1987, 1990a, 1990b; Maji and Shah 1989). It is believed that the failure of concrete results from a large system of densely distributed and randomly oriented micro-cracks. Macroscopically, this has often been observed to result in one large irregular crack. The presence of the dense system of micro-cracks can be attributed to the inhomogeneity of concrete (Swartz and Taha 1991).

The resultant crack tortuosity in three dimensions has often been idealized in analytical models for the fracture of concrete. If the analysis is to reproduce failure in such materials realistically, these idealized models of crack geometry should consider the mechanism of energy absorption. Some investigators (Swamy 1979; Hillerborg and Rots 1989; Swartz and Taha 1991) have tried to treat micro-cracking and macro-cracking of concrete as analytically analogous to yielding and fracture in metals, respectively. Cracks, in such an approach, are treated as discontinuities with idealized geometries, so as to allow simple analytical formulation. Others (Bazant and Oh 1983) modeled the region of densely distributed cracks using the properties of a softening material. Cracks, in such an approach, are treated in a smeared sense as a continuum of softer material (softer than the virgin uncracked material) (Saouma and Ingraffea 1981; Rots et al. 1985; Rots 1991).

Micro-cracks are inherent to the nature of concrete making. Even under a no-load condition, micro-cracks exist in the material. Local tensile stresses caused by combinations of multi-axial loading, temperature, creep, and shrinkage all contribute to the growth of micro-cracks. Macroscopic tensile loading may also be present in many structures. Micro-cracking and related toughening is responsible for inelastic behavior prior to peak loads in plain concrete.

Compared to the size of the specimen or structure, the size of the fracture process zone (the zone of dense micro-cracks) ahead of the crack tip is large. This makes the fracture response in concrete significantly depen-
dent on specimen size. Also, the unrestrained growth of this large process zone depends on the local stress field in the vicinity of the crack tip. This makes the fracture response in concrete dependent on geometry (Reinhardt 1984). Boundary conditions that significantly alter the stress distribution in the vicinity of the crack tip also influence the fracture process in concrete. As a result, LEFM is not suitable for studying fracture of concrete. However, cement paste has been observed to behave in an elastic-brittle manner relative to plain concrete (Mindess 1983). Consequently, Mindess (1983) used LEFM to successfully model crack growth and failure in cement paste.

### 3.4.2 Effect of reinforcement on the fracture of concrete

The presence of reinforcement (bars, tendons, fibers) significantly alters the fracture characteristics of the resultant composite. The reinforcement is significantly more ductile in tension than plain concrete and hence serves as a crack arrestor, responsible for transferring tensile stresses across the matrix crack. Consequently, failure in well-designed reinforced concrete members is not sudden or explosive as in plain concrete. The presence of reinforcement introduces additional mechanisms of energy dissipation. These include energy dissipated due to interface, debonding, and slip; extensive secondary radial cracking in the vicinity of the reinforcement (deformed bars); frictional pull-out of the reinforcement (particularly in fiber-reinforced concrete); and plastic deformation in the reinforcement (yielding in rebars, bending of misaligned fibers during pull-out). Energy absorption in these additional mechanisms should be adequately accounted for in any realistic fracture model for reinforced concrete. Reinforcement, in many instances, also alters the failure mode itself. Models for fracture of reinforced concrete should hence be able to accommodate changes in the failure mode depending on the level of reinforcement used.

In the conventional analysis of reinforced concrete used in the ACI building code (ACI Committee 318 2005), the contribution of concrete in resisting tensile loads at ultimate is neglected. While this assumption offers acceptable predictions for the ultimate capacity strength of statically determinate structural members, it leads to inaccurate deflection and crack width predictions. In situations where the post-cracking structural ductility and rotation capacity have a significant influence on stress redistributions, such as in statically indeterminate structures, the ultimate strength predictions may themselves be in error.
If, as in the fracture-mechanics-based models, the contribution of concrete in tension is not neglected in the analysis of failure in reinforced concrete, then two aspects of this contribution must be considered. The first aspect is that tensile stress can be sustained across cracks in concrete, provided the widths of these cracks are below a critical value. This stress-transferring capability is due to aggregate interlock, microcracking, crack deflection, crack face friction, crack tip blunting by voids, and crack branching and is also manifested as the post-peak softening response of concrete (Figure 17).

Concrete between primary cracks contributes to the stiffness of the cracked reinforced concrete members. This contribution in the post-cracking regime is often referred to as the tensile stiffening effect (Moosecker and Grasser 1981). Figure 18 shows the stress-deformation response of a reinforced concrete member in direct tension. In the uncracked composite, the elastic modulus, $E_c$, can be described using the law of mixtures:
where \( E_m \) and \( E_r \) are the elastic modulus of the concrete matrix and steel reinforcement, respectively, and \( V_r \) is the volume fraction of the reinforcement. If the tensile capacity of concrete is ignored in the post-cracking regime, the composite response predicted is identical to that of the steel reinforcement alone \( (E_c = E_r V_r) \). The experimental stress-deformation response is shown as the solid line in Figure 18. The contribution of the cracked concrete to the post-cracking stiffness of the composite can be determined from the difference in the stress levels between the experimental response and the response marked “steel alone.” The tensile stiffening effect is important for an accurate prediction of crack widths and deflections in reinforced concrete beams.

![Figure 18. Tensile stiffening effect in the post-cracking regime (from RILEM TC 89-FMT 1990).](image)

**3.4.3 Nonlinear fracture models for the tension response of concrete**

Micro-cracking, stable crack growth, and the presence of a large process zone ahead of the traction-free cracks are all interdependent characteristics inherent to the inhomogeneous composition of concrete. As a consequence of these inelastic processes, LEFM, as described before, cannot be directly applied to reinforced concrete.

When concrete is subjected to tensile loads in a stable, displacement-controlled test, the material behaves elastically up to about 35–50 percent of its tensile strength. The coalescence of pre-existing micro-cracks and the development of new micro-cracks are cause for the pre-peak nonlinearity observed in tension experiments. It is generally believed that these micro-cracks are uniformly distributed in the specimen prior to the peak stress level. This makes the strains in the pre-peak region essentially
independent of gage length. In the vicinity of the peak stress, the coalescence of micro-cracks produces a localized zone of cracking.

Some experimental observations indicate that the stress-displacement relationship for concrete subjected to uniaxial tension can be divided into four stages based on the initiation and propagation of internal cracks, as shown in Figure 19 (Shah et al. 1995). The first stage is before point 1, which is about 30 percent of the peak load. Initiation of internal cracks is negligible during this stage. The second stage is from point 1 to point 2, which is about 80 percent of the peak load. Internal cracks initiate and propagate during this stage. These cracks are isolated and randomly distributed over the specimen volume during this second stage. The third stage is between points 2 and 3, where the internal cracks start to localize into a major crack that propagates with increasing load. This phenomenon is known as damage localization or strain localization. The crack length at the peak load is referred to as the critical length crack. The final stage after point 3 occurs after the peak load. The major crack continuously propagates even though the load decreases (point 4). It is important to establish that the tensile strain of the material within the localized damage band increases, whereas unloading may occur for the material outside the damage band.

This critical section is capable of transferring decreasing levels of stress with further increments in displacement (Gopalaratnam and Shah 1985, 1987; Gopalaratnam and Ye 1991). The material outside the critical zone unloads elastically in the post-peak regime of the stress-displacement response. Thus, the strains in the post-peak region depend on the gage length. Displacements across the critical section, however, are relatively independent of the gage length because they are dominated by the crack width in the critical section. This localization of deformation is often referred to as strain softening or the softening response of concrete. Figure 19 illustrates the various stages in the stress-displacement behavior and the localization effect described earlier.

Nonlinear fracture models can be broadly classified into two categories. The first category incorporates the softening phenomenon explicitly. The popular models in this category include the fictitious crack model and the crack band model. The second category incorporates modifications to LEFM. The popular models in this category are the effective crack model and the two-parameter fracture model.
3.4.4 Models incorporating the softening phenomenon explicitly: fictitious crack model (FCM) or cohesive crack model (CCM)

A significant number of models have been developed to analyze the non-linear fracture mechanics. The cohesive crack model or the fictitious crack model is the simplest model that describes in full the progressive fracture mechanics (Anderson 1991). A cohesive crack is a fictitious crack able to
transfer stress from one face to another. This model was introduced by Barenblatt and Dugdale (Bazant and Planas 1998; Shah et al. 1995). They introduce this model to represent different nonlinear processes located at the front of a pre-existing crack.

Trying to extend the concept of a cohesive crack for concrete, Hillerborg et al. (1976) proposed that the cohesive crack may be assumed to develop anywhere, even if no pre-existing macro-crack exists (Anderson 1991). They call this extension of the cohesive crack model the fictitious crack model. A fictitious crack is used to describe the behavior of a pre-existing crack by using the same mathematical formalism as the classical cohesive cracks.

The Hillerborg model (Hillerborg et al. 1976) assumes that the stress displacement behavior ($\sigma$-$\omega$) observed in the damage zone of a tensile specimen is a material property. Figure 20a shows a schematic stress-displacement curve, and Figure 20b illustrates the idealization of the damage zone ahead of a growing crack (Anderson 1991). Equation 57 illustrates the Hillerborg model:

$$G_f = \int_0^{\omega_c} \sigma \times d\omega.$$

![Figure 20. Fictitious crack model for concrete (from Hillerborg et al. 1976).](image)
The main assumption for the Hillerborg model is that the $\sigma$-$w$ relationship is a unique material property, and that assumption is not strictly correct in most cases because the process zone produced during the fracture of concrete is often quite large, and interaction between the process zone and free boundaries can influence the behavior (Anderson 1991).

Hillerborg described the tensile test in the following manner (Figure 21). Up to the peak load, the load increases while the bar strain remains distributed uniformly along the specimen (arc $OP$ in Figure 21). At the peak load, a cohesive crack normal to the axis of the bar appears somewhere in the specimen (at the weakest cross section). After the peak, the crack develops a finite opening, $w$, while still transferring stress, and at the same time, the remainder of the specimen unloads, and its strain decreases uniformly along the arc $PB$. The total elongation at point $A'$ is the addition of a uniform strain corresponding to point $B$ and the crack opening, $w$ (Bazant and Planas 1998):

$$\Delta L = L \times \varepsilon_B + w$$  \hspace{1cm} (58)

where $L$ is the length of the specimen and $\varepsilon_B$ is the strain of the material elements not containing the crack that unload from the peak load. To simplify the computations while retaining the essentials of the model,

---

**Figure 21.** Idealized tensile test of Hillerborg et al. (1976) and Bazant and Planas (1998).
Hillerborg assumed that the inelastic strain in the loading-unloading path was negligible and that the behavior of the bulk material was linear elastic, so, given the softening curve (Figure 22a), the load-elongation curve is constructed as Figure 22b shows. Thus, the post-peak elongation can be computed from Equation 59 as (Bazant and Planas 1998):

$$\Delta L = L \frac{\sigma}{E} + w = L \frac{f(w)}{E} + w$$

(59)

where $E$ is the elastic modulus, and $f(w)$ is a softening function.

This model falls under the category of discrete crack approaches. The model is convenient for use with finite element codes. The input parameters for the model include the uniaxial tensile strength, $f_t$, and the post-peak softening response ($\sigma-w$ relations) of plain concrete. The model adopts a tensile strength criterion for crack initiation and propagation. It is assumed that a crack propagates when the stress at the crack tip reaches the tensile strength. It is also assumed that the relation between closing pressure and crack opening displacement is a material property. The limitation of this model during the earlier implementations was that the crack path had to be established a priori. In many simple loading configurations, this restriction was not a drawback. However, recent trends in the use of joint elements along any region suspected of potential cracking removes this limitation.

As long as the separation between the two crack faces (the crack width), $w$, does not exceed a critical value, $w_e$, the crack is assumed to be capable of transmitting stresses across the crack surfaces. A constitutive law for the
crack is given by the $\sigma-\omega$ curve. For all other parts of the structure/specimen, conventional continuum elements are typically used.

### 3.4.5 Crack band model

Bazant and Oh (1983) modeled the fracture process zone from a band of uniformly and continuously distributed micro-cracks with a fixed critical crack opening displacement, $w_c$ (Figure 23). Using a simple stress-strain relationship (Figure 24), stable crack propagation is then simulated by progressive micro-cracking within this band. The energy consumed by the crack advance per unit area of the crack band, $G_f$, is the product of the area under the stress-strain curve (Figure 24) and the width of the crack band, $w_c$. The Bazant and Oh model is described by the following equation:

\[
G_f = w_c \left( 1 + \frac{E}{E_t} \right) \times \frac{f_t^2}{2E}
\]  

(60)

where $E$ is the modulus of elasticity, $E_t$ is the strain softening modulus, and $f_t$ is the tensile strength of the material. It is seen that, in addition to $E$, three material fracture parameters, $w_c, f_t$, and $E_t$, are required in this model. An approximate function $w_c = n_a \times d_a$ has been proposed to determine the value of $w_c$, where $d_a$ is the maximum aggregate size in the concrete and $n_a$ is an empirical constant. It is suggested that the value of $n_a$ equals three for concrete and five for rock (Shah et al. 1995).

![Figure 23. Microcrack band fracture (from Bazant and Oh 1983).](image)
Since energy cannot be dissipated in a vanishing volume of material (a discrete line crack has no volume), this model treats the deformation localization as a band of distributed cracks (a continuum of softer material). The region of intensely distributed micro-cracks is assumed to be a hypothetical continuum.

The pre-peak and post-peak behaviors are both described by stress-strain relationships (pre-peak modulus $E_1$ and post-peak modulus $E_2$) (Figure 25). The width of the crack band, $w_c$, can be used to relate the stress-strain response to the fracture energy:

$$ G_f = w_c \int_{0}^{\epsilon_0} \sigma \text{d}\epsilon = \frac{w_c \times f_t^2}{2} \left[ \frac{1}{E_1} - \frac{1}{E_2} \right] $$

where $E_1$ is the pre-peak modulus of elasticity and $E_2$ is the post-peak modulus of elasticity (Figures 25a and 25b). This equation considers the energy absorbed in the pre-peak regime, while the equation developed in the cohesive crack model does not.

The critical strain energy release rate, $G_{IC}$, in Mode I fracture (LEFM) should not be confused with $G_f$ (the fracture energy). However, $G_{IC}/G_f$ is a useful measure of how well LEFM describes the fracture process in a structure. For very large concrete structures or for very high strength concrete, which exhibits a steep post-peak tensile response, this ratio approaches unity, indicating that the use of LEFM for failure analysis of the concrete structure/material may be acceptable. For many practical
applications, however, this ratio is much less than unity, indicating that the use of nonlinear fracture models or modified LEFM models may be necessary (Bazant and Cedolin 1976; Ghazavy-Khorasgany and Gopalaratnam 1993; Shah et al. 1995).

Figure 25. Basic models for concrete in tension: (a) experimentally observed stress-displacement response of concrete in tension, (b) fictitious crack model idealization, and (c) crack band model idealization (from ACI Committee 446 1992).
To realistically model the presence of extensive micro-cracks at the crack tip in a computationally convenient manner using the finite element method, Bazant and Oh (1983) proposed modeling the fracture process zone as a pseudo-continuum. This zone ahead of the crack tip is called a crack band. In this smeared crack approach, the material in the fracture process zone is assumed to have softened because of presence of the dense band of micro-cracks. Constitutive properties for the material ahead of the crack tip are changed to account for the inelastic process that takes place within the fracture process zone. Unlike the discrete crack approach, where it may be necessary to change the mesh geometry with crack growth, in the crack band model, the mesh configuration need not change during crack advance. This method greatly reduces the computations required. However, the constitutive law for the crack band should be adjusted so as to satisfy energy dissipation equivalent to $G_f$ for elements undergoing softening. Mesh size in the crack band approach thus governs the constitutive law used for obtaining objective results.

When using the crack band model, there is no displacement jump because of single discrete crack evolution. Instead, the displacement discontinuity at the crack is averaged over one row of elements in the crack band with an assumed stiffness or constitutive law. And, unlike in the fictitious crack model, the crack may grow in any direction, depending on the local stress field ahead of the crack tip and the criteria used for the direction and amount of crack growth.

### 3.4.6 Models that are modifications of LEFM effective crack model

Since cracking in concrete is geometrically complicated to model, a practical yet simple approach is needed to deal with an equivalent crack for which the structural compliance is idealized. Compliance measurement methods can be used to calculate the effective crack length at the instant of fracture in any structure/specimen. The basic principle is to equate the energy absorbed in the actual specimen/structure to the energy necessary for the formation of an equivalent stress-free crack (with no fracture process zone). Specimen/structural compliance at peak load is needed as input to the model.

Karihaloo and Nallathambi (1990) proposed an effective crack fracture model for concrete. In this method using effective crack length, $a_{eff}$, at the peak load, a critical stress intensity factor, $K_{IC}$, is calculated based on an
analytical formula available in the LEFM literature (Barsom and Rolfe 1987; Broek 1987) as:

\[ K_{IC} = \sigma_n \sqrt{a_{eff}} Y(a). \]  (62)

In this expression, \( \sigma_n \) is the bending stress, \( a_{eff} \) is the effective crack length, \( Y \) is a geometry-dependent coefficient, and \( K_{IC} \) is the fracture toughness of concrete. Karihaloo and Nallathambi (1990) observed that the fracture toughness values thus obtained for different specimens are independent of size and geometry.

Karihaloo and Nallathambi (1990) have also determined the fracture toughness of concrete by conducting compression splitting tests (Figure 28b). They obtained \( K_l \) values at crack initiation and at peak load and observed that \( K_{IC} (K_l \text{ at peak load}) \) for concrete is approximately three times the \( K_l \) value at crack initiation. They demonstrated the existence of crack growth resistance or \( R \) curve behavior for concrete.

Although the specimen (Figure 28b) is subjected to compression, the presence of the pre-existing notch produces tension-splitting failure along the specimen center. The energy absorption per unit area, \( R \), obtained by Karihaloo is given by

\[ R = \frac{1}{E I_b} \left[ \frac{P(2t - W)}{16 \cos \theta} \right]^2 \left( 1 - \theta \tan \theta \right) \]  (63)

where:

\[ \theta = \frac{Ck}{2}, \quad k = \sqrt{\frac{P}{2EI}}, \quad \text{and} \quad I = \frac{bt^3}{12} \]  (64)

where:

\( I \) = moment of inertia
\( t = b/2, \)
\( b \) = specimen width
\( C \) = notch size.
3.4.7 Two parameter fracture model

Jenq and Shah (1985) proposed a two-parameter fracture model \((K_{IC} \text { and CTOD}_c)\) where the two parameters are computed for the effective crack based on LEFM. \(K_{IC}\) is the critical stress intensity factor at the effective crack tip. To differentiate it from conventional LEFM-based \(K_{IC}\), they call the critical stress intensity factor based on the effective crack \(K^S_{IC}\). The second parameter used in their model is the critical crack tip opening displacement at the real crack tip. This is denoted as \(\text{CTOD}_C\) and is related to the crack extension at the critical load.

They postulated that at the critical (not necessarily the maximum) load, the stress intensity factor, \(K_I\), and the crack tip opening displacement, \(\text{CTOD}\), reach their critical values \((K^S_{IC} \text { and } \text{CTOD}_C, \text { respectively})\).

Cracking in the two-parameter fracture model has been divided into two stages: pre-critical crack growth and post-critical crack growth. Although the peak load level is also typically the critical point, this is not necessary for all specimen geometries. Crack opening displacement, \(\text{COD}\), at the edge of the crack is termed \(\text{CMOD}\) (crack mouth opening displacement). As long as \(P < 0.5 \, P_{max}\), the \(P-\text{CMOD}\) curve is linear and \(\text{CTOD}\) is negligible.

\(\text{CTOD}\) is assumed to reach a critical value, \(\text{CTOD}_C\), simultaneously as the stress intensity factor at the tip of effective crack reaches \(K^S_{IC}\). After \(P > 0.5 \, P_{max}\), significant slow crack growth and nonlinearity occurs until the \(\text{CTOD}\) reaches its critical value \(\text{CTOD}_C\).

Beyond the critical point the crack is assumed to propagate with \(K^S_{IC}\) remaining constant. \(K^S_{IC}\) is reported to be independent of size and geometry. The RILEM Committee on Fracture Mechanics Testing has published a draft recommendation (RILEM Draft Recommendation 1985) for obtaining \(K^S_{IC}\) and \(\text{CTOD}_C\) from tests on notched beams.

RILEM TC 89-FMT (1990), *Fracture Mechanics of Concrete Test Methods*, specifies the method to determine fracture parameters, critical stress intensity factor, critical crack tip opening displacement \((K^S_{IC} \text { and } \text{CTOD}_C \text { of plain concrete})\), and mortar using three-point bend tests. The test configuration and specimen used are similar to those used to measure the fracture energy, \(G_f\).
Based on this method, the critical stress intensity factor, $K_{IC}$, is defined as the stress intensity factor calculated at the critical effective crack tip, using the measured maximum load. The critical crack tip opening displacement, $CTOD_c$, is defined as the crack tip opening displacement calculated at the original notch tip of the specimen, using the measured maximum load and the critical effective crack length.

The recommendation further establishes that the critical stress intensity factor, $K_{IC}$, and the critical crack tip opening displacement, $CTOD_c$, along with Young's modulus, $E$, are sufficient to characterize the fracture resistance and energy dissipation of plain concrete and mortar.

The Young's modulus, $E$, is calculated from the equation:

$$E = 6Sa_0 V_1(\alpha) / (C_id^2b) \quad (65)$$

in which $C_i$ is the initial compliance calculated from the load-CMOD curve, $V_i(\alpha)$ is a function of test specimen dimensions obtained using LEFM, $S$ is the span used, $d$ and $b$ are cross-section dimensions, and $a_0$ is the notch depth.

The critical effective crack length, $S_c$ ($S_c = s_0 + \text{stable crack growth at peak load}$), is determined from Young's modulus, $E$, calculated above and the unloading compliance, $C_u$, measured at the maximum load. Using an iteration process, the critical effective crack length, $S_c$, is found when the following equation is satisfied:

$$E = \frac{6Sa_c V_1(\alpha)}{C_ud^2b} \quad (66)$$

in which $a_c$ is the critical effective crack length to be determined and $C_u$ is the unloading compliance at 95 percent of peak load.

The critical stress intensity factor is calculated using the following equation:

$$K_{IC}^S = 3(P_{\text{max}} + 0.5W) \frac{S(\pi a_c)^{1/2} F(\alpha)}{2d^2b} \quad (67)$$
in which $F(\alpha)$ is a function obtained through LEFM, where $\alpha = a_c/d$, and $P_{\text{max}}$ is the measured maximum load.

The critical crack tip opening displacement is calculated using the following equation:

$$CTOD_c = \frac{6P_{\text{max}}S\alpha_c V_1(\alpha) V_2(\alpha, \beta)}{Ed^2b}$$  \hspace{1cm} (68)

in which $V_1$ and $V_2$ are polynomial functions obtained through LEFM.

**3.4.8 Cohesive crack models applied to concrete**

The softening curve of the material is the main ingredient of the cohesive crack model (Bazant and Planas 1998). The softening curve replaces the stress-strain curve in theories such as plasticity. A softening curve valid for all concrete does not exist, because every material has its own softening curve determined from experiments. These experiments show that the softening curve is similar in shape for different mixes of ordinary concrete.

This similarity was noted by Petterson (1981), who carried out stable tensile tests to determine the softening curve of concrete. To analyze and infer general trends for particular structures, some investigators introduced analytical expressions to approximate the true softening curve. The simplest softening curve available is the rectangular softening curve (Figure 26). This curve captures the main trends of the fracture processes and the size effect, but it usually overestimates the strength of normal-sized specimens and structures. A linear softening curve was determined by Hillerborg et al. (1976) (Figure 26). This curve is not very realistic because it causes the predicted structure strength to be too high. Thus, Peterson (1981) proposed the bilinear softening curve shown in Figure 26. Although this bilinear softening curve may suffice for most practical purposes, other curves, some smooth, have also been proposed. Various mathematical forms have been proposed, like the exponential, the power law, and the trilinear softening curve (Figure 27).
Figure 26. Rectangular, linear, and Peterson's bilinear softening curves (from Bazant and Planas 1998).

Figure 27. Softening curves: (a) trilinear curve, (b) exponential curve, and (c) power curve (from Shah et al. 1995).

Figure 28. (a) Center-notched three-point bending specimen and (b) compression splitting test configuration used by Karihaloo and co-workers.
Since 1981, the bilinear curves have been accepted as reasonable approximations of the softening curve for concrete. Peterson's bilinear softening curve has its kink point fixed at (0.8, 1/3) in the dimensionless representation, and it becomes zero for $\hat{w} = \hat{w}_c = 3.6$ [i.e. for $w = 3.6(G_f/f_t)$], in which $w$ is the crack opening displacement, $G_f$ is the fracture energy, and $f_t$ is the tensile strength.

### 3.4.9 Energy absorption process and fracture energy

A material can be evaluated and analyzed using energy principles and stress-strain response. For a linear elastic material (Figure 29a), the elastic response terminates and the stress suddenly drops to zero when it reaches the fracture strength of the material (Anderson 1991). For elastic-plastic material (Figure 29b), stress approaches a constant after a certain value of strain. On the other hand, in a quasi-brittle material (Figure 29c), stress gradually decreases after the maximum stress, giving a softening curve as concrete and ceramics do (Anderson 1991).

![Figure 29. Stress-strain response in (a) a linear-elastic material, (b) an elastic-plastic material, and (c) a quasi-brittle material (from Anderson 1991).](image)

In linear elastic materials, any propagation of a crack means a catastrophic failure of the material. However, in a nonlinear material, a crack may steadily propagate until it reaches a critical length. Considering a plate with a crack subjected to a load, $P$, the necessary condition for the plate in equilibrium states during crack propagation is that the first-order derivative of the potential energy, $\Pi$, with respect to the crack length, $a$, is equal to

$$\frac{d\Pi}{da} = \frac{d}{da}(U - F + W) = 0$$

(69)
\[ \frac{d}{da}(F-U) = \frac{dW}{da} \]  

(70)

where \( U \) is the strain energy of the structure, \( F \) is the work done by the applied force, and \( W \) is the energy required for crack formation (Van Mier et al. 1992). By introducing two types of boundary conditions, the fixed load point displacement or the fixed load during crack extension becomes

\[ \varphi = R \]  

(71)

and

\[ \varphi = \frac{1}{t} \left[ \frac{\partial U(a,u)}{\partial a} \right]_u = \frac{1}{t} \left[ \frac{\partial U(a,P)}{\partial a} \right]_p \]  

(72)

where \( \varphi \) is the energy release rate for propagation per unit length of crack in a structure with unit thickness. \( R = \frac{1}{t}(\partial W/\partial a) \) is the fracture resistance of the material, and \( t \) is the thickness of the structure (Anderson 1991). In nonlinear materials, crack propagation may be stable, stationary, or unstable because of the presence of crack arrest mechanisms in the inelastic zone around the crack tip. The stable crack growth means that the crack propagates only when the applied load or displacements increase. The unstable crack growth means that the crack propagates even though the applied load decreases or remains constant. The stationary crack propagation is a critical state between the stable and unstable crack growth. Noting \( \varphi = R - \varphi \), these sufficient conditions can be expressed as (Anderson 1991)

\[ \frac{\partial^2 \varphi}{\partial a^2} = \begin{cases} 
\frac{\partial R}{\partial a} - \varphi > 0 & \text{for stable crack growth} \\
\frac{\partial R}{\partial a} - \varphi = 0 & \text{for stationary crack growth} \\
\frac{\partial R}{\partial a} - \varphi < 0 & \text{for unstable crack growth}
\end{cases} \]  

(73)

Based on the stress displacement response, the total energy dissipated by the concrete specimen, \( E_t \), can be obtained as the sum of the energy dissipated in the fracture process zone, \( E_f \), and that dissipated in the rest of the specimen, \( E_v \), where
\[ E_v = \left[ \int_0^{\delta_{\text{max}}} P d\delta \right] - \left[ \frac{E_c}{2} (\varepsilon_{\text{max}} - \varepsilon_r)^2 V \right]. \]  

(74)

where \(\varepsilon_{\text{max}}\) and \(\varepsilon_r\) are the strain and displacement, respectively, at the peak-load level, \(P_{\text{max}}\), \(E_c\) is the initial elastic modulus, and \(V\) is the volume of the specimen. The second term in Equation 74 is due to the elastic energy stored at the peak-load level (which can be recovered upon unloading). The first term can also be written in terms of stress and strain as

\[ E_v = \left[ \int_0^{\varepsilon_{\text{max}}} \sigma \partial \varepsilon - \frac{E_c}{2} (\varepsilon_{\text{max}} - \varepsilon_r)^2 \right] V. \]  

(75)

The energy dissipated in the post-peak regime is entirely due to the energy absorbed in the fracture process zone, \(E_f\). At this stage, since localization has already occurred, the energy dissipation is essentially planar in nature. The energy absorbed in the fracture process zone can be obtained as:

\[ E_f = \int_0^{\delta_{\text{max}}} p d\delta = A \int_0^{\varepsilon_{\text{max}}} \sigma dw = AG_f \]  

(76)

where \(A\) is the net cross section of the specimen, and \(G_f\) is the fracture energy. \(G_f\) in the above equation is compatible with that defined earlier for the fictitious crack model.

3.4.10 Determination of fracture energy

The fracture energy, \(G_f\), is defined by the amount of energy necessary to create a unit area of stress-free crack surface (RILEM 1985). It is the total amount of energy absorbed in a tensile test to failure, represented by the area below the load-deformation curve for the specimen. This energy can be divided in two parts, as shown in Figures 30b and 30c. The area enclosed in Figure 30b by the \(\sigma-\varepsilon\) curve represents the energy per unit volume absorbed in the whole specimen (Hillerborg et al. 1976). The area below the \(\sigma-\omega\) curve in Figure 30c represents the energy absorbed within the damage zone. This area is denoted as \(G_f\).
The fracture energy, \( G_f \), although ideally obtained in the direct tension configuration, is usually obtained from a three-point flexural test using a notched specimen (RILEM 1985). A flexural test such as the tension and compression tests yields material properties (like strength and fracture energy) that depend on specimen size. The debate over how \( G_f \) should be measured and used in analytical models is still inconclusive.

\( G_f \) is experimentally determined at the present time using the RILEM Draft Recommendation (1985). The draft recommendation specifies the dimensions of the specimen, which is a function of the maximum aggregate size. The specimen geometry used is a center-notched beam tested under three-point bending. The ratio of the notch depth to the beam depth is 0.5. The load deflection response of the beam is obtained in a stable manner using either a very stiff testing machine or closed-loop control. The area under the load deflection curve, \( W_0 \), is used to compute the fracture energy, \( G_f \), after correcting it for the contribution due to specimen weight. Thus,

\[
G = \frac{(W_0 + mg\delta_0)}{A_{lig}} \quad [N/m (J/m^2)] \quad (N/m \text{ or } lb/in.) \quad (77)
\]
where $m = m_1 + m_2$, $g$ is the acceleration due to gravity, $\delta_0$ is the final deformation of the beam at complete fracture, $A_{lig}$ is the net area of the notch, $m_1$ is the mass of the beam between supports, and $m_2$ is the mass of the part of the loading arrangement not attached to the testing machine.

The specimens should be loaded at constant (or almost constant) displacement rates. The loading rates should be such that the maximum load is reached in about five minutes. At least three identical specimens should be tested for each specimen size. All the specimens should be geometrically similar in two dimensions, with the same third dimension (thickness, $b$). This means that the ratios $l/d$, $a/d$, and $L/d$ should be the same for all specimens (Hillerborg et al. 1976; Hillerborg 1983, 1989; Hillerborg and Rots 1989).

### 3.5 Size effect and geometry-dependent behavior of concrete

The size effect is the most compelling reason for design engineers to adopt fracture mechanics (Anderson 1991). It is rigorously defined through a comparison of geometrically similar structures of different sizes. It is well recognized that micro-cracks are inherent in concrete, even before a load is applied. Their presence has been attributed to numerous causes, including differential shrinkage and water gain beneath aggregates causing interfacial debonding. It has been observed that when loading approaches approximately 30 percent of the material tensile strength, micro-cracks at the interface of aggregates and cement paste and possibly in cement paste begin to open and grow. The non-linearity in pre-peak (ascending) regime of the stress deformation curve is attributed to the stable growth of micro-cracks, which finally leads to crack bridging between the aggregates and initiation of macro-cracks. This process, which consumes energy and is responsible for pre-peak nonlinearities, has been identified as the primary cause for the size dependence of fracture behavior in concrete structures (Bazant and Gettu 1990; Gopalaratnam 1993; Bazant 1997).

Figure 31 illustrates the effect of size on structure behavior and properties. Extensive experimental data (Carpinteri 1981; Bazant and Kim 1985) show that the degree of ductility or brittleness of concrete structures depends not only on the properties of constituent materials and reinforcing parameters, but also on the size and geometry of the structure/structural components (Figure 31). Although the uniaxial tensile strength of concrete is generally assumed to be less size dependent, it has been observed that the modulus of rupture of plain concrete is significantly size dependent.
In general, large concrete structural components are more brittle than small components of the same material and quality. Very small concrete structures essentially behave in a size-independent manner, and their behavior can be predicted using a strength criterion. On the other hand, for very large concrete structures, strength is observed to be proportional to $1/\sqrt{d}$, where $d$ is a characteristic dimension of structure. In these cases, LEFM can be successfully applied.

Bazant (1992, 1997) attempted to explain the transition from strength-based limit analysis for very small concrete structures to LEFM for large concrete structures using concepts of size. Such a size effect law can also be used to extrapolate the results from fracture tests on laboratory specimens (typically small) to characterize the behavior of full-sized concrete structures or structural components.

The first study to report on the effect of size on concrete fracture behavior was undertaken by Walsh, according to Mindess (1983). Walsh suggested that valid fracture parameters could be obtained for concrete only by using large specimens. Bazant (1982, 1985, 1989, 1997) further observed that the size effect is one of the major distinguishing features of fracture-
mechanics-based failure theory for concrete compared to the presently used size-independent strength approach.

The blunting action of the micro-cracking in front of the fracture process zone in concrete structures is the reason that nominal failure stress does not obey the LEFM size effect law. The existence of the fracture process zone is also the reason why a plasticity-based analysis is inadequate. To establish the size effect and separate it from other effects, Bazant (1985, 1992) studied geometrically similar specimens of different sizes. The size effect law proposed by Bazant is schematically shown in Figure 32.

![Figure 32. Logarithmic plot of nominal strength versus normalized structural size showing predictions based on plastic-limit analysis, linear elastic fracture mechanics, and nonlinear fracture mechanics (from Bazant 1984).](image)

According to the strength criteria of plastic limit analysis, failure results when the nominal stress, \( \sigma_N \), approaches the material strength \( f_t \) (the tensile strength). This is true in practice only for very small concrete structures. The horizontal asymptote \( AB \) in Figure 32 represents this size-independent strength criterion. This horizontal line is defined by the material strength. The failure criterion based on LEFM shows a strong size dependence (the inclined asymptote with a slope of \(-1/2\)):

\[
\sigma_N = \alpha / \sqrt{d} \quad (78)
\]
where \( d \) is the characteristic dimension of the structure and \( a \) is a coefficient related to specimen and loading geometry. Using a logarithmic transformation, one can obtain

\[
\log \sigma_N = -0.5 \log d + \log a.
\] (79)

Line \( BC \) represents the LEFM failure criterion and has a slope of \(-1/2\). Curve \( AC \) represents the size effect law proposed by Bazant (1992). This law predicts a gradual transition from the size-independent strength criterion observed for small structures to a strong size-dependent behavior as in LEFM for larger structures.

The size effect on the nominal failure stress can be obtained by equating the energy released due to crack growth to the energy released by the stress relieved (the volume in the vicinity of the crack). Bazant has assumed this energy release to be a function of both the length and the area of the crack, leading to a size effect of the form of Equation 80 for geometrically similar structures or structural components. Alternatively, Equation 80 has also been obtained by dimensional analysis and arguments of similitude. The size effect law as expressed by Bazant has the following form:

\[
\sigma_N = B f_t \left(1 + \frac{d}{\lambda_0 d_o}\right)^{-0.5} = B f_t (1 + \beta)^{-0.5}
\] (80)

where \( \beta = d/(\lambda_0 d_o) \) (also denoted as \( d/d_o \)) is the brittleness number, \( d \) is a characteristic dimension of the specimen/structure, and \( \sigma_N \) is the nominal strength \([= P_u/(bd)]\). \( B \) and \( \lambda_0 \) (or \( d_o \)) are empirical coefficients that can be calculated by regression analysis of experimental data obtained from tests on geometrically similar specimens of different sizes, and \( d_o \) is the maximum aggregate size. \( P_u \) is the ultimate or maximum load, \( b \) is the specimen/structural thickness (which is constant for two-dimensional geometric similarity), and \( f_t \) is the tensile strength of the material. \( B \) and \( \lambda_0 \) can, in principle, be obtained by approximate asymptotic analysis, although this is a complicated problem even for simple test geometries.

3.5.1 Determination of parameters for the size effect law

The proposed size effect law has been successfully compared with experimental results of shear failure of reinforced concrete (Bazant and Kim
1985; Bazant and Sun 1987; Bazant and Kazemi 1991; Bazant 1992). For this comparison, Equation 80 is transformed to a linear form by defining

\[ Y = \left( \frac{\sigma_f}{\sigma_N} \right)^2 = \frac{1}{B^2} + \frac{1}{B^2} \lambda_0 x, \quad \text{or} \quad x = \frac{d}{d_a}. \]  

(81)

The intercept and slope of this linear regression analysis are used to obtain \( B \) and \( \lambda_0 \) values needed for the size effect law for the particular specimen geometry.

Modeling the reinforcement in a smeared sense, considering the bond-slip behavior of the bar-concrete interface and using energy concepts, Bazant (1982, 1989, 1992) also investigated the influence of reinforcement and bond-slip relation on the failure of reinforced concrete members. He observed that if reinforcement is present near the crack band front, and if it behaves elastically, the size dependence of the nominal stress at failure is of the same type, but the size range for which the strength criterion is valid is larger, and the transition to the LEFM criterion hence occurs at larger sizes. However, if the reinforcement yields, the size dependence of strength becomes much milder. These observations are based on analytical and numerical studies and are yet to be systematically confirmed by experiments.

Hillerborg (Bazant 1985) compared the Bazant size effect law with finite element calculations based on the fictitious crack model and observed some discrepancy. Later, with a refinement in the size effect law, Bazant found a better agreement with Hillerborg’s test results. The refinement in his size effect law is intended to take into consideration additional influences of the aggregate size. The modified size effect law is given by (Bazant and Cedolin 1982):

\[ \sigma_N = Bf_t \left[ 1 + \left( \frac{c_o}{d_a} \right) \right] \left[ 1 + \left( \frac{d}{d_a \lambda_0} \right) \right] - \frac{1}{2} r \]  

(82)

in which \( c_o \) and \( r \) are empirical constants to be determined from experimental data. For small concrete structures (where \( d/d_a \lambda_0 \ll 1 \)), limited strength criteria govern the nominal failure stress. For very large concrete structures (where \( d/d_a \lambda_0 \gg 1 \)), the nominal failure stress is proportional to \( 1/\sqrt{d} \), and LEFM governs.
3.5.2 Brittleness number

The brittleness of a structural component is not just a function of material properties, as discussed earlier, but it is also a function of the size and geometry of structural components and the loading configuration. Because all of these factors must be considered, it is not easy to find a single process that properly incorporates all these influences. Brittleness number is one parameter that can be effectively used to characterize the fracture behavior of concrete (Gopalaratnam and Shah 1985). Characteristic length ($l_{ch}$), considered a material property, is used to characterize material brittleness. Hillerborg (1983) defined it as:

$$l_{ch} = \frac{EG_f}{(f_t)^2}.$$  \hspace{1cm} (83)

The first idea in quantifying the brittleness is to look for a quantity that is as small as possible for the perfect plasticity limit and infinitely large for the elastic-brittle limit (Anderson 1991). A number with these properties is the ratio $d/l$, which appears in the general size effect law. Therefore, any variable that is proportional to it is a good candidate to be a brittleness number (Anderson 1991):

$$B \propto \frac{d}{l}.$$  \hspace{1cm} (84)

The more brittle (high-strength) or light-weight the concrete is, the lower the $l_{ch}$ values in the range 0.2–1.0 m (4–40 in.), as reported in the literature (Hillerborg et al. 1976). Hillerborg and Rots (1989) used the ratio of the structural dimension, $d$, to the characteristic length of the material as a parameter quantifying the brittleness, $B$, of a structure. Higher values of this ratio result in a more brittle structural response:

$$B = \frac{d}{l_{ch}} = \frac{df_t^2}{EG_f}$$  \hspace{1cm} (85)

in which $G_f$ refers to the fracture energy of the underlying cohesive model and $E$ is the modulus of elasticity. Carpinteri et al. (1985, 1991) used a stress-dependent brittleness number, $S$, given by

$$S = \frac{K_{IC}}{f_t \sqrt{d}}.$$  \hspace{1cm} (86)
It should be noted that $S$ is a measure similar to $B$ proposed by Hillerborg. In fact, $S = B^{0.05}$. In a later modification, Carpinteri suggested that $S_E$ be used in addition to $S$ to characterize the brittleness of a structure:

$$S_E = \frac{G_f}{f_t d}.$$  \hspace{1cm} (87)

The relation between $S$ and $S_E$ is given by

$$S_E = \frac{S^2 f_t}{E}.$$  \hspace{1cm} (88)

Smaller values of $S$ and $S_E$ similar to larger values of $B$ represent higher levels of brittleness. Threshold values of $S$ that are geometry dependent (Carpinteri et al. 1985) have been used to predict brittle fracture in concrete.

Bazant (1982) proposed the use of brittle number, $\beta$, from his size-effect law that is defined as

$$\beta = \frac{d}{d_0 \lambda_0}.$$  \hspace{1cm} (89)

The effects of both size and geometry are incorporated in such a definition of $\beta$ (Bazant 1982). It has been reported that for $\beta \leq 0.1$, strength criteria govern structural failure, while for $\beta \geq 10$, LEFM governs the structural failure. For $0.1 < \beta < 10$, nonlinear fracture mechanics and the size effect law are needed for failure analysis.

The brittleness number is used to compare various materials and sizes for a given structural shape and loading, but they cannot be used directly to compare the brittleness number of different structural geometries, because the dependence of brittleness on geometry is not included in its definition (Anderson 1991).

### 3.6 Finite element application on fracture mechanics

Since Hillerborg and his colleagues (Hillerborg et al. 1976) introduced in 1976 the Dugdale-Barrenblatt-type model to represent the fracture process zone in concrete, many investigators have attempted to define suitable
forms for that and other types of models (Vecchio and Collins 1988). Those investigations involved extensive experimental and numerical studies on the formation and propagation of the fracture process zone in concrete.

The finite element technique has the potential to play an increasingly important role in all areas of reinforced concrete research, design, and analysis. The derivations of a realistic analytical model of concrete behavior and its implementation in nonlinear finite element analysis have long been a major subject of investigation.

3.6.1 Concepts and description

A significant number of models and computer codes using finite element methods have been developed to more accurately analyze nonlinear fracture mechanics in concrete structures. The complexities in developing an analytical model for reinforced concrete are:

- The structural system is three-dimensional and composed of two different materials, concrete and steel.
- The structural system has a continuously changing character because of the cracking of the concrete under increasing load.
- The effects of dowel action in the steel reinforcement bond between the steel reinforcement and the concrete and bond slip are difficult to incorporate into a general analytical model.
- The stress-strain relationship for concrete is nonlinear and is a function of many variables.
- Concrete deformations are influenced by creep and shrinkage and are time dependent.

Ngo and Scordelis (1967) defined finite element analysis as a general method of structural analysis in which a continuous solid is replaced by a finite number of elements interconnected by a finite number of nodal points. Using that approach, a problem in solid mechanics is transformed into a related problem of an articulated structure, which can be analyzed by the standard methods of structural analysis. The key step that influences the accuracy of the finite element analysis is the determination of the finite element matrix and the complexity of the structural mesh used.
3.6.2 Nonlinear finite element analysis on concrete structures

The development of methods and models for predicting and analyzing the nonlinearity of mechanical behavior or the inelasticity of some materials has been an important goal. The effort to find these methods and models has resulted in the development of nonlinear finite element analysis (NLFEA) systems. NLFEA systems were rarely used; the fact that they successfully predicted the behavior of particular structural configurations does not necessarily imply its use.

Most NLFEA systems rely on standard numerical computer-based techniques. Significant differences always remain, mostly related to the analytical description of the various features of concrete behavior, such as the cracking process, the strength, and deformation characteristics (Van Den Berg 1962a, a962b, 1962c). The successful application of NLFEA systems to concrete structural forms predominantly depends on realistic descriptions of material characteristics, such as failure criteria and deformatonal properties of concrete and steel and fracture processes of concrete and concrete-steel iteration. Incorporating such material descriptions into an existing linear solution finite element program led to the development of a nonlinear finite element system suitable for plane stress and asymmetric analyses that yielded realistic predictions of behavior for a wide range of plain and reinforced concrete structural configurations.

Chang et al. (1987) showed a complete time-independent constitutive relation for the nonlinear analysis of reinforced concrete structures. The relation consists of an improved concrete plasticity model, a multiaxial fracture criterion for concrete, a smeared model for concrete cracking, and modeling of post-fracturing behavior via tension stiffening and shear retention effects.

Chang et al. (1987) stated that to consider the combined effect of steel and concrete, a smeared approach or an embedded model may be utilized to evaluate the stiffnesses of steel rebars and concrete in finite element calculations.

Using NLFEA, an analytical model for studying the behavior of the structure through the entire range of loading can be developed. Considering all the factors earlier mentioned, better and accurate data from the analysis of reinforced concrete are obtained for the nonlinear range using finite element methods.
3.6.3 Numerical characterization of the nonlinear fracture process in concrete

Gopalaratnam and Ye (1991) reported that the numerical formulation for nonlinear fracture processes offers a simple alternative to study the growth and development of the fracture process zone. Load deformation characteristics as well as energy absorption history can be generated for stable fracture using an incremental crack tip advance algorithm. The process zone appears to reach a steady state length that depends on both specimen size and test configuration. At peak loads, the process zone size was observed to be approximately 70 percent of its steady state for the range of material parameters, specimen sizes, and specimen geometries investigated. On further crack growth, the process zone size diminished gradually because of a combination of edge effects and compressive stress field that restrained its free growth (Gopalaratnam and Ye 1991).

3.6.4 Crack concepts and numerical modeling

The numerical aspects related to the representation of cracking have a significant influence on NLFEA predictions. These aspects are based on the smeared crack approach and the use of realistic criteria for the onset of cracking and local material failure (Van Den Berg 1962a, 1962b, 1962c). The smeared crack approach spreads the effect of the crack over the area of an element, which corresponds to one integration Gauss point (i.e., over one quarter of the eight-node isoparametric element representing concrete). After cracking has occurred at a given Gauss point, the orientation of the principal stresses does not coincide with the orientation of the existing crack (Van Den Berg 1962a, 1962b, 1962c). As a result, there may be some transfer of force across the crack surfaces.

The complexity of the finite element analysis depends very much on whether the crack path is assumed in advance or not (Zhang and Gjorv 1991). If the crack path is assumed in advance, the finite element mesh is arranged in such a way that the crack either follows along the element boundaries or follows inside the elements parallel to the element boundaries. In the former case, the fracture zone is modeled as a separation of the element along the crack path. This is a pure crack model, often referred to as the fictitious crack model. In the latter case, the fracture zone is modeled as a change in stiffness of a row of elements along the crack path. This is referred as the crack band model (Zhang and Gjorv 1991).
Figure 33 shows an application of the discrete crack approach of finite element analysis and the fictitious crack model and the crack band model.

Figure 33a illustrates the separation of elements with the introduction of closing stresses that depend on the fracture zone deformation, \( w \), which is equal to the node separation distance. Figure 33b illustrates the change of stiffness of a row of elements (Zhang and Gjørv 1991). These two models mainly differ in the finite element formulation, but the numerical results are practically identical.

Figure 33. (a) Fictitious crack model; (b) crack band model (from Hillerborg and Rots 1989).
3.6.5 Discrete crack concept model

The discrete crack approach models a crack by means of a separation between element edges (Zsutty 1968). This implies a continuous change in nodal connectivity and constrains the crack to follow a path along the element edges. For general purposes, this model is not attractive, but a number of special purposes exist for which the drawbacks can be circumvented. For comparative studies and for engineering problems whereby a mechanism of discrete cracks is preimagined in a fashion similar to yield line mechanisms, this model is effective. For such problems, one may predefine the interface elements in the original mesh, keeping the topology preserved. The initial stiffness of the elements is assigned a large dummy value in order to simulate the uncracked state with rigid connection between overlapping nodes (Zsutty 1968). Upon violating a condition of crack initiation, the element stiffness is changed, and a constitutive model for discrete cracks is mobilized.

The viewpoint of the discrete crack model is still macroscopic in principal, with the basic behavior characteristics lumped into the elements. With the cracking passing along the element boundaries, the use of simple elements, such as the constant strain triangle element, is the best suited to concept and application (Nilson 1968).

Nilson (1968) modified this approach to allow the finite element model to generate the location of the cracks. With this representation, cracking based on the average stress exceeds the tensile strength of the concrete for the flexural problems, and the elements are disconnected at their common corners. For cracks at the exterior of the beam, the outside node is separated. For cracks at the interior of the beam, both nodal points are separated. This refined method of representing discrete cracks was further improved and partially automated by Mufti (Mufti 1972). He incorporated a predefined crack using two nodes at one point connected by a linkage element (mentioned earlier). When the stresses in the elements exceed the cracking strength of the concrete, the linkage element is softened to allow the crack to open (Nilson 1968).

The use of discrete cracking representation has received only limited acceptance because of the difficulty in providing an economical redefinition of the structural topology following the formation of cracks (Nilson 1968). For those problems in which dowel forces are important, discrete cracking appears to be a natural tool. Overall, in those cases in which local
material behavior at a particular stage during the life of reinforced concre­te structures is of interest, a discrete cracking model is likely to be the representation of choice and is especially useful for investigating the stresses in a structural member with a known crack location (Nilson 1968).

3.6.6 Smeared crack model

The counterpart of the discrete crack concept is the smeared crack concept, in which a cracked solid is imagined to be a continuum with the notion of stress and strain. The necessity for a cracking model that offers automatic generation of cracks without the redefinition of the finite element topology and completes generality in the possible crack direction has led a vast majority of investigators to adopt the smeared cracking model (Nilson 1968). This procedure was introduced by Rashid (1968). He represented cracked concrete as an orthotropic material. After cracking has occurred, the modulus of elasticity of the material is reduced to zero, perpendicular to the principal tensile stress direction.

Rather than representing a single crack, this procedure has the effect of representing many finely spaced (or smeared) cracks perpendicular to the principal stress direction (Figure 34).

![Figure 34. Idealization of a single crack on the smeared crack model (from Isenberg 1991).](image-url)
The smeared cracks concept can be catalogued into fixed and rotating smeared crack concepts. The smeared cracks concept uses a fixed orientation of the crack during the entire computational process, whereas a rotating concept allows the orientation of the crack to co-rotate with the axes of principal strain (Zwoyer and Siess 1954). Many comparatives studies in this area have been devoted towards distributed fracture. The studies revealed the danger of fixed cracks with significant shear retention. Such models easily produce an overstiff response because the local stress rebuild in inclined directions leads to very severe stress-looking on a global level (Zwoyer and Siess 1954). Basically, the shear retention factor should be taken to be as low as possible, preferably as zero, to improve the fixed smeared crack results. Studies with fixed multi-directional cracks did not yet indicate significant improvements.

It has been demonstrated that even the best possible smeared crack is not free from stress-looking. This is because of the assumption of displacement continuity and the realism of a geometrical discontinuity (Zwoyer and Siess 1954). Another difficulty of this model is the danger of spurious mechanisms. These may hamper convergence and even blow up the entire iterative procedure. The discrete model does not suffer from these phenomena.
4 Non-Linear Fracture Mechanics Numerical Solution for Reinforced Concrete Deep Beams

4.1 Introduction

This chapter presents the numerical analyses of two sizes of geometrically proportionate reinforced concrete beams with normal and high compressive strengths with and without shear reinforcement. The beams were analyzed with shear-span-to-depth ratios \((a/d)\) of 2.5 and 1.5. Figure 35 and Table 1 show the beam size and loading configurations, while Tables 2 and 3 show the material properties and parameters used for the numerical computations. The numerical models were then compared to experimental data (Ghazavy-Khorasgany 1994). Further analyses were also conducted on the larger beams with shear reinforcement. Discussion of results on load displacement, cracking patterns, size effects, and concrete strength is presented in the following section.

Ghazavy-Khorasgany (1994) conducted numerical analyses of two sizes of geometrically proportionate reinforced concrete beams with normal and high compressive strengths with and without shear reinforcement. The beams were analyzed with shear-span-to-depth ratios \((a/d)\) of 2.5 and 1.5. Figure 35 and Table 1 show the beam size and loading configurations, while Tables 2 and 3 show the material properties and parameters used for the numerical computations. The numerical models were then compared to experimental data (Ghazavy-Khorasgany 1994). Further analyses were also conducted on the larger beams with shear reinforcement. Discussion of results on load displacement, cracking patterns, size effects, and concrete strength is presented in the following section.

The finite element package developed by the authors (Riveros 2005; Riveros and Gopalarathman 2008; Riveros and Gopalarathman (in preparation)) was used to performed the nonlinear fracture mechanics analysis on reinforced concrete beams. The system consists of a graphic input/output interface and analysis routines using finite element techniques. Fracture Mechanics Analysis of Reinforced Concrete Beams (FMARCB) is a two-dimensional finite element program with triangular (three and six nodes), isoparametric (four and eight nodes), bar (truss),
Figure 35. Details of beam geometry and loading configuration.

Table 1. Dimensional details of the reinforced concrete beams.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>L, mm (in.)</td>
<td>5486.4 (216.0)</td>
<td>1422.4 (56.0)</td>
</tr>
<tr>
<td>S, mm (in.)</td>
<td>4876.8 (192.0)</td>
<td>1219.2 (48.0)</td>
</tr>
<tr>
<td>H, mm (in.)</td>
<td>914.4 (36.0)</td>
<td>241.3 (9.5)</td>
</tr>
<tr>
<td>b, mm (in.)</td>
<td>152.4 (6.0)</td>
<td>152.4 (6.0)</td>
</tr>
<tr>
<td>d, mm (in.)</td>
<td>812.8 (32.0)</td>
<td>203.2 (8.0)</td>
</tr>
<tr>
<td>a [a/d], mm (in.)</td>
<td>1219.2 [1.5] (48.0)</td>
<td>2032.0 [2.5] (80.0)</td>
</tr>
</tbody>
</table>

Table 2. Material properties experimentally determined by Ghazavy-Khorasgany (1994).

<table>
<thead>
<tr>
<th>MIX</th>
<th>NSC</th>
<th>HSC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
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<td>Test</td>
</tr>
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<td>43.0 (6,238)</td>
</tr>
<tr>
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<td>29,320 (4,252,520)</td>
</tr>
<tr>
<td>$f_r$, MPa (psi)</td>
<td>4.3 (618)</td>
<td>4.6 (664)</td>
</tr>
<tr>
<td>$G_r$, N/mm (lb/in.)</td>
<td>0.10028 (0.57267)</td>
<td>0.09100 (0.51967)</td>
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Table 3a. Numerical model parameters for beams without shear reinforcement.

<table>
<thead>
<tr>
<th></th>
<th>ANW21</th>
<th>ANW11</th>
<th>BNW21</th>
<th>BNW11</th>
<th>AHW21</th>
<th>AHW11</th>
<th>BHW21</th>
<th>BHW11</th>
</tr>
</thead>
<tbody>
<tr>
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<td>31 (4.50E6)</td>
<td>29 (4.25E6)</td>
<td>31 (4.50E6)</td>
<td>29 (4.25E6)</td>
<td>31 (4.50E6)</td>
<td>29 (4.25E6)</td>
<td>31 (4.50E6)</td>
</tr>
<tr>
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<td>68.9 (10,000)</td>
<td>44.8 (6500)</td>
<td>68.9 (10,000)</td>
<td>44.8 (6500)</td>
<td>68.9 (10,000)</td>
<td>44.8 (6500)</td>
<td>68.9 (10,000)</td>
</tr>
<tr>
<td>$Y$</td>
<td>0.18</td>
<td>0.18</td>
<td>0.18</td>
<td>0.18</td>
<td>0.18</td>
<td>0.18</td>
<td>0.18</td>
<td>0.18</td>
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<td>25.4 (1.0)</td>
<td>25.4 (1.0)</td>
<td>25.4 (1.0)</td>
<td>25.4 (1.0)</td>
<td>25.4 (1.0)</td>
<td>25.4 (1.0)</td>
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<td>4.3 (625)</td>
<td>4.1 (600)</td>
<td>4.3 (625)</td>
<td>4.1 (600)</td>
<td>4.3 (625)</td>
<td>4.1 (600)</td>
<td>4.3 (625)</td>
</tr>
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<td>4#8</td>
<td>4#4</td>
<td>2#8</td>
<td>2#4</td>
<td>4#8</td>
<td>4#4</td>
</tr>
<tr>
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<td>209 (30E6)</td>
<td>209 (30E6)</td>
<td>209 (30E6)</td>
<td>209 (30E6)</td>
<td>209 (30E6)</td>
<td>209 (30E6)</td>
<td>209 (30E6)</td>
</tr>
<tr>
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<td>441 (64,000)</td>
<td>462 (67,000)</td>
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</tr>
<tr>
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<td>0.0422 (0.0016)</td>
<td>0.0484 (0.0019)</td>
<td>0.0422 (0.0016)</td>
<td>0.0484 (0.0019)</td>
<td>0.0422 (0.0016)</td>
<td>0.0484 (0.0019)</td>
<td>0.0422 (0.0016)</td>
</tr>
<tr>
<td>$T_{max}$, MPa (psi)</td>
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<td>5.5 (800)</td>
<td>5.5 (800)</td>
<td>5.5 (800)</td>
<td>5.5 (800)</td>
<td>5.5 (800)</td>
<td>5.5 (800)</td>
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<td>0.0127 (0.0005)</td>
<td>0.0127 (0.0005)</td>
<td>0.0127 (0.0005)</td>
<td>0.0127 (0.0005)</td>
<td>0.0127 (0.0005)</td>
<td>0.0127 (0.0005)</td>
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<td>1.02 (0.04)</td>
<td>1.02 (0.04)</td>
<td>1.02 (0.04)</td>
<td>1.02 (0.04)</td>
<td>1.02 (0.04)</td>
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</tr>
</tbody>
</table>

Table 3b. Numerical model parameters for beams with shear reinforcement.

<table>
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<th>AHS22</th>
<th>AHW11</th>
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</thead>
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<td>31 (4.50E6)</td>
<td>29 (4.25E6)</td>
<td>31 (4.50E6)</td>
</tr>
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<td>$f''_c$, MPa (psi)</td>
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<td>68.9 (10,000)</td>
<td>44.8 (6500)</td>
<td>68.9 (10,000)</td>
</tr>
<tr>
<td>$Y$</td>
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<td>0.18</td>
<td>0.18</td>
<td>0.18</td>
</tr>
<tr>
<td>$t$, mm (in.)</td>
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<td>25.4 (1.0)</td>
<td>25.4 (1.0)</td>
<td>25.4 (1.0)</td>
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<td>$f_t$, MPa (psi)</td>
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<td>4.3 (625)</td>
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<td>4.3 (625)</td>
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<td>4#8</td>
<td>4#4</td>
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<td>209 (30E6)</td>
<td>209 (30E6)</td>
<td>209 (30E6)</td>
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<tr>
<td>$f_s$, MPa (psi)</td>
<td>462 (67,000)</td>
<td>441 (64,000)</td>
<td>462 (67,000)</td>
<td>441 (64,000)</td>
</tr>
<tr>
<td>$w_c$, mm (in.)</td>
<td>0.0484 (0.0019)</td>
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<td>$T_{max}$, MPa (psi)</td>
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<td>1.02 (0.04)</td>
<td>1.02 (0.04)</td>
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</table>
and interface (bond-link) elements. The system uses the discrete crack approach with the fictitious crack model (FCM) (Hillerborg et al. 1976; Hillerborg 1983; Bazant 1985, 1992) to represent tensile concrete softening; a confinement concrete model (Shah et al. 1983) to characterize compression softening; a nonlinear bond-slip constitutive model for the bond-slip phenomenon, which is degraded when cracks form across the tensile reinforcement (Hayashi and Kokusho 1986; CEB-FIP 1990); and an elastic, perfectly plastic constitutive model to represent the yielding of the tensile reinforcement.

FMARCB incorporates the Delaunay refinement algorithm (Ruppert 1995) to create a triangular topology that is then transformed into a quadrilateral mesh by the quad-morphing algorithm (Owen et al. 1999). The Delaunay refinement mesh-generation algorithm constructs meshes of triangular elements. The algorithm operates by imposing a Delaunay or constrained Delaunay triangulation that is refined by inserting additional vertices until the mesh meets constraints on element quality and size. These algorithms simultaneously offer theoretical bounds on element quality, edge lengths, and spatial grading of element sizes. They also possess the ability to triangulate general straight-line domains.

Quad-morphing (Owen et al. 1999) is a relatively new technique used for generating quadrilaterals from an existing triangle mesh. Beginning with an initial triangulation, triangles are systematically transformed and combined. Quad-morphing can be categorized as an unstructured, indirect method that utilizes an advancing front algorithm to form an all-quad mesh. As an indirect method, it is able to take advantage of local topology information from the initial triangulation. Unlike other indirect methods, it is able to generate boundary-sensitive rows of elements with few irregular nodes.

The analysis begins with the definition of the finite element model of the continuum in the elastic state. Once the elastic analysis of the system is completed for the first load step, and the principal stresses are extrapolated at the nodes, cracking criteria based on the principal tensile stresses are verified. If the principal tensile stress exceeds the tensile strength, a fictitious crack is incorporated at the location, and automatic remeshing is achieved. Once the system has cracked, the nonlinear solver (Secant Method) is activated. If new cracks and extensions are required after the nonlinear problem satisfies equilibrium for an unbalanced tolerance, the
system is remeshed with the new cracks and the existing crack extensions. It is then calibrated again for the same load step until no new cracks or extensions are required. The process is repeated for each load.

4.2 Experimental evaluation

Ghazavy-Khorasgany and Gopalaratnam (1993) and Ghazavy-Khorasgany (1994) conducted systematic experiments to characterize the structural and material responses of over 150 reinforced concrete deep beams with and without shear reinforcement. The focus of their investigation was to study the effect of material and structural brittleness on shear failure in reinforced concrete beams. The primary variables in their investigation were concrete compressive strength, shear-span-to-depth ratio, and size of the beams. They also studied the effect of lateral reinforcement on the failure of the beams.

The compressive response of concrete was obtained using specimens cored [75-mm (3-in.) diameter, 150-mm (6-in.) length] and tested under specimen displacement controlled conditions to obtain the complete (including post-peak softening) stress-strain response (Table 2) (Gopalaratnam and Shah 1985). Mode I fracture parameters were obtained by testing notched beams on a three-point loading configuration. The fracture energy for normal-strength concrete (NSC) and high-strength concrete (HSC) reported by Ghazavy-Khorasgany (1994) is presented in Table 2.

Ghazavy-Khorasgany tested size A beams in an inverted configuration (tension face up) to allow easier detection of cracks as well as to study the influence of self-weight. The beam was inverted using hooks provided for handling the beam, special fabric slings, and an overhead crane. The inverted beam was placed over temporary supports provided by two mechanical jacks.

He placed the tension face of the beam on two pairs of lubricated loading plates to minimize friction. The mechanical jacks were then used to raise the beam so as to contact the supports. The loading plates were then attached to the compression face of the beam using Hydrostone before the actuators were moved up to contact the loading plates.

For beam size A, Ghazavy-Khorasgany used two ram-displacements that were controlled using a dual ramp command function. An initial slope of 1 in./hr was used for the first half hour. The second slope was 3 in./hr and
was used until failure. Ghazavy-Khorasgany reported that flexural cracking and diagonal crack initiation both occurred well within the first ramp. He preloaded the beams up to approximately 10 percent of the expected flexural cracking load and then unloaded them to check the instrumentation as well as to reduce nonlinearities due to initial seating effects. He used two video cameras, one on each shear span, to record crack growth, failure mode, and failure sequence. Careful visual monitoring of crack growth was also conducted at various loading stages. Since the specimen response was monitored continuously during the test, Ghazavy-Khorasgany was successful in correlating crack initiation and growth with stiffness changes recorded in the load deflection behavior. He typically paused the test at several levels of loading to allow visual monitoring of crack growth.

Data were acquired at 3 s/point/channel for size A beams. Two loads (two load points), two LVDT displacements (net deflections at midspan and one load point), two resistance pot displacements (net deflections at midspan and one load point), and strain (compression face at midspan) were typically monitored.

The test procedures were similar for beam sizes B, C, and D. The major differences were:

- These beams were tested in an upright configuration.
- Only one load (the sum of two load-point loads) was measured.
- The ramp rate used allowed peak loads to occur at approximately the same test duration.

For tests on beam size B, the two ramp rates used were 0.28 in./hr for the first half hour. The second ramp rate was 1 in./hr. Ram displacement control was used for tests on beam size B. The parameters monitored for these tests included total load, ram displacement, two LVDT displacements (net deflections at beam midspan and one load point), and strain (compression face at midspan). A data acquisition rate of 3 s/point/channel was used.

For tests on beam size C, the two ramp rates used were 0.09 in./hr for the first half hour. The second ramp rate was 0.27 in./hr. Similar values for beam size D were 0.036 in./hr and 0.1 in./hr, respectively. Gross specimen displacement at beam midspan was used to control the tests on beam size
C and size D. The parameters monitored for these tests (sizes C and D) included total load ram displacement, two LVDT displacements (net deflections at beam midspan and one load point), and strain (compressive face at midspan). A data acquisition rate of 3 s/point/channel was used for tests on beam sizes C and D.

Ghazavy-Khorasgany used whitewash on the surface of the beams to detect cracks on beam sizes B, C, and D. Since the crack widths were smaller than in beam size A and since the same level of magnification was used for crack detection, the surface coating was observed to be very useful for tests on these three beam sizes.

The effective depths, $d$, used were 50, 100, 200, and 800 mm (2, 4, 8, and 32 in.). Shear-span-to-depth ratios, $a/d$, of 1.5 and 2.5 were also used to characterize different failure modes. Data from the two larger beams with and without shear reinforcement from this study were used in the current numerical investigation (Figure 35 and Table 1). Size A and B beams used commonly available steel reinforcement while the smaller sizes in the earlier study (Ghazavy-Khorasgany 1994) used model reinforcement with significantly different stress-strain response. Load displacement behaviors obtained in the experiments were then used in the numerical model.

### 4.3 Numerical solution without shear reinforcement

#### 4.3.1 Typical load deflection response

Load deformation responses are discussed for sizes A and B. The overall load deformation behavior observed in size A beams is representative of the other sizes. Aspects of the response unique to size B members are also discussed. In the analysis presented here, the results shown include the weight of each member.

Results from the numerical analysis of an NSC beam without lateral reinforcement (Beam ANW21) and a shear-span-to-depth ratio of 2.5 indicated a diagonal tension failure after yielding of the longitudinal steel reinforcement. This type of failure was driven by the unstable crack growth of a flexure shear crack. It is suspected that for beams that fail in this manner (post-yielding shear failure), the shear capacity and flexure capacity are nearly equal (Riveros 2005; Riveros and Gopalaratnam 2008; Riveros and Gopalarathman (in preparation)).
Typically, the load deflection response is linear until the first flexural crack appears in the tension face (Point 1 in Figure 36). Flexural cracks in the inner span of the beam grow in number and size with continued loading. Further loading produces diagonal cracks at the midheight of the beam.

Figure 36. Left and right load displacement curves for ANW21.
This stage in the load-deflection response is denoted as point 2 in Figure 36. At this load level, debonding of the steel begins. With additional load, the bonding capacity deteriorates, reflecting another nonlinear behavior that causes deflections to increase. Also, some flexural cracks that develop in the shear span curve toward midspan at beam midheight. This is shown as point 3 in Figure 36. Longitudinal steel yielding initiates at point 4 in Figure 36.

Figure 37 shows results from the test of an HSC beam without lateral reinforcement (Beam AHW22). The shear-span-to-depth ratio used in the numerical model equals 2.5. The beam failed due to diagonal tension failure. Once again, the failure was driven by the unstable crack growth of a flexure shear crack combined with debonding of the longitudinal reinforcement. However, no yielding of the longitudinal reinforcement was observed (Riveros 2005; Riveros and Gopalaratnam 2008; Riveros and Gopalarathman (in preparation)).

Initial stiffness differences result from the higher modulus for the HSC matrix and the larger steel content used in the HSC beam. The increased load and deflection capacity between diagonal cracking and ultimate capacity depends on the geometry and the material characteristics. For the beam geometry and material properties used in this investigation, the ultimate capacity in all the modes of failure and for all beam sizes was distinct from diagonal cracking.

In the case of NSC and HSC beams without stirrup reinforcement analyzed at an $a/d$ ratio of 1.5, multiple diagonal tension cracks in each shear span (as shown in Figure 43, to be discussed later) were calculated. A combination of ultimate diagonal tension failure and shear compression failures results from the catastrophic growth of these diagonal cracks. Shear compression failure occurred when diagonal cracks penetrated the compression region and compressive strength was reached (Figure 43). Reinforcement yielding began prior to the ultimate failure of the members (Figures 38 and 39, point 1).

Figures 40 and 41 show the load displacement responses for size B. For these beams analyzed with $a/d$ ratios of 1.5 and 2.5, a diagonal compression and a diagonal shear failure similar to the one discussed for size A beam were observed. However, fewer cracks were produced.
Figure 37. Left and right load displacement curves for AHW22.
Figure 38. Left and right load displacement curves for ANW11.
Figure 39. Left and right load displacement curves for AHW11.
Figure 40. Load displacement curves for BNW21 and BHW21.
Figure 41. Load displacement curves for BNW11 and BHW11.
4.3.2 General observations on the crack patterns

Figure 42 includes numerical and experimental cracking patterns of the two beam sizes without stirrup reinforcement for NSC beams with an \( a/d \) ratio of 2.5 and HSC beams with an \( a/d \) ratio of 1.5. NSC beams with an \( a/d \) ratio of 1.5 and HSC beams with an \( a/d \) ratio of 2.5 had more similar cracking patterns than the beams with similar \( a/d \) ratios.

Figure 43 shows a unique type of failure: shear compression after yielding of longitudinal steel (AWH11). For the \( a/d \) ratio used (1.5) and the material parameters chosen for HSC (Table 2), the shear capacity and the flexural capacity are nearly comparable. The diagonal crack penetrated the compression region, and compressive strength was reached. Failure in NSC beams without lateral reinforcement was observed to be often accompanied by debonding of the longitudinal reinforcement (Figure 42).

For both the NSC and HSC beams, the general inclination of diagonal cracks was influenced by the \( a/d \) ratios used. For an \( a/d \) ratio of 1.5, the main diagonal cracks appeared to span from the support to the load point in each shear span (Figure 43). For an \( a/d \) ratio of 2.5, the diagonal cracks were generally z-shaped, often connected with debonding of the longitudinal reinforcement (Figure 42). Debonding starts when the first flexural crack crosses the longitudinal reinforcement and ends with catastrophic diagonal tension shear failure. The general crack and failure patterns obtained from the numerical analysis in each case are comparable with those obtained from the experiments of Ghazavy-Khorasgany (1994).

Crack patterns and failure modes in the smaller (size B) NSC and HSC beams without stirrup reinforcement for \( a/d \) ratios of 1.5 and 2.5 are similar to those for size A beams analyzed at the same \( a/d \) ratios. Generally, the numerical model predicted fewer cracks, which for smaller beams (size B) is similar to experimental observations (Ghazavy-Khorasgany 1994).
Figure 42. Final cracking pattern for (a) ANW21 numerical model, (b) ANW21 experiment, (c) BNW21 numerical model, and (d) BNW21 experiment.
4.3.3 Influence of specimen size

Diagonal crack initiation has been reported to be less size dependent than ultimate failure in shear failure (Walraven 1990). This observation is based on the analysis made for this investigation and it was also reported by Ghazavy-Khorasgany (1994). A closer examination of the numerical models showed that the size effect at diagonal crack initiation was only
marginally less size dependent than that at ultimate failure for very deep beams without stirrup reinforcement. The strength at diagonal crack initiation dropped by 42 percent for an increase in effective depth from 0.16 to 0.93 m compared to an approximately 47 percent drop in the ultimate capacity for a corresponding increase in specimen depth (struts and tie action in the postdiagonal cracking regime reported for these specimens). In light of these observations and difference in size effect that is well within the range for concrete specimens, an equally plausible conclusion is that the size effect can be documented in both the diagonal crack initiation and the ultimate strength for beams that fail in shear. It should be noted that conclusions on the extent of the size effect at ultimate capacity are strongly dependent on the failure mode (Riveros 2005; Riveros and Gopalaratnam 2008; Riveros and Gopalarathman (in preparation)).

As reported by Ghazavy-Khorasgany (1994) and also found in this investigation, the size effect is milder for an $a/d$ ratio of 1.5 than for one of 2.5 for both the NSC and HSC beams. The strength reduction due to shear failure as a function of $a/d$ ratios can be treated as a geometry- or structural-configuration-related brittleness. If brittleness and the size effect are implicitly related, as implied in fracture mechanics analysis, it is not surprising that an $a/d$ ratio of 2.5 would exhibit a stronger size effect. The ultimate shear strength presents a marginally milder size effect than that observed for diagonal cracking. It should be pointed out that these observations are from failures where yielding of longitudinal steel preceded ultimate failure in shear, implying that the shear capacity may be comparable to the flexural capacity. The ultimate shear strength for HSC beams also exhibits only a mild size effect in spite of the fact that most failures are shear failures that occur prior to yielding of the longitudinal steel. The size effect is not significantly different from that observed at diagonal crack initiation. This is somewhat similar to the deep beam test reported by Walraven (1990). Caution should be exercised in making generalizations regarding the influence of the size effect on the ultimate capacity of reinforced concrete beams, particularly when comparing failure types that are not identical.

### 4.3.4 Influence of concrete strength

The stresses at flexural cracking, diagonal crack initiation, and ultimate capacity are all larger for the HSC beams than for NSC beams. It was expected that the size effect would be stronger for the HSC beams than for the NSC beams. No conclusive observations could be made with respect to
the size effect either in diagonal crack initiation or at ultimate capacity. The size effect with regard to diagonal crack initiation was observed to be comparable in the two concrete materials. The size effect at ultimate capacity, even with the slightly different failure modes (for beams without stirrup reinforcement), was again comparable. One possible explanation for the lack of distinct difference in size effect between the two concrete materials is that, even though the compressive strength ratio was 1.7, the tensile strength ratio was approximately 1.3. Perhaps if the compressive strengths differed by a greater amount, one could have seen a stronger size effect for the HSC material.

4.4 Numerical solution with shear reinforcement

Numerical analyses were conducted for the size A beams with shear reinforcement. The spacing and shear reinforcement content follows that specified in Ghazavy-Khorasgany (1994). Because of the confinement introduced by the shear reinforcement, a plane strain assumption was utilized in the analysis. The beam geometry is shown in Figure 35 and Table 1, and the material properties for size A beams are shown in Table 2.

The confinement induced by the shear reinforcement was applied to the numerical model by means of an equivalent confinement pressure. It was assumed that the efficiency of the confinement pressure induced by the rectangular hoops was 50 percent of that provided by spirals.

4.4.1 Typical load deflection response

The results from the numerical analysis of an NSC beam with lateral reinforcement (Beam ANS22) and a shear-span-to-depth ratio of 2.5 indicated a flexural failure after yielding of the longitudinal steel reinforcement. One of the factors of this ductile type of failure is the confinement pressure provided by the shear reinforcement, which reduces the initiation and growth of tension shear cracks. Furthermore, the confinement pressure provides additional bonding capacity, limiting the debonding of the tensile reinforcement. Load displacement responses for size A beams with shear reinforcement are shown in Figures 44–47. The capacity of this beam increased by 10 percent compared with the beam without shear reinforcement; however, the main contribution was that the failure mode changed from brittle to ductile.
Figure 44. Left and right load displacement curves for ANS22.
Figure 45. Left and right load displacement curves for AHS22.
Figure 46. Left and right load displacement curves for ANS11.
Figure 47. Left and right load displacement curves for AHS11.
In the case of NSC and HSC beams with stirrup reinforcement analyzed at an $a/d$ ratio of 1.5, flexure failure occurred after yielding of the tensile reinforcement prior to crushing of the concrete. Once again, the confinement pressure provided by shear reinforcement delayed the initiation and catastrophic propagation of the diagonal tension cracks in each shear span (Figures 46 and 47).

In general, the numerical results show that the presence of confinement pressure equivalent to the shear reinforcement does not make a significant difference in the performance of the size A beams until the initiation of the diagonal cracks. Furthermore, the amount of confinement pressure equivalent to the stirrup spacing will greatly alter the failure mode in reinforced concrete beams. The general load deflection curves obtained from the numerical analysis in each case are comparable with those obtained from the experiments.

4.4.2 General observations on the crack patterns

Figure 48 includes numerical cracking patterns of size A beams with stirrup reinforcement (confined pressure) for NSC and HSC beams with $a/d$ ratios of 2.5 and 1.5, respectively.

All size A beams with shear reinforcement failed in flexure. For the $a/d$ ratio of 2.5 and the material parameters chosen for NSC and HSC (Table 1), a reduction of the amount of debonding was observed in addition to a delay in the formation of the flexure shear cracks. For both the NSC and HSC beams, the general inclination of diagonal cracks was influenced by the $a/d$ ratios used; however, diagonal cracks did not propagate in an unstable manner, allowing the tensile reinforcement to yield prior to the crushing of the concrete. For an $a/d$ ratio of 1.5, the main diagonal cracks appeared to span from the support to the load point in each shear span (Figure 48). For an $a/d$ ratio of 2.5, the diagonal cracks were generally z-shaped, often connected with a reduced amount of debonding of the longitudinal reinforcement (Figure 48). Debonding started when the first flexural crack crossed the longitudinal reinforcement; however, the bonding capacity was larger because of the confinement pressure provided by the shear reinforcement, which allows a ductile type of failure.
Figure 48. Final numerical cracking pattern for beams with shear reinforcement.
5 Conclusions and Recommendations

5.1 Conclusions

The nonlinear fracture-mechanics-based numerical model developed here has unique features such as automated crack initiation and propagation, automated remeshing, and solution of the nonlinear problems (concrete softening in tension and compression, bond slip, and yielding of reinforcement).

The system was developed to study the shear behavior of reinforced concrete deep beams and has been validated with eight beams of two sizes with different material properties and loading geometries. The model successfully predicted the ultimate capacity of these beams. The model shows good correlation between the predicted cracking pattern and the experimental cracking pattern. It also predicted the load displacement response successfully. Bond-slip characteristics exert a significant influence on the load deflection characteristics of the reinforced concrete deep beams. The model also shows no need to use the shear capacity for tension softening.

5.1.1 Size effect in strength and deformation capacity

Brittle shear failure in NSC and HSC beams without stirrup reinforcement exhibited measurable effects of size on strength, as well as corresponding deflections for effective beam depths of 0.2 and 0.8 m (8 and 32 in.). The stress at diagonal crack initiation was observed to be size dependent. A size effect was also observed for the ultimate shear capacity. In this investigation, the size effect at the ultimate shear capacity was not significantly different from that observed for diagonal crack initiation. The size effect on the deflection capacity at diagonal crack initiation found in the numerical analysis is of practical relevance in design. Although a direct comparison of the size effect on the deflection value at the ultimate capacity was not made because of the differences in failure mechanisms, a size effect similar to that at diagonal crack initiation was observed at this loading. The flexural strength of plain concrete and the corresponding deflection capacity were observed to be size dependent for the range of beam depths investigated [0.24–0.9 m (9.5–36 in.)]. These size effect conclusions follow those reported by Ghazavy-Khorasgany (1994).
5.1.2 Shear-span-to-depth ratio and geometry-related brittleness

For the two shear-span-to-depth ratios investigated (1.5 and 2.5), the failure in beams without stirrup reinforcement was due predominantly to diagonal tension and shear compression. The reduction in shear capacity compared to the flexural capacity in all cases investigated was more severe for the \( a/d \) ratio of 2.5. This is in line with Kani’s shear valley concept. Distinct changes in crack patterns and the resultant mode of failure also accompanied changes in the \( a/d \) ratio. The size effect was greater at an \( a/d \) ratio of 2.5. This observation can be treated as geometry-related brittleness in analytical models. Once again, these conclusions follow those reported by Ghazavy-Khorasgany (1994).

5.1.3 Concrete compressive strength

The shear strength of HSC [compressive strength of 70 MPa (10,000 psi)] was markedly higher than that of NSC [compressive strength 43 MPa (6,250 psi)] at diagonal crack initiation and at ultimate capacity. The HSC was more brittle than the NSC, but no noticeable differences in the size effect on failure loads were observed. Ghazavy-Khorasgany (1994) reported similar findings in his experimental program.

5.1.4 Influence of stirrup reinforcement

When shear reinforcement is provided, and these stirrups work appropriately, the potential for brittle shear failure is reduced. In the numerical evaluations, it was found that the failure mode changed from brittle shear to a ductile flexure. However, shear reinforcement becomes effective after diagonal crack initiation. Therefore, the presence of stirrups does not modify the size effect in strength and deflection capacity observed at this stage of loading. In the experimental evaluation, Ghazavy-Khorasgany (1994) found similar results.

5.1.5 Brittleness characterization and its application

Fracture-mechanics-based brittleness characterization offers promise as far as design for shear capacity is concerned. It is possible to incorporate the effect of material and structural brittleness in an elegant and rational manner through the use of the size effect law proposed by Bazant. Even while good correlation was observed for the size effect law predictions of the ultimate capacity based on results from this investigation, refinements to better model the influence of reinforcement (longitudinal as well as
lateral) are needed before specific proposals for changes to the ACI shear design equation can be made.

5.2 Recommendations for Further Research

It is timely to initiate the process of incorporating into design practice many of the observations with regard to brittle failure made here as well as in other similar recent studies. An energy-based design approach as a complement to existing strength-based concepts should be developed for use in design applications where strength may not be of primary concern or where serviceability, ductility requirements, and energy absorption capacity become important.

Numerical investigations on how shear failure is influenced by steel concrete bond characteristics, especially when multiple layers of longitudinal steel reinforcement are used and different bond-slip characteristics are implemented. These investigations, in addition to different degrees of confinement pressure, may help in the determination of an enhance shear formulation.

Also, the numerical system can be used to study different loadings and geometrical configurations to predict the shear capacity in any type of reinforced concrete system. This type of analysis will provide a better understanding of size effects and loading geometry for shear design approach.
References


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A significant number of failures in reinforced concrete structures initiate in tension regions promoted by stress risers such as areas of high-stress concentrations or pre-existing cracks. Stable growth of these tensile cracks until peak loads are reached is associated with the development of large zones of fracture [fracture process zone (FPZ)]. The growth of the FPZ introduces the effect of structure size on the failure loads. An energy approach based on fracture mechanics concepts can be used to rationally analyze and design for size effects in brittle failures. Current design equations were developed based on strength analysis (as in the current American Concrete Institute code) where the margin of safety is higher for smaller structures than for larger ones. It is also conceivable that this approach would lead to unconservative designs for some very large structures (e.g., deep slabs for underground storage tanks). Since the empirical formulations of the code are based on data for concrete of normal strength, it places a limit on the maximum strength that can be effectively used in the design equations. As a result, promising high-performance concrete cannot be used to its fullest potential. Revisions to the shear design formulations are needed to ensure a uniform margin of safety for members of all sizes, strength, and geometries. This report describes a finite element analysis of (Continued)
14. ABSTRACT (Concluded)

reinforced concrete deep beams using nonlinear fracture mechanics. The development of a numerical model that incorporates compression and tension softening of concrete, bond slip between concrete and reinforcement, and the yielding of the longitudinal steel reinforcement is presented and discussed. The development also incorporates the Delaunay refinement algorithm to create a triangular topology that is then transformed into a quadrilateral mesh by the quad-morphing algorithm. These two techniques allow automatic remeshing using the discrete crack approach. Nonlinear fracture mechanics is incorporated using the fictitious crack model and the principal tensile strength for crack initiation and propagation. The model has been successful in reproducing the load deflections, cracking patterns and size effects observed in experiments of normal- and high-strength concrete deep beams with and without shear reinforcement.

15. SUBJECT TERMS (Concluded)

Bond slip
High compressive strength
Non-linear finite element analysis
Non-linear fracture mechanics
Normal compressive strength
Reinforced concrete deep beams
Shear
Size effects